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ON TYPE I BLOW UP FOR THE NAVIER-STOKES EQUATIONS NEAR THE BOUNDARY

Abstract. For suitable weak solutions to the Navier-stokes equations a new sufficient condition for the uniform boundeness of the scale invariant energy functionals near a boundary point is established.

§1. Introduction and Main Results

Denote $\mathcal{C} := \{ \ x \in \mathbb{R}^3 : \ x_1^2 + x_2^2 < 1, \ |x_3| < 1 \ \}$, and $\mathcal{Q} := \mathcal{C} \times (-1, 0)$. We consider the Navier-Stokes equations in $\mathcal{Q}$

$$\left\{ \begin{array}{c} \partial_t u - \Delta u + (u \cdot \nabla) u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{array} \right. \quad \text{in} \ \ \mathcal{Q}. \tag{1.1}$$

Here $u : \mathcal{Q} \to \mathbb{R}^3$ is the velocity field and $p : \mathcal{Q} \to \mathbb{R}$ is pressure. Together with the system (1.1) we consider the Navier-Stokes equations near the boundary:

$$\left\{ \begin{array}{c} \partial_t u - \Delta u + (u \cdot \nabla) u + \nabla p = 0 \\ \operatorname{div} u = 0 \\ u|_{x_3=0} = 0 \end{array} \right. \quad \text{in} \quad \mathcal{Q}^+. \tag{1.2}$$

We use notation $\mathcal{C}^+ := \mathcal{C} \cap \{x_3 > 0\}$ and $\mathcal{Q}^+ := \mathcal{C}^+ \times (-1, 0)$.

In this paper we are interested in the local regularity for weak solutions to the systems (1.2) satisfying the estimate

$$|u(x,t)| \leqslant \frac{C}{\sqrt{x_1^2 + x_2^2}} \tag{1.3}$$

for a.e. $(x, t) \in \mathcal{Q}$ with some positive constant C.

Our interest is partly motivated by the study of the possible behavior of axially symmetric solutions to the Navier–Stokes equations near the boundary (that is why we use cylinders $\mathcal{C}$, $\mathcal{C}^+$ etc rather than standard

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balls). We remind that that the solution u and p of (1.1) or (1.2) is called axially symmetric if

$$u(x,t) = u_r(r,z,t)\mathbf{e}_r + u_\varphi(r,z,t)\mathbf{e}_\varphi + u_r(r,z,t)\mathbf{e}_z, \quad p(x,t) = p(r,z,t),$$

where $\mathbf{e}_r$, $\mathbf{e}_\varphi$, $\mathbf{e}_z$ is the cylindrical basis in $\mathbb{R}^3$, $r = \sqrt{x_1^2 + x_2^2}$, $z = x_3$. We say the solution is axi-symmetric without swirl if

$$u(x,t) = u_r(r,z,t)\mathbf{e}_r + u_r(r,z,t)\mathbf{e}_z, \qquad p(x,t) = p(r,z,t).$$

For axially symmetric solutions the condition (1.3) is one of the scale invariant conditions which characterize so called Type I blow up at the axis of symmetry, see terminology in [15] or [20].

It is well-known that in the internal case axi-symmetric solutions without swirl are locally regular, see [7,9] and [6]. In contrast, in the boundary case the corresponding result is unknown and an axi-symmetric solution without swirl potentially can have a singularity near the origin (i.e. at the point of the intersection of the axis of symmetry with the boundary of the domain, see, for example, [5]).

On the other hand in the internal case in [6] and [15] it was proved that axi-symmetric weak solutions with swirl satisfying (1.3) are regular. The analogues result near the boundary is unknown.

In our approach we replace the condition (1.3) by a more general condition

$$\sup_{r<1} A_w(u,r) \leqslant C_0, \tag{1.4}$$

where

$$A_w(u,r) := \frac{1}{\sqrt{r}} \ \operatorname{esssup}_{t\in(-r^2,0)} \|u(\cdot,t)\|_{L_{2,w}(\mathcal{C}^+(r))}.$$

We denote $\mathcal{C}^+(r) := \{ \ x \in \mathbb{R}^3 : \ \sqrt{x_1^2 + x_2^2} < r, \ 0 < x_3 < r \ \}$ and for any domain $\Omega \subset \mathbb{R}^3$ $L_{2,w}(\Omega)$ is a weak Lebesgue space equipped with the quasinorm

$$\|f\|_{L_{2,w}(\Omega)} := \sup_{\lambda>0} \ \lambda \ |\{x \in \Omega : |f(x)| > \lambda\}|^{1/2}$$

Note that every measurable function u satisfying (1.3) meets the condition (1.4) as well.

To formulate our main results we remind the notion of the boundary suitable weak solutions. The notion of suitable weak solutions to the Navier–Stokes system was introduced in the celebrated paper [2]. For the boundary case we use the following definition, see, for example, [18] (the notation for functional spaces are explained at the end of this section):

Definition 1.1. *We say that a pair of functions u and p are a boundary suitable weak solution to the Navier–Stokes system in $\mathcal{Q}^+$ if*

- $u \in L_{2,\infty}(\mathcal{Q}^+) \cap W_2^{1,0}(\mathcal{Q}^+)$, $p \in L_{\frac{3}{2}}(\mathcal{Q}^+)$
- $u|_{x_3=0} = 0$ *in the sense of traces*
- *u and p satisfy the Navier–Stokes system in $\mathcal{Q}^+$ in the sense of distributions*
- *for a.a. $t \in (-1,0)$, the pair u and p satisfies the local energy inequality in Q^+*

$$\begin{aligned} &\int\limits_{\mathcal{C}^+} \zeta(x,t)|u(x,t)|^2\, dx + 2\int\limits_{-1}^{t}\int\limits_{\mathcal{C}^+} \zeta|\nabla u|^2\, dxdt \\ \leqslant &\int\limits_{-1}^{t}\int\limits_{\mathcal{C}^+} |u|^2\, (\partial_t\zeta + \Delta\zeta)\, dxdt + \int\limits_{-1}^{t}\int\limits_{\mathcal{C}^+} u\cdot\nabla\zeta\, \big(|u|^2 + 2p\big)\, dxdt \end{aligned} \tag{1.5}$$

for any non-negative test function $\zeta \in C^\infty(\mathbb{R}^3\times\mathbb{R})$ vanishing near the parabolic boundary $\partial'\mathcal{Q} = (\partial\mathcal{C}\times[-1,0]) \cup (\bar{\mathcal{C}}\times\{t=-1\})$.

To formulate our results we also introduce scale invariant functionals:

$$\begin{aligned} A(u,r) &= \operatorname{esssup}_{t\in(-r^2,0)} \Big(\frac{1}{r}\int\limits_{\mathcal{C}^+(r)} |u(x,t)|^2\, dx \Big)^{1/2}, \\ C(u,r) &= \Big(\frac{1}{r^2}\int\limits_{\mathcal{Q}^+(r)} |u(x,t)|^3\, dxdt \Big)^{1/3}, \\ E(u,r) &= \Big(\frac{1}{r}\int\limits_{\mathcal{Q}^+(r)} |\nabla u(x,t)|^2\, dxdt \Big)^{1/2}, \\ D(p,r) &= \Big(\frac{1}{r^2}\int\limits_{\mathcal{Q}^+(r)} |p(x,t) - [p]_{\mathcal{C}^+(r)}(t)|^{3/2}\, dxdt \Big)^{2/3}. \end{aligned} \tag{1.6}$$

The main result of the present paper is the following theorem:

Theorem 1.1. *Assume u and p are a boundary suitable weak solution to the system* (1.2). *Assume there exists $C_0 > 0$ such that the condition* (1.4) *holds. Then*

$$\sup_{r<1}\Big(A(u,r) + C(u,r) + E(u,r) + D(p,r)\Big) < +\infty. \tag{1.7}$$

Theorem 1.1 implies that suitable weak solutions with $A_w(u,r)$–norm uniformly bounded in r can have only Type I singularities at the origin (see the terminology in [20]).

It is interesting to compare our result with other known results in the area. The first important result was obtained in [14] in the internal case. Namely, in [14] it was shown that if

$$\min\Big\{\sup_{r<1} A(u,r),\ \sup_{r<1} C(u,r),\ \sup_{r<1} E(u,r)\Big\} < +\infty$$

then (1.7) holds. In [10] the same result was proved near the boundary. In [13] (see also [21]) an analogues result was established in the internal case under the condition

$$\sup_{r<1} C_{s,l}(u,r) < +\infty, \quad \max\Big\{2-\frac{1}{l}, \frac{3}{2}+\frac{1}{2l}\Big\} < \frac{3}{s}+\frac{2}{l} < 2,$$

where $s \in (3,+\infty)$, $l \in (2,+\infty)$ and

$$C_{s,l}(u,r) := r^{1-\frac{3}{s}-\frac{2}{l}}\Big(\int\limits_{-r^2}^{0}\Big(\int\limits_{B(r)} |u|^s\,dx\Big)^{l/s} dt\Big)^{1/l}$$

Under the assumption (1.3) the statements (1.7) in the internal case was proved in [15].

The condition (1.4) also can be interpreted as a condition

$$\operatorname{esssup}_{t\in(-1,0)} \|u(\,\cdot\,,t)\|_X < +\infty \tag{1.8}$$

where X is some Morrey-type class with the scale-invariant quasi-norm

$$\|w\|_X = \sup_{r<1} \frac{1}{\sqrt{r}}\|w\|_{L_{2,w}(\mathcal{C}^+(r))}$$

The statement (1.7) is known in the internal case if (1.8) is satisfied and X is one of the following spaces: $X = L_3(\mathcal{C})$, $L_{3,w}(\mathcal{C})$, $BMO^{-1}(\mathcal{C})$ (for the explanation of the notation at the end of this section). Namely, in the case $X = L_3$ the condition (1.8) implies Hölder continuity of u near the origin both in the internal and in the boundary cases, see [3] and [12]. In the case of $X = L_{3,w}$ the regularity of u is unknown and only the estimate (1.7) is available (this result follows easily from our Theorem 1.1). In the case of $X = BMO^{-1}$ the estimate (1.7) was obtained in in the internal case in [8] and [16]. Moreover, in the paper of G. Seregin and D. Zhou [22] a similar result is proved in the internal case if X is (globally defined) Besov space $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$.

A simple consequence of our approach is the following ε-regularity condition. Similar conditions in the boundary case can be found in [17]:

Theorem 1.2. *There exists an absolute constant $\varepsilon > 0$ such that if a boundary suitable suitable weak solution u and p to* (1.2) *satisfies the condition* (1.3) *with $C_0 < \varepsilon$ then u is Hölder continuous in some neighborhood of the origin.*

Our paper is organized as follows. In Section 2 we recall some known facts from the theory of functions. In Section 3 we prove Theorems 1.1 and 1.2.

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We use the following notation:

- $\mathbb{R}^3_+ := \{x \in \mathbb{R}^3 : \ x_3 > 0\}$
- $\mathcal{C}(r) := \{x \in \mathbb{R}^3 : \ \sqrt{x_1^2 + x_2^2} < r, \ |x_3| < r\}, \ \ \mathcal{C} := \mathcal{C}(1)$
- $\mathcal{C}^+(r) := \mathcal{C}^+(r) \cap \mathbb{R}^3_+, \quad \mathcal{C}^+ := \mathcal{C} \cap \mathbb{R}^3_+$
- $L_q(\Omega)$, $W^k_q(\Omega)$, $\overset{\circ}{W}{}^k_q(\Omega)$ are standard Lebesgue and Sobolev spaces
- for any measurable $f : \Omega \to \mathbb{R}$ we define

$$d_f(\lambda) := |\{\, x \in \Omega : \ |f(x)| \geqslant \lambda \,\}|$$

- for $q \in [1, +\infty)$ $\ L_{q,w}(\Omega)$ is a weak Lebesgue space equipped with the quasi-norm

$$\|f\|_{L_{q,w}(\Omega)} := \sup_{\lambda > 0} \ \lambda \, d_f(\lambda)^{1/q}$$

  For $q = \infty$ we take $L_{\infty,w}(\Omega) := L_\infty(\Omega)$
- for $q \in (0, +\infty)$ and $s \in (0, +\infty)$ we denote by $L^{q,s}(\Omega)$ the Lorentz space equipped with the quasi-norm

$$\|f\|_{L^{q,s}(\Omega)} \ := \ q^{\frac{1}{s}} \ \left(\int\limits_0^{+\infty} \lambda^{s-1} d_f(\lambda)^{\frac{s}{q}} \ d\lambda \right)^{\frac{1}{s}} \tag{1.9}$$

  If $s = \infty$ we put $\ L^{q,\infty}(\Omega) := L_{q,w}(\Omega)$.

- $BMO(\Omega)$ is the space of functions with bounded mean oscillation in Ω equipped with the norm

$$\|f\|_{BMO(\Omega)} := \sup_{B(x_0,R)\subset\Omega} \frac{1}{|B(R)|} \int\limits_{B(x_0,R)} |f - [f]_{B(x_0,R)}|\, dx,$$

$$[f]_{B(x_0,R)} := \frac{1}{|B(R)|} \int\limits_{B(x_0,R)} f dx$$

$BMO^{-1}(\Omega) := \{\operatorname{div} F \in \mathcal{D}'(\Omega) :\ F \in BMO(\Omega)\}$
- $\mathcal{Q}(r) := \mathcal{C}(r) \times (-r^2, 0)$, $\mathcal{Q} := \mathcal{Q}(1)$
- By $[p]_{\mathcal{C}}$ and $(p)_{\mathcal{Q}}$ we denote the spatial and the total averages of the function $p(x,t)$:

$$[p]_{\mathcal{C}}(t) := \frac{1}{|\mathcal{C}|} \int\limits_{\mathcal{C}} p(x,t)\ dx, \qquad (p)_{\mathcal{Q}} := \frac{1}{|\mathcal{Q}|} \int\limits_{\mathcal{Q}} p(x,t)\ dxdt$$

- $\mathcal{Q}^+(r) := \mathcal{C}^+(r) \times (-r^2, 0)$, $\mathcal{Q}^+ := \mathcal{Q}^+(1)$
- $L_{q,l}(\mathcal{Q}(r))$ is the anisotropic Lebesgue space equipped with the norm

$$\|f\|_{L_{q,l}(\mathcal{Q}(r))} := \Big(\int\limits_{-r^2}^{0} \|f(\cdot,t)\|^l_{L_q(\mathcal{C}(r))} dt \Big)^{1/l},$$

in the case $l = \infty$ $\quad L_{q,\infty}(\mathcal{Q}(r)) := L_\infty(-r^2, 0; L_q(\mathcal{C}(r)))$,

$$\|f\|_{L_{q,\infty}(\mathcal{Q}(r))} := \operatorname{esssup}_{t\in(-r^2,0)} \|f(\cdot,t)\|_{L_q(\mathcal{C}(r))}$$

- $W^{1,0}_{q,l}(\mathcal{Q}(r)) := \{u \in L_{q,l}(\mathcal{Q}(r)) :\ \nabla u \in L_{q,l}(\mathcal{Q}(r))\}$,

$$\|u\|_{W^{1,0}_{q,l}(\mathcal{Q}(r))} := \|u\|_{L_{q,l}(\mathcal{Q}(r))} + \|\nabla u\|_{L_{q,l}(\mathcal{Q}(r))}$$

- $W^{2,1}_{q,l}(\mathcal{Q}(r)) := \{u \in W^{1,0}_{q,l}(\mathcal{Q}(r)) :\ \nabla^2 u \in L_{q,l}(\mathcal{Q}(r)),\ \partial_t u \in L_{q,l}(\mathcal{Q}(r))\}$,

$$\|u\|_{W^{2,1}_{q,l}(\mathcal{Q}(r))} := \|u\|_{W^{1,0}_{q,l}(\mathcal{Q}(r))} + \|\nabla^2 u\|_{L_{q,l}(\mathcal{Q}(r))} + \|\partial_t u\|_{L_{q,l}(\mathcal{Q}(r))}$$

In the case of $q = l$ we denote $W^{1,0}_q(\mathcal{Q}) := W^{1,0}_{q,q}(\mathcal{Q})$ etc
- $L_{q,w;\infty}(\mathcal{Q}(r)) := L_\infty(-r^2, 0; L_{q,w}(\mathcal{C}(r)))$

§2. Some results from the function theory

First we recall an interpolation result concerning Lorentz spaces, see [1, Theorem 5.3.1]:

Lemma 2.1. *Assume* $1 \leqslant q_1 < q < q_2 \leqslant \infty$ *and* $\theta \in (0,1)$ *satisfy*

$$\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}.$$

Then for any $0 < s \leqslant \infty$ *there is a constant* $c = c(q_1, q_2, q, s) > 0$ *such that for any domain* $\Omega \subset \mathbb{R}^n$ *if* $u \in L_{q_1,w}(\Omega) \cap L_{q_2,w}(\Omega)$ *then* $u \in L^{q,s}(\Omega)$ *and the estimate*

$$\|u\|_{L^{q,s}(\Omega)} \ \leqslant \ c \ \|u\|^{1-\theta}_{L_{q_1,w}(\Omega)} \|u\|^{\theta}_{L_{q_2,w}(\Omega)} \tag{2.1}$$

holds.

The next result is a trivial combination of Lemma 2.1 and Sobolev's imbedding theorem:

Lemma 2.2. *Assume* $1 \leqslant q \leqslant p \leqslant 6$ *and* $\theta \in [0,1]$ *satisfy*

$$\frac{1}{p} = \frac{1-\theta}{q} + \frac{\theta}{6}.$$

Then for any $f \in L_{q,w}(\mathcal{C}^+(r)) \cap W^1_2(\mathcal{C}^+(r))$ *the inclusion* $f \in L_p(\mathcal{C}^+(r))$ *holds and there exists a positive constant* $c = c(p,q)$ *(independent on* $r > 0$*) such that if* f *additionally satisfies* $f|_{x_3=0} = 0$ *then*

$$\|f\|_{L_p(\mathcal{C}^+(r))} \leqslant c \ \|f\|^{1-\theta}_{L_{q,w}(\mathcal{C}^+(r))} \|\nabla f\|^{\theta}_{L_2(\mathcal{C}^+(r))} \tag{2.2}$$

Next we recall the well known O'Neils inequality, see [4, Exercise 1.4.19]:

Lemma 2.3. *If* $q_1,\ q_2,\ q \in (1, +\infty]$ *and* $s_1,\ s_2,\ s \in (0, +\infty]$ *satisfy*

$$\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q} \quad \textit{and} \quad \frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{s}$$

then

$$\|fg\|_{L^{q,s}(\mathcal{C}^+(r))} \leqslant c(q_1, q_2, s_1, s_2) \ \|f\|_{L^{q_1,s_1}(\mathcal{C}^+(r))} \|g\|_{L^{q_2,s_2}(\mathcal{C}^+(r))}$$

We will use the following modification of the O'Neils inequality for three functions:

Lemma 2.4. *If q_1, q_2, q_3, $q \in (1, +\infty]$ and s_1, s_2, s_3, $s \in (0, +\infty]$ satisfy*

$$\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = \frac{1}{q} \quad \textit{and} \quad \frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3} = \frac{1}{s}$$

then

$$\|fgh\|_{L^{q,s}(\mathcal{C}^+(r))} \leqslant c(q_i, s_i)\ \|f\|_{L^{q_1,s_1}(\mathcal{C}^+(r))}\|g\|_{L^{q_2,s_2}(\mathcal{C}^+(r))}\|h\|_{L^{q_3,s_3}(\mathcal{C}^+(r))} \tag{2.3}$$

§3. Proof of the Main Results

We start with the following interpolation inequality. Below we denote $A_w(r) := A_w(u, r)$, $C(r) := C(u, r)$ etc, see the definition of the functionals in (1.6).

Theorem 3.1. *Let u and p be a boundary suitable weak solution to the Navier–Stokes equations in Q^+. Then the following inequality holds:*

$$C(r) \leqslant c\ A_w^{\frac{1}{2}}(r)\ E^{\frac{1}{2}}(r) \tag{3.1}$$

Proof. We apply (2.2) with $p = 3,\ q = 2$, and $\theta = \frac{1}{2}$, which gives the result. □

Our next result is the estimate of the pressure. This estimate is the crucial point of our approach and it is different from the analogues estimate in the internal case as it involves the stronger (energy-type) norms in the right-hand side. This leads to additional technical difficulties which do not arise in the internal case. To obtain the result we adopt the technique developed in [11] for the study of the boundary regularity to the Navier–Stokes equations.

Theorem 3.2. *For any $\delta \in (0, 1)$ there exist positive constants c_1, c_2 such that for any boundary suitable weak solution u and p to the Navier–Stokes equation in $\mathcal{Q}^+$ the following inequality holds:*

$$D(\theta r) \leqslant c_1\theta^{\frac{4}{3}}\big(C(r) + D(r)\big) + c_2\theta^{-\frac{4}{3}}E^{1+\delta}(r)A_w^{1-\delta}(r), \tag{3.2}$$

for any $r \in (0, 1)$ and any $\theta \in (0, \frac{1}{2})$.

Proof. First we prove (3.2) for $r = 1$. We decompose $p = p_1 + p_2$ and $u = u_1 + u_2$ where u_1 and p_1 are the solution to the initial-boundary value problem for the following linear system:

$$\begin{cases} \partial_t u_1 - \Delta u_1 + \nabla p_1 = (u\cdot\nabla)u \\ \operatorname{div} u_1 = 0 \qquad\qquad \text{in} \quad \mathcal{Q}^+. \\ u_1|_{\partial\mathcal{Q}^+} = 0 \end{cases} \tag{3.3}$$

Then $u_2 = u - u_1$ and $p_2 = p - p_1$ satisfy following system

$$\begin{cases} \partial_t u_2 - \Delta u_2 + \nabla p_2 = 0 \\ \operatorname{div} u_2 = 0 \qquad\qquad \text{in} \quad \mathcal{Q}^+. \\ u_2|_{x_3=0} = 0 \end{cases}$$

Moreover, we can assume that for a.e. $t \in (-1,0)$ $[p]_{\mathcal{C}^+} = [p_1]_{\mathcal{C}^+} = [p_2]_{\mathcal{C}^+} = 0$. The right-hand side $(u\cdot\nabla)u$ in the system (3.3) belongs to $L_{\frac{9}{8},\frac{3}{2}}(\mathcal{Q})$. Applying the coercive estimate of solutions to the Stokes problem in anisotropic Sobolev spaces (see [23]) for any $\varepsilon \in (0,\frac{1}{8}]$ we obtain

$$\|u_1\|_{W^{2,1}_{1+\varepsilon,\frac{3}{2}}(\mathcal{Q}^+)} + \|\nabla p_1\|_{L_{1+\varepsilon,\frac{3}{2}}(\mathcal{Q}^+)} \leqslant c\,\|(u\cdot\nabla)u\|_{L_{1+\varepsilon,\frac{3}{2}}(\mathcal{Q}^+)}$$

To estimate the right-hand side of the last inequality we split

$$|(u\cdot\nabla)u| \leqslant |u|^{\frac{1}{3}}|u|^{\frac{2}{3}}|\nabla u|$$

and apply (2.3) with exponents $q_1 = 2,\ q_2 = 3,\ q_3 = \frac{6(1+\varepsilon)}{1-5\varepsilon}$ and $r_1 = 2,\ r_2 = \infty,\ r_3 = \frac{2(1+\varepsilon)}{1-\varepsilon}$:

$$\frac{1}{1+\varepsilon} = \frac{1}{2} + \frac{1}{3} + \frac{1-5\varepsilon}{6(1+\varepsilon)}, \qquad \frac{1}{1+\varepsilon} = \frac{1}{2} + \frac{1}{\infty} + \frac{1-\varepsilon}{2(1+\varepsilon)}$$

For a.e. $t \in (-1,0)$ we obtain

$$\|(u\cdot\nabla)u\|_{L_{1+\varepsilon}(\mathcal{C}^+)} \leqslant c\,\|\nabla u\|_{L_2(\mathcal{C}^+)}\||u|^{\frac{2}{3}}\|_{L_{3,w}(\mathcal{C}^+)}\||u|^{\frac{1}{3}}\|_{L^{\frac{6(1+\varepsilon)}{1-5\varepsilon},\frac{2(1+\varepsilon)}{1-\varepsilon}}(\mathcal{C}^+)}$$

Taking into account the property of the Lorentz norm $\big\||u|^\theta\big\|_{L^{q,s}(\mathcal{C}^+)} = \|u\|^\theta_{L^{\theta q,\theta s}(\mathcal{C}^+)}$, where $q,\theta \in (0,+\infty)$ and $s \in (0,+\infty]$, we obtain

$$\|(u\cdot\nabla)u\|_{L_{1+\varepsilon}(\mathcal{C}^+)} \leqslant c\,\|\nabla u\|_{L_2(\mathcal{C}^+)}\|u\|^{\frac{2}{3}}_{L_{2,w}(\mathcal{C}^+)}\|u\|^{\frac{1}{3}}_{L^{\frac{2(1+\varepsilon)}{1-5\varepsilon},\frac{2(1+\varepsilon)}{3(1-\varepsilon)}}(\mathcal{C}^+)}$$

Applying the Hölder inequality with exponents $l_1 = 2$, $l_2 = \infty$, $l_3 = 6$,

$$\frac{2}{3} = \frac{1}{2} + \frac{1}{\infty} + \frac{1}{6},$$

we arrive at

$$\|(u\cdot\nabla)u\|_{L_{1+\varepsilon,\frac32}(\mathcal{Q}^+)}$$
$$\leqslant c\,\|\nabla u\|_{L_2(\mathcal{Q}^+)}\,\|u\|^{\frac23}_{L_{2,w;\infty}(\mathcal{Q}^+)}\,\|u\|^{\frac13}_{L_2\left(-r^2,0;\,L^{\frac{2(1+\varepsilon)}{1-5\varepsilon},\frac{2(1+\varepsilon)}{3(1-\varepsilon)}}(\mathcal{C}^+)\right)}$$

Using (2.1) we get:

$$\|u\|_{L^{\frac{2(1+\varepsilon)}{1-5\varepsilon},\frac{2(1+\varepsilon)}{3(1-\varepsilon)}}(\mathcal{C}^+)}\leqslant c\,\|u\|^{1-\delta'}_{L_{2,w}(\mathcal{C}^+(r))}\|u\|^{\delta'}_{L_6(\mathcal{C}^+)}$$

Here $\frac{1-5\varepsilon}{2(1+\varepsilon)}=\frac{1-\delta'}{2}+\frac{\delta'}{6}$ and $\delta'=\frac{3\varepsilon}{1-5\varepsilon}$. Now using Sobolev embedding theorem we obtain

$$\|u\|_{L^{\frac{2(1+\varepsilon)}{1-5\varepsilon},\frac{2(1+\varepsilon)}{3(1-\varepsilon)}}(\mathcal{C}^+)}\leqslant c\,\|u\|^{1-\delta'}_{L_{2,w}(\mathcal{C}^+)}\|\nabla u\|^{\delta'}_{L_2(\mathcal{C}^+)}$$

Therefore

$$\begin{aligned}\|u\cdot\nabla u\|_{L_{1+\varepsilon,\frac32}(\mathcal{Q}^+)}&\leqslant c\,\|\nabla u\|^{1+\frac{\delta'}{3}}_{L_2(\mathcal{Q}^+)}\|u\|^{\frac23+\frac{1-\delta'}{3}}_{L_{2,w;\infty}(\mathcal{Q}^+)}\\&=c\,\|\nabla u\|^{1+\delta}_{L_2(\mathcal{Q}^+)}\|u\|^{1-\delta}_{L_{2,w;\infty}(\mathcal{Q}^+)}\end{aligned}$$

where $\delta:=\frac{\delta'}{3}=\frac{\varepsilon}{1-5\varepsilon}$. So, we finally obtain

$$\|u_1\|_{W^{2,1}_{1+\varepsilon,\frac32}(\mathcal{Q}^+)}+\|\nabla p_1\|_{L_{1+\varepsilon,\frac32}(\mathcal{Q}^+)}\leqslant c\,\|\nabla u\|^{1+\delta}_{L_2(\mathcal{Q}^+)}\|u\|^{1-\delta}_{L_{2,w;\infty}(\mathcal{Q}^+)}$$

Now we turn to the derivation of the estimate for p_2. From the local regularity theory for the linear Stokes system near the boundary (see, for example, [18, Theorem 2.3]) for any $m\in(1,+\infty)$, we conclude $p_2\in W^{1,0}_{m,\frac32}(\mathcal{Q}^+(\frac12))$ and for any $\rho<\frac12$ the following estimate holds:

$$\begin{aligned}&\|\nabla p_2\|_{L_{m,\frac32}(\mathcal{Q}^+(\frac12))}\leqslant c\left(\|u_2\|_{L_{1+\varepsilon,\frac32}(\mathcal{Q}^+)}+\|p_2\|_{L_{1+\varepsilon,\frac32}(\mathcal{Q}^+)}\right)\\&\leqslant c\left(\|u_1\|_{L_{1+\varepsilon,\frac32}(\mathcal{Q}^+)}+\|u\|_{L_{1+\varepsilon,\frac32}(\mathcal{Q}^+)}+\|p\|_{L_{1+\varepsilon,\frac32}(\mathcal{Q}^+)}+\|p_1\|_{L_{1+\varepsilon,\frac32}(\mathcal{Q}^+)}\right)\\&\leqslant c\left(\|u\|_{L_{1+\varepsilon,\frac32}(\mathcal{Q}^+)}+\|p\|_{L_{1+\varepsilon,\frac32}(\mathcal{Q}^+)}+\|\nabla u\|^{1+\delta}_{L_2(\mathcal{Q}^+)}\|u\|^{1-\delta}_{L_{2,w;\infty}(\mathcal{Q}^+)}\right)\quad(3.4)\end{aligned}$$

Taking any $\theta<\frac12$ and using Poincare inequality we obtain

$$\|p_2-[p_2]_{\mathcal{C}^+(\theta)}\|_{L_{\frac32}(\mathcal{Q}^+(\theta))}\leqslant c\,\theta^{\beta}\|\nabla p_2\|_{L_{m,\frac32}(\mathcal{Q}^+(\frac12))}$$

where $\beta > 0$ depends on $m \in (1, +\infty)$. Choosing $m = 9$ we get $\beta = \frac{8}{3}$. Finally, we can estimate $p = p_1 + p_2$:

$$\|p - [p]_{\mathcal{C}^+(\theta)}\|_{L_{\frac{3}{2}}(\mathcal{Q}^+(\theta))} \leqslant 2\|p_1\|_{L_{\frac{3}{2}}(\mathcal{Q}^+(\theta))} + \|p_2 - [p_2]_{C^+(\theta)}\|_{L_{\frac{3}{2}}(\mathcal{Q}^+(\theta))}$$

$$\leqslant c\Big(\|\nabla u\|^{1+\delta}_{L_2(\mathcal{Q}^+)} \|u\|^{1-\delta}_{L_{2,w;\infty}(\mathcal{Q}^+)} + \theta^{\frac{8}{3}} \|\nabla p_2\|_{L_{9,\frac{3}{2}}(\mathcal{Q}^+(\frac{1}{2}))}\Big)$$

Taking into account (3.4) with $m = 9$ we obtain

$$\|p - [p]_{\mathcal{C}^+(\theta)}\|_{L_{\frac{3}{2}}(\mathcal{Q}^+(\theta))}$$

$$\leqslant c\|\nabla u\|^{1+\delta}_{L_2(Q^+)} \|u\|^{1-\delta}_{L_{2,w;\infty}(\mathcal{Q}^+)} + c\theta^{\frac{8}{3}}\Big(\|u\|_{L_{1+\varepsilon,\frac{3}{2}}(\mathcal{Q}^+)} + \|p\|_{L_{\frac{3}{2}}(\mathcal{Q}^+)}\Big) \quad (3.5)$$

Using the definition of the functionals $D(r) := D(p, r)$ etc we arrive at the estimate

$$D(\theta) \leqslant c_1\, \theta^{\frac{4}{3}}(C(1) + D(1)) + c_2\theta^{-\frac{4}{3}} E^{1+\delta}(1) A_w^{1-\delta}(1)$$

To finish the proof we use the standard scaling arguments and get (3.2) for any $r \in (0, 1)$ and any $\theta \in (0, \frac{1}{2})$. □

Now we can give the proof of our Theorem 1.1:

Proof. As u and p are a boundary suitable weak solution and (3.1) holds we have:

$$\sup_{r<1} A_w(r) \leqslant C_0, \qquad A\Big(\frac{3}{4}\Big) + E\Big(\frac{3}{4}\Big) \leqslant C_1 < \infty$$

From (3.1) we obtain

$$C(r) \leqslant c(C_0)\; E^{\frac{1}{2}}(r) \quad (3.6)$$

Denote $\mathcal{E}(r) = E(r) + A(r) + D(r)$. Using the local energy inequality (1.5) for any $\theta \in (0, \frac{1}{16})$ we get:

$$\mathcal{E}(\theta r) \leqslant C(2\theta r) + C^{\frac{3}{2}}(2\theta r) + C^{\frac{1}{2}}(2\theta r) D^{\frac{1}{2}}(2\theta r) + D(\theta r)$$

applying Young's inequality we get:

$$\mathcal{E}(\theta r) \leqslant c\,(C(2\theta r) + C^{\frac{3}{2}}(2\theta r) + D(2\theta r)) \quad (3.7)$$

Taking in (3.2) $\delta = \frac{1}{7}$ with the help of (3.6) we obtain

$$D(\theta r) + D(2\theta r) \leqslant c\,\theta^{\frac{4}{3}}\Big[C\Big(\frac{r}{4}\Big) + D\Big(\frac{r}{4}\Big)\Big] \;+\; c(C_0)\theta^{-\frac{4}{3}} E^{\frac{8}{7}}\Big(\frac{r}{4}\Big) \quad (3.8)$$

Combining (3.6), (3.7) and (3.8) we get

$$\mathcal{E}(\theta r) \leqslant c(C_0)\Big[E^{\frac12}(2\theta r) + E^{\frac34}(2\theta r) + \theta^{-\frac43}E^{\frac87}\Big(\frac{r}{4}\Big)\Big]$$
$$+ c\theta^{\frac43}\Big(C\Big(\frac{r}{4}\Big) + D\Big(\frac{r}{4}\Big)\Big) \leqslant c(C_0)\Big[\theta^{-\frac14}\mathcal{E}^{\frac12}(r) + \theta^{-\frac38}\mathcal{E}^{\frac34}(r)$$
$$+ \theta^{-\frac43}E^{\frac87}\Big(\frac{r}{4}\Big)\Big] + c\theta^{\frac43}\mathcal{E}\Big(\frac{r}{4}\Big) \tag{3.9}$$

One of the terms in the right hand side of (3.9) has an exponent $\frac87 > 1$. So, to estimate it we use (3.6) and (3.7) again:

$$E^{\frac87}(\frac{r}{4}) \leqslant \Big(C\Big(\frac{r}{2}\Big) + C^{\frac32}\Big(\frac{r}{2}\Big) + D\Big(\frac{r}{2}\Big)\Big)^{\frac87}$$
$$\leqslant c(C_0)\Big(\mathcal{E}^{\frac12}(r) + \mathcal{E}^{\frac34}(r)\Big)^{\frac87} \leqslant c(C_0)\Big(\mathcal{E}^{\frac47}(r) + \mathcal{E}^{\frac67}(r)\Big)$$

Combining the last estimate with (3.9) we arrive at

$$\mathcal{E}(\theta r) \leqslant c(C_0)\Big[\theta^{-\frac14}\mathcal{E}^{\frac12}(r) + \theta^{-\frac38}\mathcal{E}^{\frac34}(r) + \theta^{-\frac43}\Big(\mathcal{E}^{\frac47}(r) + \mathcal{E}^{\frac67}(r)\Big)\Big] + c\theta^{\frac43}\mathcal{E}(r).$$

Taking $\varepsilon > 0$ and using Young's inequality $\theta^\beta\mathcal{E}^\alpha(r) \leqslant \varepsilon\mathcal{E}(r) + c(\varepsilon,\theta,\alpha,\beta)$ for any $\alpha < 1$, $\beta \in \mathbb{R}$, we proceed to

$$\mathcal{E}(\theta r) \leqslant \mathcal{E}(r)(\varepsilon + c\theta^{\frac43}) + F(\varepsilon, C_0, \theta)$$

where $F(\varepsilon, C, \theta)$ is some continuous function which in nondecreasing with respect to C and has the property

$$\text{for any fixed } \ \varepsilon,\theta \in (0,1) \quad F(\varepsilon, C, \theta) \to 0 \quad \text{as} \quad C \to +0.$$

Let us fix $\theta \in (0, \frac{1}{16})$ and after that fix $\varepsilon \in (0,1)$ in such a way that $\varepsilon + c\theta^{\frac43} \leqslant \frac12$. Then we obtain the estimate

$$\mathcal{E}(\theta r) \leqslant \frac12\,\mathcal{E}(r) + F(C_0), \qquad \forall\; r \in (0,1).$$

Using the standard iteration technique we can conclude that

$$\sup_{r<1}\mathcal{E}(r) \leqslant c\,F(C_0) < +\infty. \tag{3.10}$$

Theorem 1.1 is proved. □

We finish the paper with the proof of Theorem 1.2.

Proof. Assume $C_0 \leqslant \varepsilon$. As the function $F(C)$ in (3.10) is continuous, nondecreasing and tends to zero as $C \to +0$ we can fix $\varepsilon > 0$ in such a way that

$$\sup_{r<1} \mathcal{E}(r) \leqslant \varepsilon_*$$

where $\varepsilon_* > 0$ is the absolute constant from the boundary analogue of Caffarelli–Kohn–Nirenberg theorem, see [11]. Then Theorem 1.2 follows from results of [11], see also [19] and [18]. $\square$

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