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ON SPECTRAL ASYMPTOTICS OF THE STURM–LIOUVILLE PROBLEM WITH SELF-CONFORMAL SINGULAR WEIGHT WITH STRONG BOUNDED DISTORTION PROPERTY

Abstract. Spectral asymptotics of the Neumann problem for the Sturm-Liouville equation with a singular self-conformal weight measure is considered under the assumption of a stronger version of the bounded distortion property for the conformal iterated function system corresponding to the weight measure. The power exponent of the main term of the eigenvalue counting function asymptotics is obtained. This generalizes the result obtained by T. Fujita (Taniguchi Symp. PMMP Katata, 1985) in the case of self-similar (self-affine) measure.

§1. Introduction

We consider spectral asymptotics for the Neumann problem

$$\begin{cases} -y'' = \lambda\mu y, \\ y'(0) = y'(1) = 0, \end{cases} \tag{1}$$

where the weight μ is a self-conformal measure on a line.

The history of this problem goes back to the works of M. G. Krein. In [1] it is shown, that if the measure μ contains absolutely continuous component, its singular component does not influence the main term of the spectral asymptotic, and in the case of singular measure μ the eigenvalue counting function $N : (0, +\infty) \to \mathbb{N}$ of the problem (1) admits the estimate $o(\lambda^{\frac{1}{2}})$ instead of the usual asymptotics $N(\lambda) \sim C\lambda^{\frac{1}{2}}$ in the case of measure containing a regular component.

The problem is comparatively well-studied in the case of a self-similar (self-affine) measure. Exact power exponent in the case of self-similar measure was obtained in [2]. It is shown in [3] and [4] that the eigenvalues

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counting function of problem (1) for the self-similar weight has the asymptotics

$$N(\lambda) = \lambda^D \cdot \big(s(\ln\lambda) + o(1)\big), \quad \lambda \to +\infty, \tag{2}$$

where $D \in (0, \frac{1}{2})$ and s is a continuous T-periodic function, dependent on the choice of the weight μ (see also [5] for similar asymptotics in the case of an arbitrary even order differential operator, and [6] for similar results for problems containing two self-similar measures). A series of works [7, 8, 9] is dedicated to the fine properties of the function s for incrementally generalized classes of self-similar measures.

The aim of this paper is to find the power exponent D in the case of self-conformal measure with some special properties.

This paper has the following structure. Sec. 2 provides the necessary definitions of self-conformal measures, their properties and defines some restrictions. Sec. 3 introduces the formal boundary value problem and defines the spectrum under consideration. Sec. 4 gives the definition of the deformed self-similar measure, establishes the spectral asymptotics for them and formulates the strong bounded distortion property, which is the main restriction on the self-conformal measures considered in this paper. It further shows the connection between self-conformal measures with strong bounded distortion property and deformed self-similar measures, thus extending the spectral asymptotics to them.

§2. Self-conformal measures on a line

Let $m \geqslant 2$. We say $/ = (\varphi_1, \dots, \varphi_m; \rho_1, \dots, \rho_m)$ is a *conformal iterated function system* on $[0, 1]$, if:

(1) $\varphi_i : [0,1] \to \varphi_i([0,1])$ is a $C^{1+\gamma}$ diffeomorphism for $\gamma > 0$ and all $i = 1, \dots, m$.

(2) $\varphi_i((0,1)) \subset (0,1)$ and $\varphi_i((0,1)) \cap \varphi_j((0,1)) = \varnothing$ for all $i, j = 1, \dots, m$, $i \neq j$.

(3) $0 < |\varphi_i'(x)| < 1$ for all $i = 1, \dots, m$ and all $x \in [0,1]$.

(4) Positive numbers ρ_i are such, that $\sum\limits_{i=1}^{m} \rho_i = 1$.

Without loss of generality we assume, that φ_i are numbered in ascending order, i.e., $\varphi_i(x) \leqslant \varphi_{i+1}(y)$ for all $x, y \in [0,1]$, $i = 1, \dots, m-1$. We define boolean values e_i:

$$e_i = \begin{cases} 0, & \varphi_i(0) < \varphi_i(1), \\ 1, & \varphi_i(0) > \varphi_i(1). \end{cases}$$

As such, $e_i = 1$ when φ_i changes the orientation of the segment.

We define the operator $\mathcal{S}$ on the space $L_\infty[0,1]$ as follows:

$$\mathcal{S}(f) = \sum_{i=1}^{m} \big(\chi_{\varphi_i([0,1])}(e_i + (-1)^{e_i} f \circ \varphi_i^{-1}) + \chi_{\{x>\varphi_i(1-e_i)\}}\big)\, \rho_i.$$

Lemma 1. *$\mathcal{S}$ is a contraction mapping on $L_\infty[0,1]$.*

Hence, by the Banach fixed-point theorem there exists a (unique) function $C \in L_\infty[0,1]$ such that $\mathcal{S}(C) = C$. Function $C(t)$ could be found as the uniform limit of the sequence $\mathcal{S}^k(f)$ for $f(t) \equiv t$, which allows us to assume that it is continuous and monotone, and also $C(0) = 0$, $C(1) = 1$. The derivative of the function $C(t)$ in the sense of distributions is a measure μ without atoms, invariant with respect to / in the sense of Hutchinson (see [11]), i.e., it satisfies the relation $\mu(E) = \sum\limits_{i=1}^{m} \rho_i \cdot \mu(\varphi_i^{-1}(E))$ for any measurable set E.

Definition 1. *We call μ a self-conformal measure and denote it*

$$\mu := \mu(\varphi_1, \ldots, \varphi_m; \rho_1, \ldots, \rho_m).$$

Remark 1. For a fixed measure μ the choice of iterated function system is not unique. Also, the definition does not require the function system to be conformal. For example, we will define measures using W_∞^1 diffeomorphisms later. However, we call measure μ self-conformal only when it is possible to choose appropriate $C^{1+\gamma}$ diffeomorpisms to define it.

Lemma 2. *Let us define $\Phi(E) := \bigcup\limits_{i=1}^{m} \varphi_i(E)$, and let $|\Phi^k([0,1])| \to 0$ as $k \to \infty$. Then measure μ is singular with respect to Lebesgue measure.*

Lemma 3. *Let* $\operatorname{Lip} \sum\limits_{i=1}^{m} |\varphi_i - \varphi_i(0)| < 1$. *Then $|\Phi^k([0,1])| \to 0$ as $k \to \infty$.*

Corollary 1. *Denote $\alpha_i := \operatorname{Lip} \varphi_i = \|\varphi_i'\|_\infty$ and let $\sum\limits_{i=1}^{m} \alpha_i < 1$. Then measure μ is singular with respect to Lebesgue measure.*

Hereafter we always assume, that

$$\sum_{i=1}^{m} \alpha_i = \sum_{i=1}^{m} \|\varphi_i'\|_\infty < 1. \tag{3}$$

Remark 2. If all diffeomorphisms φ_i are linear functions, we call μ a *self-similar measure*. For self-similar measures Lemma 3 means, that if $\Phi([0,1]) \neq [0,1]$, then μ is singular with respect to the Lebesgue measure. More general ways to construct self-similar functions on a line are described in [10].

§3. Sturm–Liouville problem with self-similar weight

We consider the formal boundary value problem

$$\begin{cases} -y'' = \lambda\mu y, \\ y'(0) = y'(1) = 0. \end{cases} \tag{4}$$

We call the function $y \in W_2^1[0,1]$ its generalized solution if it satisfies the integral equation

$$\int_0^1 y'\eta'\,dx = \lambda \int_0^1 y\eta\,d\mu(x)$$

for any $\eta \in W_2^1[0,1]$. It is clear, that the derivative y' is a primitive of a singular measure without atoms $\lambda\mu y$, thus $y \in C^1[0,1]$.

We denote by $\lambda_n(\mu)$ the eigenvalues of the problem (4) numbered in ascending order, and by $N(\lambda,\mu) := \#\{n : \lambda_n(\mu) < \lambda\}$ their counting function.

§4. Deformed self-similar measures. Spectral asymptotics

Consider $S_i : [0,1] \to I_i$ — a set of affine (linear) contractions of $[0,1]$ onto non-intersecting subsegments I_i of $[0,1]$. Denote by

$$\mu_0 := \mu_0(S_1,\dots,S_m;\rho_1,\dots,\rho_m)$$

the self-similar measure generated by them. Let $g : [0,1] \to [0,1]$ be a W_∞^1 diffeomorphism.

Proposition 1. [2, Theorem 3.6] *$N(\lambda,\mu_0) \asymp \lambda^D$, i.e., there exist constants $C_1, C_2 > 0$, such that $C_1\lambda^D \leqslant N(\lambda,\mu_0) \leqslant C_2\lambda^D$ for all $\lambda \geqslant 0$, where $D \in (0,\frac{1}{2})$ is the only solution of $\sum\limits_{i=1}^{m} (\rho_i|I_i|)^D = 1$.*

Definition 2. *We define a deformed self-similar measure μ as*

$$\mu(E) := \mu_0(g(E))$$

for every measurable set E.

Lemma 4. *Let $\mu = \mu_0 \circ g$ be a deformed self-similar measure and let g be a $C^{1+\gamma}$ diffeomorphism for some $\gamma > 0$. Then μ is a self-conformal measure.*

Theorem 1. *Let μ be a deformed self-similar measure. Then*

$$N(\lambda, \mu) \asymp N(\lambda, \mu_0).$$

Corollary 2. *Let $\mu = \mu_0 \circ g$ be a deformed self-similar measure, where g is a C^1 diffeomorphism. Then $N(\lambda, \mu) \asymp \lambda^D$, where $D \in (0, \frac{1}{2})$ is the only solution of $\sum\limits_{i=1}^{m} (\rho_i |I_i|)^D = 1$.*

Definition 3. *Let's introduce the following notations*:

$$\Sigma_k = \{1, \ldots, m\}^k, \quad \Sigma_* = \bigcup_{i=0}^{\infty} \Sigma_i,$$

for a word $w = (i_1, i_2, \ldots, i_k) \in \Sigma_k$ we say $|w| = k$, denote $\varphi_w = \varphi_{i_1} \circ \varphi_{i_2} \circ \ldots \circ \varphi_{i_k}$.

It is known, that the conformal iterated function system fulfils the bounded distortion property (*see* [12, Lemma 2.1]), *i.e., there exists a constant $C \geqslant 1$, such that*

$$C^{-1} \leqslant \left| \frac{\varphi'_w(x)}{\varphi'_w(y)} \right| \leqslant C$$

for all $x, y \in [0, 1]$ and all $w \in \Sigma_*$.

We say that the conformal iterated function system fulfils the strong bounded distortion property, if there exists a constant $C \geqslant 1$, such that

$$C^{-1} \leqslant \left| \frac{\varphi'_w(x)}{\varphi'_{\sigma w}(y)} \right| \leqslant C$$

for all $x, y \in [0, 1]$, all $w \in \Sigma_*$ and all permutations $\sigma \in \mathcal{S}_{|w|}$.

Lemma 5. *Let self-conformal measure*

$$\mu = \mu(\varphi_1, \ldots, \varphi_m; \rho_1, \ldots, \rho_m)$$

satisfy the relation (3) *and the strong bounded distortion property. Then μ is a deformed self-similar measure.*

Theorem 2. *Let self-conformal measure*

$$\mu = \mu(\varphi_1, \dots, \varphi_m; \rho_1, \dots, \rho_m)$$

satisfy the relation (3) *and the strong bounded distortion property. Then* $N(\lambda, \mu) \asymp \lambda^D$*, where* $D \in (0, \frac{1}{2})$ *is the only solution of*

$$\sum_{i=1}^{m} (\rho_i |\varphi_i'(x_i)|)^D = 1$$

and x_i *is the unique fixed point of* φ_i*.*

Remark 3. Every conformal iterated function system fulfils the bounded distortion property (see [12, Lemma 2.1]), but there are examples of conformal iterated function systems that do not fulfil the strong bounded distortion property.

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