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REGULARITY OF SOLUTIONS TO THE NAVIER–STOKES EQUATIONS IN $\dot{B}^{-1}_{\infty,\infty}$

Abstract. We prove that if u is a suitable weak solution to the three dimensional Navier–Stokes equations from the space $L_\infty(0,T;\dot{B}^{-1}_{\infty,\infty})$, then all scaled energy quantities of u are bounded. As a consequence, it is shown that any axially symmetric suitable weak solution u, belonging to $L_\infty(0,T;\dot{B}^{-1}_{\infty,\infty})$, is smooth.

§1. Introduction

The main aim of this paper is to show that suitable weak solutions to the Navier–Stokes equations, whose $\dot{B}^{-1}_{\infty,\infty}$-norm is bounded, have the Type I singularities (or Type I blowups) only. To be more precise in the statement of our results, we need to define certain notions.

Definition 1.1. *Let Ω be a domain in $\mathbb{R}^3$ and let $Q_T := \Omega\times]0,T[$. It is said that a pair of functions v and q is a suitable weak solution to the Navier–Stokes equations in Q_T if the following conditions are fulfilled:*

(i)

$$v\in L_\infty(\delta,T;L_{2,\rm loc}(\Omega))\cap L_2(\delta,T;W^1_{2,\rm loc}(\Omega)),\quad q\in L_{\frac32}(\delta,T;L_{\frac32,\rm loc}(\Omega))$$

for any $\delta\in]0,T]$;

(ii) *v and q satisfy the Navier–Stokes equations*

$$\partial_t v+v\cdot\nabla v-\Delta v=-\nabla q,\qquad \operatorname{div} v=0$$

in Q_T in the sense of distributions;

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(iii) *for* $Q(z_0,R)\subset\Omega\times]0,T[$, *the local energy inequality*

$$\int\limits_{B(x_0,R)}\varphi|v(x,t)|^2dx+2\int\limits_{t_0-R^2}^{t}\int\limits_{B(x_0,R)}\varphi|\nabla v|^2dxd\tau$$

$$\leqslant\int\limits_{t_0-R^2}^{t}\int\limits_{B(x_0,R)}\Big(|v|^2(\partial_t\varphi+\Delta\varphi)+v\cdot\nabla\varphi(|v|^2+2q)\Big)dxd\tau$$

holds for a.a. $t\in]t_0-R^2,t_0[$ *and for all non-negative test functions* $\varphi\in C_0^\infty(B(x_0,R)\times]t_0-R^2,t_0+R^2[)$.

Let us introduce the following scaled energy quantities:

$$\begin{aligned}
A(z_0,r)&:=\sup_{t_0-r^2<t<t_0}\frac{1}{r}\int\limits_{B(x_0,r)}|v(x,t)|^2dx,\\
E(z_0,r)&:=\frac{1}{r}\int\limits_{Q(z_0,r)}|\nabla v|^2dxdt,\\
C(z_0,r)&:=\frac{1}{r^2}\int\limits_{Q(z_0,r)}|v|^3dxdt,\\
D(z_0,r)&:=\frac{1}{r^2}\int\limits_{Q(z_0,r)}|q|^{\frac{3}{2}}dxdt,\\
G(z_0,r)&:=\max\{A(z_0,r),E(z_0,r),C(z_0,r)\},\\
g(z_0,r)&:=\min\{A(z_0,r),E(z_0,r),C(z_0,r)\}.
\end{aligned}$$

Here, $Q(z_0,r):=B(x_0,r)\times]t_0-r^2,t_0[$ and $B(x_0,r)$ is the ball of radius r centred at a point $x_0\in\mathbb{R}^3$.

The important feature of the above quantities is that all of them are invariant with respect to the Navier–Stokes scaling.

Our main result is as follows.

Theorem 1.2. *Let* $\Omega=\mathbb{R}^3$. *Assume that a pair* v *and* q *is a suitable weak solution to the Navier–Stokes equations in* Q_T. *Moreover, it is supposed that*

$$v\in L_\infty(0,T;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)). \tag{1.1}$$

Then, for any $z_0 \in \mathbb{R}^3 \times]0,T]$, *we have the estimate*

$$\sup_{0<r<r_0} G(z_0,r) \leqslant c[r_0^{\frac{1}{2}} + \|v\|^2_{L_\infty(0,T;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))} + \|v\|^6_{L_\infty(0,T;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))}], \quad (1.2)$$

where $r_0 \leqslant \frac{1}{2}\min\{1,t_0\}$ *and* c *depends on* $C(z_0,1)$ *and* $D(z_0,1)$ *only.*

Let us recall one of definitions of the norm in the space

$$\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3) = \{f \in S' : \|f\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)} < \infty\},$$

which is the following:

$$\|f\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)} := \sup_{t>0} t^{\frac{1}{2}} \|w\|_{L_\infty(\mathbb{R}^3)},$$

where S' is the space of tempered distributions, w is the solution to the Cauchy problem for the heat equation with initial datum f.

Definition 1.3. *Assume that* $z_0 = (x_0, t_0)$ *is a singular point of* v, *i.e., there is no parabolic vicinity of* z_0 *where* v *is bounded. We call* z_0 *Type* I *singularity (or Type* I *blowup) if there exists a positive number* r_1 *such that*

$$\sup_{0<r<r_1} g(z_0,r) < \infty.$$

According to Definition 1.3, any suitable weak solution, satisfying assumption (1.1), has Type I singularities only. In particular, arguments, used in paper [21], show that axially symmetric suitable weak solutions to the Navier–Stokes equations have no Type I blowups. This is an improvement of what has been known so far, see papers [14] and [21], where condition (1.1) is replaced by stronger one

$$v \in L_\infty(0,T; BMO^{-1}(\mathbb{R}^3)).$$

Regarding other regularity results on axially symmetric solutions to the Navier–Stokes equations, we refer to papers [2–5, 9–12, 15–20, 22–24].

Another important consequence is that the smallness of

$$\|v\|_{L\infty(0,T;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))}$$

implies regularity, see also [1, 7].

§2. Proof of the Main Result

In this section, Theorem 1.2 is proved. First, we recall the known multiplicative inequality, see [6].

Lemma 2.1. *For any* $u \in \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3)$*, the following is valid:*

$$\|u\|_{L_4(\mathbb{R}^3)} \leqslant c\|u\|^{\frac{1}{2}}_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)}\|\nabla u\|^{\frac{1}{2}}_{L_2(\mathbb{R}^3)}, \qquad (2.1)$$

where $\dot{H}^1(\mathbb{R}^3)$ *is a homogeneous Sobolev space.*

In fact, a weaker version of (2.1) with $\|u\|_{L^{4,\infty}}$ instead of $\|u\|_{L_4(\mathbb{R}^3)}$ is needed. Here, $L^{4,\infty}(\mathbb{R}^3)$ is a weak Lebesgue space. An elementary proof of a weaker inequality is given in [13].

The second auxiliary statement is about cutting-off in the space $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$.

Lemma 2.2. *Let* $u \in \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$ *and* $\phi \in C^\infty_0(\mathbb{R}^3)$*. Then*

$$\|u\phi\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)} \leqslant c(|\mathrm{spt}\,\phi|)\|u\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)},$$

where $\mathrm{spt}\,\phi$ *is the support of* ϕ*.*

We have not found out a proof of Lemma 2.2 in the literature and presented it in Appendix. Our proof is elementary and based on typical PDE's arguments. A scaled version of the previous lemma is as follows.

Lemma 2.3. *For any* $u \in \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3) \cap H^1(B(2))$*, the estimate*

$$\|u\|_{L^{4,\infty}(B)} \leqslant c\|u\|^{\frac{1}{2}}_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)}\|u\|^{\frac{1}{2}}_{H^1(B(2))}. \qquad (2.2)$$

is valid for a universal constant c. Moreover, if

$$u \in \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3) \cap H^1(B(x_0, 2R)),$$

then

$$\|u\|_{L^{4,\infty}(B_R(x_0))} \leqslant c\|u\|^{\frac{1}{2}}_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)}\left(\|\nabla u\|_{L_2(B_{2R}(x_0))} + \frac{1}{R}\|u\|_{L_2(B_{2R}(x_0))}\right)^{\frac{1}{2}} \qquad (2.3)$$

with a universal constant c.

Here, we use notation for the ball centred at the origin $B(R) = B(0, R)$ and $B = B(1)$.

Proof. It follows from Lemma 2.1 that for all $\phi \in C^\infty_0(\mathbb{R}^3)$,

$$\|u\phi\|_{L^{4,\infty}(\mathbb{R}^3)} \leqslant c\|u\phi\|^{\frac{1}{2}}_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)}\|u\phi\|^{\frac{1}{2}}_{\dot{H}^1(\mathbb{R}^3)}.$$

Taking a cut-off function ϕ such that $\phi = 1$ in B, $\phi = 0$ out of $B(2)$, and $0 \leqslant \phi \leqslant 1$ for $1 \leqslant |x| \leqslant 2$, we get inequality (2.2) from Lemma 2.2.

To prove inequality (2.3), one can use scaling and shift $x = x_0 + Ry$, $x \in B(x_0, 2R)$, $y \in B(2)$ in (2.2). □

In order to prove the main result, we need the following auxiliary inequalities for $C(z_0, r)$.

Lemma 2.4. *For any* $0 < r \leqslant R < \infty$*, we have*

$$C(z_0,r) \leqslant c\|u\|^{\frac{3}{2}}_{L_\infty(0,T;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))}\left(A^{\frac{3}{4}}(z_0,2r)+E^{\frac{3}{4}}(z_0,2r)\right), \qquad (2.4)$$

and

$$C(z_0,r) \leqslant c\|u\|^{\frac{3}{2}}_{L_\infty(0,T;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))}\left(\frac{R}{r}\right)^{\frac{3}{4}}\left(A^{\frac{3}{4}}(z_0,R)+E^{\frac{3}{4}}(z_0,R)\right). \qquad (2.5)$$

Proof. Obviously, (2.5) easily follows from (2.4). So, we need to prove the first inequality only. By the Hölder inequality, we have

$$\|u(\cdot,t)\|_{L_3(B(x_0,r)}\leqslant cr^{\frac{1}{4}}\|u(\cdot,t)\|_{L^{4,\infty}(B(x_0,r)}$$

and thus, by (2.3),

$$\begin{aligned}
C(z_0,r) &= \frac{1}{r^2}\int_{t_0-r^2}^{t_0}\|u(\cdot,t)\|^3_{L_3(B(x_0,r))}dt\\
&\leqslant c\frac{1}{r^{\frac{3}{4}}}\|u\|^{\frac{3}{2}}_{L_\infty(0,T;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))}\Big(\int_{t_0-(2r)^2}^{t_0}\|\nabla u(\cdot,t)\|^2_{L_2(B(x_0,2r))}\\
&\qquad\qquad + \frac{1}{r^2}\|u(\cdot,t)\|^2_{L_2(B(x_0,2r))}dt\Big)^{\frac{3}{4}}\\
&\leqslant c\frac{1}{r^{\frac{3}{4}}}\|u\|^{\frac{3}{2}}_{L_\infty(0,T;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))}\Big(\int_{t_0-(2r)^2}^{t_0}\|\nabla u(\cdot,t)\|^2_{L_2(B(x_0,2r))}dt\\
&\qquad\qquad + \sup_{-(2r)^2+t_0\leqslant t<t_0}\|u(\cdot,t)\|^2_{L_2(B(x_0,2r))}\Big)^{\frac{3}{4}}.
\end{aligned}$$

This completes the proof of inequality (2.4). □

Now we are going to jusify our main result.

Proof of Theorem 1.2. From the local energy inequality, it follows that, for any $0 < r < \infty$,

$$A(z_0,r)+E(z_0,r)\leqslant c\left(C^{\frac{2}{3}}(z_0,2r)+C(z_0,2r)+D(z_0,2r)\right). \qquad (2.6)$$

For the pressure q, we have the decay estimate

$$D(z_0,r)\leqslant c\left(\frac{r}{R}D(z_0,R)+\left(\frac{R}{r}\right)^2C(z_0,R)\right). \tag{2.7}$$

which is valid for any $0<r<R<\infty$.

Assume that $0<r\leqslant\frac{\rho}{4}<\rho\leqslant 1$. Combining (2.7) and (2.6), we find

$$A(z_0,r)+E(z_0,r)+D(z_0,r)$$
$$\leqslant c\left(C^{\frac{2}{3}}(z_0,2r)+C(z_0,2r)+\left(\frac{\rho}{r}\right)^2C\Big(z_0,\frac{\rho}{2}\Big)+\frac{r}{\rho}D\Big(z_0,\frac{\rho}{2}\Big)\right).$$

Now, let us estimate each term on the right hand side of the last inequality. From (2.4), (2.5), and Young's inequality with an arbitrary positive constant δ, we can derive

$$C(z_0,2r)\leqslant c\delta\left(A(z_0,\rho)+E(z_0,\rho)\right)+c\delta^{-3}\|u\|^6_{L^\infty(0,T;\dot B^{-1}_{\infty,\infty}(\mathbb{R}^3))}\left(\frac{\rho}{r}\right)^3.$$

Similarly,

$$C^{\frac{2}{3}}(z_0,2r)\leqslant c\delta\left(A(z_0,\rho)+E(z_0,\rho)\right)+c\delta^{-1}\|u\|^2_{L^\infty(0,T;\dot B^{-1}_{\infty,\infty}(\mathbb{R}^3))}\left(\frac{\rho}{r}\right),$$

and

$$(\frac{\rho}{r})^2C(z_0,\frac{\rho}{2})\leqslant c\delta\big(A(z_0,\rho)+E(z_0,\rho)\big)+c\delta^{-3}\|u\|^6_{L^\infty(0,T;\dot B^{-1}_{\infty,\infty}(\mathbb{R}^3))}\left(\frac{\rho}{r}\right)^8.$$

Denote $\mathcal{E}(r)=A(z_0,r)+E(z_0,r)+D(z_0,r)$. By a simple inequality $D(z_0,\rho/2)\leqslant cD(z_0,\rho)$,

$$\mathcal{E}(r)\leqslant c(\delta+\frac{r}{\rho})\mathcal{E}(\rho)+c\Big\{\|u\|^2_{L_\infty(0,T;\dot B^{-1}_{\infty,\infty}(\mathbb{R}^3))}\left(\frac{\rho}{r}\right)\delta^{-1}$$
$$+\|u\|^6_{L_\infty(0,T;\dot B^{-1}_{\infty,\infty}(\mathbb{R}^3))}\left[\left(\frac{\rho}{r}\right)^3+\left(\frac{\rho}{r}\right)^8\right]\delta^{-3}\Big\}.$$

Letting $r=\theta\rho$ and $\delta=\theta$ and picking up θ such that $2c\theta^{1/2}\leqslant 1$, we find

$$\mathcal{E}(\theta\rho)\leqslant\theta^{1/2}\mathcal{E}(\rho)$$
$$+c\left\{\|u\|^2_{L_\infty(0,T;\dot B^{-1}_{\infty,\infty}(\mathbb{R}^3))}\theta^{-2}+\|u\|^6_{L_\infty(0,T;\dot B^{-1}_{\infty,\infty}(\mathbb{R}^3))}\theta^{-11}\right\}.$$

Standard iteration gives us that for $0<r\leqslant\frac{1}{2}$,

$$\mathcal{E}(r)\leqslant c\left(r^{\frac{1}{2}}\mathcal{E}(1)+\|u\|^2_{L_\infty(0,T;\dot B^{-1}_{\infty,\infty}(\mathbb{R}^3))}+\|u\|^6_{L_\infty(0,T;\dot B^{-1}_{\infty,\infty}(\mathbb{R}^3))}\right).$$

Taking into account (2.4), we get in addition that

$$C(z_0,r)\leqslant c\left(r^{\frac12}\mathcal{E}(1)+\|u\|^6_{L_\infty(0,T;\dot B^{-1}_{\infty,\infty}(\mathbb{R}^3))}\right).$$

This completes the proof of Theorem 1.2. □

Appendix §A. Proof of Lemma 2.2.

We let $w(\,\cdot\,,t)=S(t)f(\,\cdot\,)$ and $w_\varphi(\,\cdot\,,t)=S(t)\varphi f(\,\cdot\,)$, where $S(t)$ is a solution operator of the Cauchy problem for the heat equation with the initial data f and φf, respectevely. Then $u:=w\varphi-w_\varphi$ satisfies the equation

$$\partial_t u-\Delta u=-2\mathrm{div}\,(\nabla\varphi w)+w\Delta\varphi$$

and the initial condition $u(\cdot,0)=0$. A unique solution to the problem is as follows:

$$u(x,t)=I+J,$$

where

$$I=-\int\limits_0^t\int\limits_{\mathbb{R}^3}\Gamma(x-y,t-\tau)(2\mathrm{div}\,(\nabla\varphi w))(y,\tau)dyd\tau,$$

$$J=\int\limits_0^t\int\limits_{\mathbb{R}^3}\Gamma(x-y,t-\tau)(w\Delta\varphi)(y,\tau)dyd\tau,$$

and Γ is the heat kernel.

Let us evaluate I. We abbreviate

$$A:=\|f\|_{\dot B^{-1}_{\infty,\infty}(\mathbb{R}^3)}=\sup_{t>0}\sqrt{t}\|w(\cdot,t)\|_{L_\infty(\mathbb{R}^3)}$$

and $\Omega=\mathrm{spt}\,\varphi$. Then we have

$$\begin{aligned}\sqrt{t}|I|&\leqslant 2A\sqrt{t}\int\limits_0^t\frac{1}{\sqrt{\tau}}\int\limits_\Omega\frac{1}{(4\pi(t-\tau))^{\frac32}}\exp\Big\{-\frac{|x-y|^2}{4(t-\tau)}\Big\}\frac{|x-y|}{t-\tau}dyd\tau\\ &\leqslant cA\int\limits_0^t\sqrt{\frac{t}{\tau}}\frac{1}{(t-\tau)^2}\int\limits_\Omega\exp\Big\{-\frac{|x-y|^2}{4(t-\tau)}\Big\}\frac{|x-y|}{\sqrt{t-\tau}}dyd\tau\end{aligned}$$

$$= cA\Big[\int\limits_0^{\frac{t}{2}} \dots + \int\limits_{\frac{t}{2}}^{t} \dots\Big] = cA(I_1 + I_2).$$

Regarding I_1, consider first the case $0 < t < 1$. By the standard change of variables, we have

$$I_1 \leqslant cAC_0 \int\limits_0^{\frac{t}{2}} \sqrt{\frac{t}{\tau}} \frac{1}{\sqrt{t-\tau}} d\tau$$

with

$$C_0 = \int\limits_{\mathbb{R}^3} \exp\{-|u|^2\}|u|du.$$

And thus $I_1 \leqslant cA$. In the second case $t \geqslant 1$,

$$I_1 \leqslant c|\Omega| \int\limits_0^{\frac{t}{2}} \sqrt{\frac{t}{\tau}} \frac{1}{(t-\tau)^2} d\tau \leqslant c|\Omega| \frac{1}{t} \leqslant c|\Omega|.$$

Now, let us evaluate I_2. Obviously,

$$I_2 \leqslant c \int\limits_{\frac{t}{2}}^{t} \frac{1}{(t-\tau)^2} \int\limits_{\Omega} \exp\Big\{-\frac{|x-y|^2}{4(t-\tau)}\Big\} \frac{|x-y|}{\sqrt{t-\tau}} dy d\tau.$$

Make change of variables $\vartheta = t - \tau$, then

$$I_2 \leqslant c \int\limits_0^{\infty} \frac{1}{\vartheta^2} \int\limits_{\Omega} \exp\Big\{-\frac{|x-y|^2}{4\vartheta}\Big\} \frac{|x-y|}{\sqrt{\vartheta}} dy d\vartheta$$

$$= c\int\limits_0^1 \dots + c\int\limits_1^{\infty} \dots = J_1 + J_2.$$

For J_1, we have

$$J_1 \leqslant cC_0 \int\limits_0^1 \frac{1}{\sqrt{\vartheta}} d\vartheta \leqslant cC_0.$$

Finally, J_2 is bounded as follows:

$$J_2 \leqslant c \int\limits_1^\infty \frac{1}{\vartheta^2} d\vartheta \leqslant c|\Omega|.$$

The quantity J is estimated in the same way. Lemma 2.2 is proved.

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