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SOME REMARKS ON THE PARTIAL REGULARITY OF A SUITABLE WEAK SOLUTION TO THE NAVIER–STOKES CAUCHY PROBLEM

Abstract. The aim of the paper is to investigate on some questions of local regularity of a suitable weak solution to the Navier–Stokes Cauchy problem. The results are obtained in the wake of the ones, well known, by Caffarelli–Kohn–Nirenberg.

§1. Introduction

We deal with the Navier–Stokes Cauchy problem

$$\begin{aligned} &u_t + u \cdot \nabla u + \nabla \pi_u = \Delta u, \ \nabla \cdot u = 0, \ \text{in } (0,T) \times \mathbb{R}^3, \\ &u(0,x) = u_0(x) \text{ on } \{0\} \times \mathbb{R}^3. \end{aligned} \tag{1.1}$$

In system (1.1) u is the kinetic field, π_u is the pressure field, $u_t := \frac{\partial}{\partial t} u$ and $u \cdot \nabla u := u_k \frac{\partial}{\partial x_k} u$. We investigate on the partial regularity of a suitable weak solution, and we detect a new sufficient condition for the existence of a regular solution. Our results are in the wake of the ones obtained in [1] and, for small data, in [3]. As in [2,3,6], our study attempts to highlight what is possible to obtain, without extra condition, in the setting of the L^2-theory. In this connection, although it is not our chief aim, we like to point out that our results could lead to a sort of structure theorem in the space-time cylinder. To be more precise in the claim we recall the well known Leray's structure theorem related to a weak solution. Leray's theorem claims that there exist an interval of regularity of the kind (θ, ∞) and a sequence of intervals of regularity included in $(0,\theta)$ whose complementary set on $(0,\theta)$ is a set of zero $\frac{1}{2}$-Hausdorff measure. *Mutatis mutandis*, the results of [1] (see below Theorem 1.4) and of this note give a sort of structure theorem for a suitable weak solution related to the Cauchy problem. More precisely,

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under a suitable assumption for the initial data, in Theorem 1.4 it is proved that a suitable weak solution is regular for all $t > 0$ in the exterior of a ball with radius R_0. In this note we prove that, almost everywhere, a point $(t, x) \in (0, \theta) \times B(R_0)$ is the center of a parabolic neighborhood of regularity for a suitable weak solution. Hence in $(0, \theta) \times B(R_0)$ there is at most a sequence of open sets of regularity, whose complementary set in $(0, \theta) \times B(R_0)$ has at most zero 1-Hausdorff measure.

To better state the details of our main results, we split the introduction in two short subsections. In the first one we recall some definitions and notation following the ones in [1]. Then we recall two fundamental regularity results obtained in [1], and, with an alternative proof, in [11], and their consequences. In the second subsection we give the statement of our results.

1.1. Suitable weak solutions. We start by recalling the following:

Definition 1.1. *Let $u_0 \in J^2(\mathbb{R}^3)$. A pair (u, π_u), such that $u : (0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ and $\pi_u : (0, \infty) \times \mathbb{R}^3 \to \mathbb{R}$, is said a weak solution to problem* (1.1) *if*

i) *for all* $T > 0$, $u \in L^2(0, T; J^{1,2}(\mathbb{R}^3))$ *and* $\pi_u \in L^{\frac{5}{3}}((0, T) \times \mathbb{R}^3)$
ii) $\lim\limits_{t \to 0} \|u(t) - u_0\|_2 = 0$,
iii) *for all* $t, s \in (0, T)$, *the pair* (u, π_u) *satisfies the equation:*

$$\int_s^t \Big[(u, \varphi_\tau) - (\nabla u, \nabla \varphi) + (u \cdot \nabla \varphi, u) + (\pi_u, \nabla \cdot \varphi)\Big] d\tau + (u(s), \varphi(s)) = (u(t), \varphi(t)) \qquad \text{for all} \quad \varphi \in C_0^1([0, T] \times \mathbb{R}^3).$$

In [1] in order to investigate on the regularity of a weak solution it is introduced an energy relation having a local character:

Definition 1.2. *A pair (u, π_u) is said a suitable weak solution if it is a weak solution in the sense of the Definition 1.1 and, moreover,*

$$\begin{aligned} &\int_{\mathbb{R}^3} |u(t)|^2 \phi(t) dx + 2 \int_\sigma^t \int_{\mathbb{R}^3} |\nabla u|^2 \phi \, dx d\tau \leqslant \int_{\mathbb{R}^3} |u(\sigma)|^2 \phi(\sigma) dx \\ &+ \int_\sigma^t \int_{\mathbb{R}^3} |u|^2 (\phi_\tau + \Delta \phi) dx d\tau + \int_\sigma^t \int_{\mathbb{R}^3} (|u|^2 + 2\pi_u) u \cdot \nabla \phi dx d\tau, \end{aligned} \tag{1.2}$$

for all $t \geqslant \sigma$, *for* $\sigma = 0$ *and a.e. in* $\sigma \geqslant 0$, *and for all nonnegative* $\phi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^3)$.

In [1] and [7] the following existence result is proved:

Theorem 1.1. *For all* $u_0 \in J^2(\mathbb{R}^3)$ *there exists a suitable weak solution.*

As a consequence of the inequality (1.2) and of the existence theorem one gets.

Corollary 1.1. *A suitable weak solution enjoys the strong energy inequality:*

$$\|u(t)\|_2^2 + 2\int\limits_s^t \|\nabla u(\tau)\|_2^2 d\tau \leqslant \|u(s)\|_2^2,$$
$$\textit{for all } t \geqslant s, \textit{ for } s = 0 \textit{ and a.e. in } s \geqslant 0\,. \quad (1.3)$$

Moreover for all s *such that* (1.3) *holds we get*

$$\lim_{t \to s^+} \|u(t) - u(s)\|_2 = 0\,. \quad (1.4)$$

Let us recall the definition of singular point for a weak solution.

Definition 1.3. *We say that* (t, x) *is a singular point for a weak solution* (u, π_u) *if* $u \notin L^\infty$ *in any neighborhood of* (t, x)*; the remaining points, where* $u \in L^\infty(I(t, x))$ *for some neighborhood* $I(t, x)$*, are called regular.*

Definition 1.4. *We say that* u *is a regular solution in* $(t_0, t_1) \times \Omega \subseteq (0, T) \times \mathbb{R}^3$ *if* u *is a weak solution, for some* $q > 1$, $u_t \in L^q_{\ell oc}((t_0, t_1) \times \Omega))$ *and, for all* $\delta > 0$, $u \in L^\infty((t_0 + \delta, t_1 - \delta) \times \Omega)$.

It is known that a regular solution in $(t_0, t_1) \times \Omega$ is smooth on compact subsets contained in $(t_0, t_1) \times \Omega$, see e.g. [10].

Following [1] we introduce the parabolic cylinders

$$Q_r = Q_r(t, x) := \{(\tau, y) : t - r^2 < \tau < t \text{ and } |y - x| < r\}, \quad (1.5)$$

and

$$Q_r^* := Q_r^*(t, x) := \{(\tau, y) : t - \frac{7}{8}r^2 < \tau < t + \frac{1}{8}r^2 \text{ and } |y - x| < r\}, \quad (1.6)$$

and, for $r \in (0, t^{\frac{1}{2}})$, we set

$$M(r) = M(t,x,r) := r^{-2} \iint\limits_{Q_r} (|u|^3 + |u||\pi_u|) dy d\tau + r^{-\frac{13}{4}} \int\limits_{t-r^2}^{t} \Bigg(\int\limits_{|x-y|<r} |\pi_u| dy \Bigg)^{\frac{5}{4}} d\tau \,, \quad (1.7)$$

with Q_r as in (1.5).

In paper [1], in connection with the regularity of a suitable weak solution, the authors furnish two regularity criteria. The first is Proposition 1 (or Corollary 1, p. 776) on p. 775:

Proposition 1.1. *Let (u, π_u) be a suitable weak solution in some parabolic cylinder $Q_r(t,x)$. There exist $\varepsilon_1 > 0$ and $c_0 > 0$ independent of (u, π_u) such that, if*

$$M(t,x,r) \leqslant \varepsilon_1, \quad (1.8)$$

then

$$|u(\tau,y)| \leqslant c_1^{\frac{1}{2}} r^{-1}, \text{ a.e. in } (\tau,y) \in Q_{\frac{r}{2}}(t,x), \quad (1.9)$$

where $c_1 := c_0 \varepsilon_1^{\frac{2}{3}}$. In particular, a suitable weak solution u is regular in $Q_{\frac{r}{2}}(t,x)$.

In [1] this result is used to prove another regularity criterion, that is Proposition 2 on p. 776:

Proposition 1.2. *There is a constant $\varepsilon_3 > 0$ with the following property. If (u, π_u) is a suitable weak solution in some parabolic cylinder $Q_r^*(t,x)$ and*

$$\limsup_{r \to 0} r^{-1} \iint\limits_{Q_r^*} |\nabla u|^2 dy d\tau \leqslant \varepsilon_3 \,,$$

then (t,x) is a regular point.

The above criterion is employed to get the following two main results (respectively, Theorem B on page 772 and Theorem D on page 774 in [1]).

Theorem 1.2. *For any suitable weak solution the set $\mathbb{S}$ of singular points has one-dimensional parabolic Hausdorff measure equal to zero.*

Theorem 1.3. *There exists an absolute constant* $L_0 > 0$ *with the following property. If* $u_0 \in J^2(\mathbb{R}^3)$, *and if*

$$\|u_0|x|^{-\frac{1}{2}}\|_2 = L < L_0\,, \tag{1.10}$$

then there exists a suitable weak solution to (1.1) *which is regular in the region*

$$\{(t,x) : |x|^2 < t(L_0 - L)\}\,.$$

There is a difference in the meaning of the above theorems. Theorem 1.2 gives a geometric measure of the possible set $\mathbb{S}$ of singular points. Theorem 1.3 furnishes the existence of a suitable weak solution to (1.1) having finite the following scaling invariant metric:

$$\sup_{0<\tau<t} \int\limits_{\{\tau\}\times\mathbb{R}^3} |u|^2|x|^{-1}dx < \infty\,, \quad \int\limits_0^t \int\limits_{\mathbb{R}^3} |\nabla u|^2|x|^{-1}dxd\tau < \infty \;\; t > 0\,, \tag{1.11}$$

hence $x = 0$ is regular for $t > 0$.

Finally, as a corollary of the latter result, in [1] the authors prove the following (Corollary p. 820 in [1]):

Theorem 1.4. *Let* (u, π_u) *be a suitable weak solution assuming initial data* u_0. *Suppose that* $\|\nabla u_0\|_{L^2(|x|>R)} < \infty$. *Then, there exists a* $R_0 > R$ *such that, for all* $\delta > 0$, $u \in L^\infty((\delta,\infty) \times \{x : |x| > R_0\})$.

1.2. The aims of this note. We work in the setting of the results of Theorem 1.3 and Theorem 1.7 (below) already proved in [3]. Both these theorems work with a scaling invariant norm that leads to (1.11) provided that at the initial instant the weighted norm, that is (1.10),

$$\mathscr{E}(u_0, x) := \int\limits_{\mathbb{R}^3} |u_0|^2|x-y|^{-1}dy\,,\; x \in \mathbb{R}^3\,, \tag{1.12}$$

is small in a suitable sense. The consequence of the smallness is the existence of a regular solution global in time.

In this note we study the existence of a suitable weak solution that, at least locally in time satisfies the regularity criterion of Proposition 1.1 and, as a consequence, is locally a regular solution. Also in this case the result follows from the assumption that the weighted norm (1.12) of the initial data is finite, but, contrary to Theorem 1.3 and Theorem 1.7, we do not require smallness. As a consequence we are able to deduce the regularity only locally in time.

Theorem 1.5. *Let $u(t,x)$ be a suitable weak solution. Assume that for $x \in \mathbb{E} \subseteq \mathbb{R}^3$ there exists $v_0 \in J^{1,2}(\mathbb{R}^3)$ such that*

$$\psi(x) := \int\limits_{\mathbb{R}^3} \frac{|u_0(y) - v_0(y)|^2}{|x-y|} dy < \frac{1}{(4c)^2}, \tag{1.13}$$

where the constant c is independent of u_0, x and v_0. Then there exists a $t(x) > 0$ such that

$$u \in L^\infty(Q_{(\frac{s}{4})^{\frac{1}{2}}}(\tfrac{7}{6}s, x)), \text{ for all } s \in (0, t(x)). \tag{1.14}$$

In particular, if $(\tau, y) \in Q_{(\frac{s}{4})^{\frac{1}{2}}}(\frac{7}{6}s, x)$ is a Lebesgue point, then

$$|u(\tau, y)| \leqslant c\tau^{-\frac{1}{2}}. \tag{1.15}$$

Corollary 1.2. *Let $u(t,x)$ be a suitable weak solution. Then, for all σ of validity of the weighted energy inequality (1.2) there exists a set $\mathbb{E} \subseteq \mathbb{R}^3$, with $\mathbb{R}^3 - \mathbb{E}$ having zero Lebesgue measure, enjoying the property: for all $x \in \mathbb{E}(\sigma)$, there exists a $t(x) > 0$ such that*

$$u \in L^\infty(Q_{(\frac{s}{4})^{\frac{1}{2}}}(\sigma + \tfrac{7}{6}s, x)), \text{ for all } s \in (0, t(x)). \tag{1.16}$$

In particular, if $(\tau, y) \in Q_{(\frac{s}{4})^{\frac{1}{2}}}(\sigma + \frac{7}{6}s, x)$ is a Lebesgue point, then

$$|u(\tau, y)| \leqslant c(\tau - \sigma)^{-\frac{1}{2}}, \tag{1.17}$$

with c independent of τ.

We give some comments.

Firstly we observe that Theorem 1.5 seems similar to Theorem 1.3. The difference is in the fact that we do not require condition (1.10) to the initial data, but the weaker condition (1.13), that is almost everywhere satisfied by means of $u_0 \in J^2(\Omega)$. The theorem establishes a result of local regularity for a suitable weak solution of (1.1). The local character is expressed in (1.14) either by the fact that the solution is L^∞ just on the parabolic cylinder, and by the fact that the height of the cylinder depends on x, through $t(x)$.

Estimate (1.15) (resp. (1.17)) expresses in what way the solution can be singular in $t = 0$ (resp. in σ) provided that $x \in \mathbb{E}$ (resp. $x \in \mathbb{E}(\sigma)$).

In the way specified below, the set $\mathbb{E}$ represents the new aspect of our result of local regularity stated with an initial data in $J^2(\mathbb{R}^3)$. Actually, if we consider $u_0 \in J^2(\mathbb{R}^3)$, then the Riesz potential

$$\mathscr{E}(u_0, x) := \int\limits_{\mathbb{R}^3} \frac{u_0^2(y)}{|x-y|} dy \tag{1.18}$$

is well posed a.e. in $x \in \mathbb{R}^3$. This claim is consequence of the fact that, by the Hardy–Littlewood–Sobolev theorem, the following transformation is well defined:

$$u_0^2 \in L^1(\mathbb{R}^3) \to \mathscr{E}(u_0, x) := \int\limits_{\mathbb{R}^3} \frac{u_0^2(y)}{|x-y|} dy \in L(3,\infty)(\mathbb{R}^3). \tag{1.19}$$

Then, for all $q \in [1,3)$ and for any compact $\mathbb{K} \subset \mathbb{R}^3$, the function $\mathscr{E}(u_0, x) \in L^q(\mathbb{K})$. Hence it is almost everywhere finite. Denoting by $\{u_0^k\}$ a sequence of smooth functions converging to u_0 in $L^2(\mathbb{R}^3)$, for example the mollified of u_0, for $x \in \mathbb{R}^3$ and $k \in \mathbb{N}$ we define the sequence

$$\psi^k(x) := \int\limits_{\mathbb{R}^3} \frac{|u_0 - u_0^k|^2}{|x-y|} dy\,. \tag{1.20}$$

By Hardy–Littlewood–Sobolev theorem (see Lemma 2.6), it is easy to verify that the sequence $\{\psi^k\}$ converges to zero almost everywhere in $x \in \mathbb{E} \subseteq \mathbb{R}^3$. This makes satisfied almost everywhere in x the assumption (1.13) and $\mathbb{E}$ is the set indicated in Corollary 1.2. We prove that for any $x \in \mathbb{E}$ there exists a $t(x) > 0$ such that $M(\frac{7}{6}s, x, r) \leqslant \varepsilon_1$ for suitable r and for any $s \in (0, t(x))$. This result, by means of Proposition 1.1, ensures the regularity in $Q_{\frac{r}{2}}(\frac{7}{6}s, x)$, for any $s \in (0, t(x))$. Therefore, if we denote by $\mathbb{S}_x$ the projection onto $\mathbb{R}^3$ of the set $\mathbb{S}$ of singular points given in Theorem 1.2 (whose one-dimensional Hausdorff measure is zero from the same theorem), throughout Corollary 1.5 we can claim that $\mathbb{S} \subseteq \mathbb{R}^3 \setminus \mathbb{E}$. This last claim makes clear that we do not improve the regularity exhibited in [1] (according with the result proved in [8]), but we investigate on the existence of a size, as function of x belonging to $\mathbb{E}$, of the parabolic neighborhood of regularity of a weak solution. In Corollary 1.2 it is claimed a dependence on σ of the set $\mathbb{E}$: this is due to the fact that we have to employ both (1.2) and the continuity on the right in L^2-norm of the weak solution.

The following results are two main consequences of Theorem 1.5.

Theorem 1.6. *Let $u(t,x)$ be a suitable weak solution. Assume the existence of $\Omega \subseteq \mathbb{R}^3$ and $v_0 \in J^{1,2}(\mathbb{R}^3)$ such that*

$$\psi(x) < \frac{1}{(4c)^2} \text{ uniformly in } x \in \Omega\,. \tag{1.21}$$

Then there exists a T_0 such that (1.14), *and* (1.15), *hold for all $(s,x) \in (0,T_0) \times \Omega$.*

We observe that if $\Omega \equiv \mathbb{R}^3$ then Theorem 1.6 gives the existence of a regular solution (u, π_u) on $(0,T_0) \times \mathbb{R}^3$.

Corollary 1.3. *Let $u(t,x)$ be a suitable weak solution. For any $B(R)$ and for any $\varepsilon > 0$, there exists a set $\Omega_\varepsilon \subset B(R)$, with $meas(B(R)\backslash\Omega_\varepsilon) < \varepsilon$, and there exists a $T_0(\varepsilon) > 0$ such that* (1.14) *holds for all $(s,x) \in (0,T_0(\varepsilon)) \times \Omega_\varepsilon$.*

Theorem 1.7. *Let $u(t,x)$ be a suitable weak solution, and assume also that $\operatorname{ess\,sup}_x \mathscr{E}(u_0,x)$ is sufficiently small. Then, (u,π_u) is regular for all $t > 0$ and it is unique up to a function $c(t)$ for the pressure field.*

The last theorems are the regular solutions counterpart of Theorem 1.5 and Corollary 1.2, provided that the assumptions on the data are stronger than the simple assumption $u_0 \in J^2(\mathbb{R}^3)$. The theorems work in the light of the scaling invariant weighted norm (1.18).

Theorem 1.6 establishes a local existence result stated by requiring a "suitable closeness", in the weighted norm (1.18), of the initial data $u_0 \in L^2(\mathbb{R}^3)$ to a smooth function v_0. As the existence is achieved on the element v_0 of the approximation which is close to u_0 in the metric (1.18), we are not able to give a size of T_0 by means of u_0, but $(0,T_0)$ is just (*a priori*) a subinterval of existence of the smooth solution (v,π_v) corresponding to v_0. In this connection we point out that the above question on the size of T_0 is the same that we meet assuming the data u_0 in $J^3(\Omega)$ or in $\mathbb{L}^3(\Omega) \subset L(3,\infty)$, respectively completion of $\mathscr{C}_0(\Omega)$ in $L^3(\Omega)$ and in $L(3,\infty)(\Omega)$. Both these spaces are scaling invariant and in order to prove the existence local in time we need an auxiliary function, say $\overline{u}_0$ which is close to u_0 in the metric of L^3 or $L(3,\infty)$ and $\overline{u}_0 \in X$, where X is a function space adequate to ensure the existence of a regular solution on some interval $(0,T_0)$. This is an aspect developed with details in [5]. We conclude that in the statement of Theorem 1.6 we can substitute $J^{1,2}$ with any space X which is suitable to ensure the existence of a regular solution corresponding to v_0.

Corollary 1.3 makes operational condition (1.21) on a suitable subdomain. Indeed the existence of the domain $\Omega_\varepsilon \subseteq B(R)$ follows from the construction of a sequence $\{\psi^k\}$ almost everywhere converging to zero and the Severini–Egorov theorem.

Theorem 1.7 furnishes a global existence result just requiring a smallness condition. It is also an immediate consequence of our previous result in [3].

§2. Preliminaries

Below we recall some results which are fundamental for our aims.

Lemma 2.1. *Suppose that $|x|^\beta u \in L^2(\mathbb{R}^3)$ and $|x|^\alpha \nabla u \in L^2(\mathbb{R}^3)$. Also*

i) $r \geqslant 2$, $\gamma + \frac{3}{r} > 0$, $\alpha + \frac{3}{2} > 0$, $\beta + \frac{3}{2} > 0$, *and* $a \in [\frac{1}{2}, 1]$,

ii) $\gamma + \frac{3}{r} = a(\alpha + \frac{1}{2}) + (1-a)(\beta + \frac{3}{2})$ *(dimensional balance),*

iii) $a(\alpha - 1) + (1-a)\beta \leqslant \gamma \leqslant a\alpha + (1-a)\beta$.

Then, with a constant c independent of u, the following inequality holds:

$$\||x|^\gamma u\|_r \leqslant c\||x|^\alpha \nabla u\|_2^a \||x|^\beta u\|_2^{1-a}. \tag{2.1}$$

Proof. See [1] Lemma 7.1 . □

Lemma 2.2. *Assume that $\mathbb{K}$ is a singular bounded transformation from L^p into L^p, $p \in (1, \infty)$, of Calderón–Zigmund kind. Then, $\mathbb{K}$ is also a bounded transformation from L^p into L^p with respect to the measure $(\mu + |x|)^\alpha dx, \mu \geqslant 0$, provided that $\alpha \in (-n, n(p-1))$.*

Proof. [9] Theorem 1. □

Lemma 2.3. *Assume that (u, π_u) is a suitable weak solution. Then the pressure field admits the following representation formula*

$$\pi_u(t,x) = -D_{x_i} D_{x_j} \int_{\mathbb{R}^3} \mathcal{E}(x-y) u^i(y) u^j(y) dy, \text{ a.e. in } (t,x) \in (0,\infty) \times \mathbb{R}^3, \tag{2.2}$$

and the following holds:

$$\pi_u(t,x) \in L^{\frac{5}{3}}(0,T; L^{\frac{5}{3}}(\mathbb{R}^3)) \,. \tag{2.3}$$

Proof. See [3] Lemma 2.4. Moreover, since $u^2 \in L^{\frac{5}{3}}(0,T; L^{\frac{5}{3}}(\mathbb{R}^3))$ estimate (2.3) easily follows. □

Lemma 2.4. *For all $v_0 \in J^{1,2}(\mathbb{R}^3)$ there exists a unique regular solution (v, π_v) to problem (1.1) on some interval $(0,T)$ such that*

$$v \in C([0,T); J^{1,2}(\mathbb{R}^3)),\ v_t,\ D^2 v,\ \nabla \pi_v \in L^2(0,T; L^2(\Omega)) \,, \tag{2.4}$$

where $T \geqslant c\|\nabla u_0\|_2^{-4}$.

Proof. The result is due to Leray, see [4]. □

For $\mu \geqslant 0$ we define the functionals

$$\begin{aligned} \mathscr{E}(v,t,x,\mu) &:= \int\limits_{\mathbb{R}^3} \frac{|v(t,y)|^2}{(|x-y|^2+\mu^2)^{\frac{1}{2}}} dy\,, \\ \mathscr{D}(v,t,x,\mu) &:= \int\limits_{\mathbb{R}^3} \frac{|\nabla v(t,y)|^2}{(|x-y|^2+\mu^2)^{\frac{1}{2}}} dy, \end{aligned} \tag{2.5}$$

and set

$$p(y) := (|x-y|^2+\mu^2)^{-\frac{1}{2}}.$$

When no confusion arises, we omit some or all the dependences on (v, t, x, μ). For $\mu \geqslant 0$, we call

$$\mathscr{E}(v,t,x,\mu) + \int\limits_0^t \mathscr{D}(v,t,x,\mu) d\tau \tag{2.6}$$

weighted energy.

Lemma 2.5. *Let* (v, π_v) *be the regular the solution of Lemma 2.4. Then, for all* $\mu > 0$*, the following weighted energy relation and weighted energy inequality hold:*

$$\begin{aligned} &\mathscr{E}(v,t,x,\mu) + 2\int\limits_0^t \mathscr{D}(v,\tau,x,\mu)d\tau + 3\mu^2 \int\limits_0^t\int\limits_{\mathbb{R}^3} \frac{v^2(\tau,y)}{(|x-y|^2+\mu^2)^{\frac{5}{2}}} dy d\tau \\ &= \mathscr{E}(v,0,x,\mu) + \int\limits_0^t\int\limits_{\mathbb{R}^3} v\otimes v\cdot v\otimes\nabla p dy d\tau + 2\int\limits_0^t\int\limits_{\mathbb{R}^3} \pi_v v\cdot\nabla p dy d\tau, \end{aligned} \tag{2.7}$$

$$\begin{aligned} &\mathscr{E}(t,x,\mu) + \int\limits_0^t \mathscr{D}(\tau,x,\mu)d\tau + 3\mu^2 \int\limits_0^t\int\limits_{\mathbb{R}^3} \frac{v^2(\tau,y)}{(|x-y|^2+\mu^2)^{\frac{5}{2}}} dy d\tau \\ &\leqslant \mathscr{E}(0,x,\mu) + c\int\limits_0^t \mathscr{E}(\tau,x,\mu)\|\nabla v(\tau)\|_2^4 d\tau\,, \end{aligned} \tag{2.8}$$

for all $t \in [0,T)$ *and* $x \in \mathbb{R}^3$.

Proof. Identity (2.7) can be formally obtained by multiplying equation $(1.1)_1$ by vp and integrating by parts on $(0,t)\times\mathbb{R}^3$. Let us show that it is well posed for any $\mu>0$. We start by remarking that in our hypotheses on v_0 we get $\mathscr{E}(0,x,\mu)<\infty$ for all $x\in\mathbb{R}^3$ and $\mu\geqslant 0$. By multiplying equation $(1.1)_1$ by vp and integrating by parts on $(0,t)\times\mathbb{R}^3$, we obtain

$$\begin{aligned}
&\mathscr{E}(t,x,\mu)+2\int_0^t\mathscr{D}(\tau,x,\mu)d\tau+3\mu^2\int_0^t\int_{\mathbb{R}^3}\frac{v^2(\tau,y)}{(|x-y|^2+\mu^2)^{\frac52}}dyd\tau\\
&=\mathscr{E}(0,x,\mu)-2\int_0^t\int_{\mathbb{R}^3}(v\cdot\nabla v)\cdot vpdyd\tau-2\int_0^t\int_{\mathbb{R}^3}\nabla\pi_v\cdot v\cdot pdyd\tau \qquad (2.9)\\
&=:\mathscr{E}(0,x,\mu)+2\int_0^t(J_1+J_2)d\tau\,.
\end{aligned}$$

Let us show that the right-hand side is well defined. Applying Hölder's inequality and inequality (2.1), we get

$$|J_1|\leqslant\|v(|x-y|^2+h^2)^{-\frac14}\|_4^2\|\nabla v\|_2\leqslant\mathscr{E}^{\frac14}\mathscr{D}^{\frac34}\|\nabla v\|_2\leqslant\frac14\mathscr{D}+c\mathscr{E}\|\nabla v\|_2^4\,.$$

From the representation formula (2.2), after integrating by parts, we get

$$\nabla\pi_v(t,x)=\nabla\int_{\mathbb{R}^3}D_{y_j}\mathcal{E}(x-y)v^i(y)D_{y_i}v^j(y)dy\,.$$

Hence, applying Hölder's inequality and employing Lemma 2.2, we deduce

$$\begin{aligned}
|J_2|&\leqslant\|\nabla\pi_v(|x-y|^2+\mu^2)^{-\frac14}\|_{\frac43}\|v(|x-y|^2+\mu^2)^{-\frac14}\|_4\\
&\leqslant c\|v\cdot\nabla v(|x-y|^2+\mu^2)^{-\frac14}\|_{\frac43}\|v(|x-y|^2+\mu^2)^{-\frac14}\|_4\,.
\end{aligned}$$

Applying again Hölder's inequality and subsequently (2.1), we deduce the following estimate:

$$|J_2|\leqslant c\|v(|x-y|^2+\mu^2)^{-\frac14}\|_4^2\|\nabla v\|_2\leqslant c\mathscr{E}^{\frac14}\mathscr{D}^{\frac34}\|\nabla v\|_2\leqslant\frac14\mathscr{D}+c\mathscr{E}\|\nabla v\|_2^4.$$

Hence from (2.9) and via estimates for terms J_1 and J_2 we obtain the integral inequality (2.8), from which, thanks to the regularity of v, it is easy to deduce that (2.7) holds for all $\mu>0$ and for all $t\in[0,T)$. □

Lemma 2.6. *Let $u_0 \in J^2(\mathbb{R}^3)$. There exists a set $\mathbb{E}$ such that $\mathbb{R}^3 - \mathbb{E}$ has zero Lebesgue measure, and for all $x \in \mathbb{E}$ and for all $\eta > 0$ there exists a $\overline{u}_0 \in J^{1,2}(\mathbb{R}^3)$ such that*

$$\int\limits_{\mathbb{R}^3} \frac{|u_0 - \overline{u}_0|^2}{|x-y|}\, dy < \eta\,. \tag{2.10}$$

Moreover, for all $R > 0$ and $\varepsilon > 0$ there exists $\Omega_\varepsilon \subseteq \mathbb{E}$ such that $meas(B(R) - \Omega_\varepsilon) < \varepsilon$ and

$$\int\limits_{\mathbb{R}^3} \frac{|u_0 - \overline{u}_0|^2}{|x-y|}\, dy < \eta \ \text{ uniformly in } x \in \Omega_\varepsilon\,. \tag{2.11}$$

Proof. We denote by $\{u_0^k\}$ the mollified functions of u_0. It is known that $\{u_0^k\} \subset C^\infty(\mathbb{R}^3) \cap J^{1,2}(\mathbb{R}^3)$, and $\{u_0^k\}$ converges to u_0 in L^2-norm. For all $k \in \mathbb{N}$, we define (1.20), that is

$$\psi^k(x) := \int\limits_{\mathbb{R}^3} \frac{|u_0 - u_0^k|^2}{|x-y|}\, dy < \infty\,.$$

By the Hardy-Littlewood-Sobolev theorem we get, for $r \in [1,3)$,

$$\|\psi^k\|_{L^r(\mathbb{K})} \leqslant c(r,\mathbb{K})\|u_0^k - u_0\|_2^2, \text{ for all compact set } \mathbb{K} \subset \mathbb{R}^3.$$

Hence, the sequence $\{\psi^k\}$ converges to zero in $L^r(\mathbb{K})$, for all $r \in [1,3)$. In particular, there exists a subsequence $\{\psi^{k_j}\}$ which converges to zero almost everywhere in $x \in \mathbb{K}$. We denote by $\{\mathbb{K}_\nu\}$ a sequence of compact sets such that $\mathbb{K}_\nu \subset \mathbb{K}_{\nu+1}$ and $\underset{\nu\in\mathbb{N}}{\cup} \mathbb{K}_\nu = \mathbb{R}^3$. By virtue of the above convergence, we denote $\mathbb{E}_\nu \subseteq \mathbb{K}_\nu$ the set of the convergence almost everywhere of the sequence $\{\psi^{k_j}\}$. Then, by means of Cantor's diagonal method, we construct a sequence $\{\psi^\ell\}$ which converges to 0 for all $x \in \mathbb{E} := \underset{\nu\in\mathbb{N}}{\cup} \mathbb{E}_\nu$. Hence for all $x \in \mathbb{E}$ and $\eta > 0$ there exists a $\psi^{\overline{\ell}} \in \{\psi^\ell\}$ such that $\overline{u}_0 := u_0^{\overline{l}}$ verifies (2.10). Property (2.11) is a consequence of the above construction and of the Severino–Egorov theorem. The lemma is completely proved. □

§3. Local in time weighted energy inequality for a suitable weak solution

In this section we prove that any suitable weak solution admits at least locally in time a weighted energy inequality with $\mu = 0$. Actually, the following lemma holds

Lemma 3.1. *Let (u,π_u) be a suitable weak solution. Let x, v_0 and c as in Theorem* 1.5. *Then there exists a $t^*(x)>0$ such that*

$$\mathscr{E}(u,t,x)+\frac{1}{2}\int_0^t \mathscr{D}(u,\tau,x)d\tau \leqslant N<\infty,\ \textit{for all } t\in[0,t^*(x))\,, \tag{3.1}$$

with $\mathscr{E}(u,t,x)$ and $\mathscr{D}(u,\tau,x)$ defined in (2.5).

Proof. The proof of estimate (3.1) reproduces in a suitable way an idea employed in [2]. This idea follows the Leray-Serrin arguments employed for the proof of the energy inequality in strong form. The proof is achieved by means of five steps. We set $w:=u-v$ and $\pi_w:=\pi_u-\pi_v$, where (u,π_u) is the suitable weak solution and (v,π_v) the regular solution corresponding to v_0 and furnished by Lemma 2.4. The first four steps are devoted to prove the following inequality

$$\mathscr{E}(w,t,x,\mu)+\frac{1}{2}\int_0^t \mathscr{D}(w,\tau,x,\mu)d\tau \leqslant \frac{1}{8c^2},$$
$$\text{for all } t\in[0,t^*(x)) \text{ and } \mu>0\,. \tag{3.2}$$

Step 1. We start proving that for all $t>0$

$$\mathscr{E}(t,x,\mu)+2\int_0^t\int_{\mathbb{R}^3}\mathscr{D}(\tau,x,\mu)d\tau+3\mu^2\int_0^t\int_{\mathbb{R}^3}\frac{|u(\tau)|^2}{(|x-y|^2+\mu^2)^{\frac{5}{2}}}dyd\tau$$
$$\leqslant \mathscr{E}(0,x,\mu)+\int_0^t\int_{\mathbb{R}^3}\frac{|u(\tau)|^2\,u\cdot(x-y)}{(|x-y|^2+\mu^2)^{\frac{3}{2}}}dyd\tau+2\int_0^t\int_{\mathbb{R}^3}\frac{\pi_u(\tau)u(\tau)\cdot(x-y)}{(|x-y|^2+\mu^2)^{\frac{3}{2}}}dyd\tau. \tag{3.3}$$

In the energy inequality (1.2) we set $\phi(\tau,y):=(|x-y|^2+\mu^2)^{-\frac{1}{2}}h_R(y)k(\tau)\in C_0^\infty(\mathbb{R}\times\mathbb{R}^3)$, with h_R and k such that

$$h_R(y):=\begin{cases}1 & \text{if } |y|\leqslant R\\ \in(0,1) & \text{if } |y|\in(R,2R)\\ 0 & \text{for } |y|\geqslant 2R\,,\end{cases}\quad \text{and } k(\tau):=\begin{cases}1 & \text{if } |\tau|\leqslant t\\ \in(0,1) & \text{if } |\tau|\in(t,2t)\\ 0 & \text{for } |\tau|\geqslant 2t\,.\end{cases}$$

We get

$$\begin{aligned}
&\int\limits_{\mathbb{R}^3} \frac{|u(t)|^2 h_R}{(|x-y|^2+\mu^2)^{\frac{1}{2}}} dy + 2\int\limits_0^t \int\limits_{\mathbb{R}^3} \frac{|\nabla u(\tau)|^2 h_R}{(|x-y|^2+\mu^2)^{\frac{1}{2}}} dy d\tau \\
&+3\mu^2 \int\limits_0^t \int\limits_{\mathbb{R}^3} \frac{|u(\tau)|^2 h_R}{(|x-y|^2+\mu^2)^{\frac{5}{2}}} dy d\tau \leqslant \int\limits_{\mathbb{R}^3} \frac{|u_0|^2 h_R}{(|x-y|^2+\mu^2)^{\frac{1}{2}}} dy \\
&+ \int\limits_0^t \int\limits_{\mathbb{R}^3} \frac{|u(\tau)|^2 h_R\, u\cdot(x-y)}{(|x-y|^2+\mu^2)^{\frac{3}{2}}} dy d\tau + 2\int\limits_0^t \int\limits_{\mathbb{R}^3} \frac{\pi_u(\tau) h_R u(\tau)\cdot(x-y)}{(|x-y|^2+\mu^2)^{\frac{3}{2}}} dy d\tau \\
&+F(t,R) := \int\limits_{\mathbb{R}^3} \frac{|u_0|^2 h_R}{(|x-y|^2+\mu^2)^{\frac{1}{2}}} dy + I_1(t,x) + I_2(t,x) + F(t,R),
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
F(t,R) := \int\limits_0^t \int\limits_{\mathbb{R}^3} |u|^2 \Big[2\nabla h_R \cdot \nabla(|x-y|^2+\mu^2)^{-\frac{1}{2}} + \frac{\Delta h_R}{(|x-y|^2+\mu^2)^{\frac{1}{2}}} \\
+ \frac{u\cdot\nabla h_R}{(|x-y|^2+\mu^2)^{\frac{1}{2}}} \Big] dy d\tau + \int\limits_0^t \int\limits_{\mathbb{R}^3} \frac{\pi_u u\cdot\nabla h_R}{(|x-y|^2+\mu^2)^{\frac{1}{2}}} dy d\tau \,.
\end{aligned}$$

Since $\pi_u,\, u^2 \in L^{\frac{5}{3}}(0,T;L^{\frac{5}{3}}(\mathbb{R}^3))$, applying Hölder's inequality and employing the decay of ∇h_R, Δh_R, for all $t>0$, we get $F(t,R)=o(R)$. We estimate the terms I_i, $i=1,2$. Since $\mu>0$, by virtue of the integrability properties of a suitable weak solution, applying Lemma 2.1 we get

$$\begin{aligned}
|I_1(t,x)| &\leqslant \int\limits_0^t \Big\| \frac{u}{(|x-y|^2+\mu^2)^{\frac{1}{3}}} \Big\|_3^3 d\tau \\
&\leqslant c \int\limits_0^t \Big\| \frac{u}{(|x-y|^2+\mu^2)^{\frac{1}{4}}} \Big\|_2 \Big\| \frac{\nabla u}{(|x-y|^2+\mu^2)^{\frac{1}{4}}} \Big\|_2^2 d\tau.
\end{aligned}$$

For I_2, applying the Hölder's inequality and Lemma 2.2, we obtain

$$|I_2(t,x)| \leqslant c\int\limits_0^t \|\frac{u}{(|x-y|^2+\mu^2)^{\frac{1}{3}}}\|_3\|\frac{\pi_u}{(|x-y|^2+\mu^2)^{\frac{2}{3}}}\|_{\frac{3}{2}}d\tau$$
$$\leqslant c\int\limits_0^t \|\frac{u}{(|x-y|^2+\mu^2)^{\frac{1}{3}}}\|_3^3 d\tau.$$

Hence, as in the previous case, applying Lemma 2.1, we get

$$|I_2(t,x)| \leqslant c\int\limits_0^t \|\frac{u}{(|x-y|^2+\mu^2)^{\frac{1}{4}}}\|_2\|\frac{\nabla u}{(|x-y|^2+\mu^2)^{\frac{1}{4}}}\|_2^2 d\tau.$$

Employing the estimates obtained for $I_i, i=1,2$, via the Lebesgue dominated convergence theorem, in the limit as $R\to\infty$, for all $t>0$ we deduce the inequality (3.3).

Step 2. In this step we derive a sort of Green's identity between solutions (u,π_u) and (v,π_v), where (v,π_v) is the regular solution given in Lemma 2.4, corresponding to the initial data $v_0\in J^{1,2}(\mathbb{R}^3)$. In the following $(0,T)$ is the interval of existence of (v,π_v). We also recall that the regular solution (v,π_v) is smooth for $t>0$. We denote by $\lambda(\tau)$ a smooth cutoff function such that $\lambda(\tau)=1$ for $\tau\in[s,t]$ and $\lambda(\tau)=0$ for $\tau\in[0,\frac{s}{2}]$.

For all $t,s\in(0,T)$, we consider the weak formulation iii) of Definition 1.1 written with $\varphi=\lambda vp$:

$$\int\limits_s^t\Big[(pu,v_\tau)-(p\nabla u,\nabla v)+(pu\cdot\nabla v,u)+(\pi_u,v\cdot\nabla p)\Big]d\tau+(pu(s),v(s))$$
$$=(pu(t),v(t))+\int\limits_s^t\Big[(\nabla u,v\otimes\nabla p)+(u\otimes u,v\otimes\nabla p)\Big]d\tau\,. \quad (3.5)$$

We multiply equation $(1.1)_1$ written for (v,π_v) by up. After integrating by parts on $(s,t)\times\mathbb{R}^3$, we get

$$\begin{aligned}&\int_s^t \Big[(pu, v_\tau) + (p\nabla u, \nabla v) + (pv \cdot \nabla v, u) - (\pi_v, u \cdot \nabla p)\Big] d\tau \\ &= \int_s^t \Big[(\nabla u, v \otimes \nabla p) + (u \cdot v, \Delta p)\Big] d\tau \,.\end{aligned} \tag{3.6}$$

making the difference between formulas (3.5) and (3.6) we get

$$\int_s^t \Big[-2(p\nabla u,\nabla v)+(pu \cdot \nabla v, u)-(pv \cdot \nabla v, u)+(\pi_u, v \cdot \nabla p)+(\pi_v, u \cdot \nabla p)\Big] d\tau$$

$$= (pu(t), v(t)) - (pu(s), v(s)) + \int_s^t \Big[(u \otimes u, v \otimes \nabla p) - (u \cdot v, \Delta p)\Big] d\tau \,,$$

Since in a suitable neighborhood of 0 all the terms of the last integral equation are continuous on the right, letting $s \to 0^+$, we get

$$\begin{aligned}&\int_0^t \Big[-2(p\nabla u,\nabla v)+(pu \cdot \nabla v, u)-(pv \cdot \nabla v, u)+(\pi_u, v \cdot \nabla p)+(\pi_v, u \cdot \nabla p)\Big] d\tau \\ &= (pu(t), v(t)) - (pu(0), v(0)) + \int_0^t \Big[(u \otimes u, v \otimes \nabla p) - (u \cdot v, \Delta p)\Big] d\tau,\end{aligned} \tag{3.7}$$

which furnishes the wanted Green's identity.

Step 3. Setting $w := u - v$ and $\pi_w := \pi_u - \pi_v$, let us derive the following estimate

$$\begin{aligned}&\mathscr{E}(w, t, x, \mu) + \int_0^t \mathscr{D}(w, \tau, x, \mu) d\tau \\ &\leqslant \mathscr{E}(w, 0, x, \mu) + c \int_0^t \mathscr{E}^{\frac{1}{2}}(w, \tau, x, \mu)\mathscr{D}(w, \tau, x, \mu) d\tau + H(v, t, x, \mu),\end{aligned} \tag{3.8}$$

for all $t \in [0, T),\ x \in \mathbb{R}^3,\ \mu > 0,$

with

$$H(v,t,x,\mu):=c\int_0^t\|\nabla v(\tau)\|_2^4 d\tau+c\int_0^t\mathscr{E}(v,\tau,x,\mu)\mathscr{D}(\tau,v,x,\mu)d\tau.$$

We remark that from the representation formula (2.2) and regularity of v we get that

$$\pi_w=\pi^1+\pi^2\,,\quad \pi^1:=D_{x_j}\int_{\mathbb{R}^3}D_{y_i}\mathcal{E}(x-y)w^i(y)w^j(y)dy$$
$$\text{and }\ \pi^2:=2\int_{\mathbb{R}^3}D_{y_j}\mathcal{E}(x-y)w(y)\cdot\nabla v^j(y)dy. \tag{3.9}$$

We sum estimates (2.7) and (3.3), then we add twice formula (3.7). written for $s=0$. Recalling the definition of (w,π_w) and formula (3.9), after a straightforward computation we get

$$\mathscr{E}(w,t,x,\mu)+2\int_0^t\mathscr{D}(w,\tau,x,\mu)d\tau+3\mu^2\int_0^t\int_{\mathbb{R}^3}\frac{w^2(\tau,x)}{(|x-y|^2+\mu^2)^{\frac{5}{2}}}d\tau$$
$$\leqslant\mathscr{E}(w,0,x,\mu)+F_1(w,t,x,\mu)+F_2(w,v,t,x,\mu), \tag{3.10}$$

where

$$F_1:=F_1(w,t,x,\mu):=\int_0^t(w\otimes w,w\otimes\nabla p)d\tau+2\int_0^t(\pi_1,w\cdot\nabla p)d\tau$$

$$F_2:=F_2(w,v,t,x,\mu):=2\int_0^t(\pi_2,w\cdot\nabla p)d\tau$$
$$-2\int_0^t(w\cdot\nabla v,wp)d\tau+\int_0^t(v\cdot\nabla p,w^2)\,.$$

The term F_1 admits the same estimate as I_1 and I_2 given in *Step* 1, hence we get

$$|F_1|\leqslant c\int_0^t\mathscr{E}^{\frac{1}{2}}(\tau,w,x,\mu)\mathscr{D}(\tau,w,x,\mu)d\tau\ \text{ for all } t\in(0,T),\ x\in\mathbb{R}^3,\ \mu>0.$$

For term F_2 we estimate the first two terms in a different way from the last. Taking the representation formula of π_2 into account, we get

$$\Big|\int_0^t(\pi_2, w\cdot\nabla p)d\tau - 2\int_0^t(w\cdot\nabla v, wp)d\tau\Big| = \Big|\int_0^t p\nabla\pi_2\cdot w dy d\tau + 2\int_0^t(w\cdot\nabla v, wp)dy d\tau\Big|.$$

Hence, applying the same arguments employed in Lemma 2.5 to estimate J_1 and J_2, we get

$$\Big|\int_0^t(\pi_2, w\cdot\nabla p)d\tau - 2\int_0^t(w\cdot\nabla v, wp)d\tau\Big| \leqslant \int_0^t \|wp^{\frac{1}{2}}\|_4^2\|\nabla v\|_2 d\tau$$
$$\leqslant \int_0^t \mathscr{E}^{\frac{1}{3}}(w,\tau,x,\mu)\mathscr{D}(w,\tau,x,\mu)d\tau + c\int_0^t \|\nabla v(\tau)\|_2^4 d\tau\,,$$

for all $t\in[0,T)$, $x\in\mathbb{R}^3$, $\mu>0$.

For the last term in F_2, applying Hölder's inequality, we get

$$\Big|\int_0^t(v\cdot\nabla p, w^2)d\tau\Big| \leqslant \int_0^t \|wp^{\frac{1}{2}}\|_4^2\|vp\|_2^2 d\tau\,.$$

By virtue of estimate (2.1), applying Young's inequality we deduce:

$$\Big|\int_0^t(v\cdot\nabla p, w^2)d\tau\Big|$$
$$\leqslant c\int_0^t \mathscr{E}^{\frac{1}{4}}(w,\tau,x,\mu)\mathscr{D}^{\frac{3}{4}}(w,\tau,x,\mu)\mathscr{E}^{\frac{1}{4}}(v,\tau,x,\mu)\mathscr{D}^{\frac{1}{4}}(v,\tau,x,\mu)d\tau$$
$$\leqslant \int_0^t \mathscr{E}^{\frac{1}{3}}(w,\tau,x,\mu)\mathscr{D}(w,\tau,x,\mu)d\tau + c\int_0^t \mathscr{E}(v,\tau,x,\mu)\mathscr{D}(v,\tau,x,\mu)d\tau.$$

Hence, we obtain

$$|F_2| \leqslant 2\int_0^t \mathscr{E}^{\frac{1}{3}}(w,\tau,x,\mu)\mathscr{D}(w,\tau,x,\mu)d\tau + c\int_0^t \|\nabla v(\tau)\|_2^4 d\tau$$

$$+c\int_0^t \mathscr{E}(v,\tau,x,\mu)\mathscr{D}(v,\tau,x,\mu)d\tau, \text{ for all } t\in[0,T),\, x\in\mathbb{R}^3,\, \mu>0.$$

Finally, applying Young's inequality, we get

$$|F_2| \leqslant \int_0^t \mathscr{D}(w,\tau,x,\mu)d\tau + c\int_0^t \mathscr{E}^{\frac{1}{2}}(w,\tau,x,\mu)\mathscr{D}(w,\tau,x,\mu)d\tau$$

$$+c\int_0^t \|\nabla v(\tau)\|_2^4 d\tau + c\int_0^t \mathscr{E}(v,\tau,x,\mu)\mathscr{D}(v,\tau,x,\mu)d\tau,$$

for all $t\in[0,T)$, $x\in\mathbb{R}^3$, $\mu>0$.

Estimates for F_1, F_2 and (3.10) furnish the integral inequality (3.8).

Step 4. Deduction of estimate (3.2).

Under our assumptions on x, v_0 and c, we have, a fortiori,

$$\mathscr{E}(w,0,x,\mu) < \frac{1}{(4c)}2, \text{ for all } \mu>0. \tag{3.11}$$

Moreover by virtue of the regularity of the solution (v,π_v), see Lemma 2.4 and Lemma 2.5, there exists a t^* such that

$$H(t^*) < \frac{1}{(4c)}2, \text{ for all } \mu>0. \tag{3.12}$$

Let us deduce (3.2) that for convenience of the reader we rewrite:

$$\mathscr{E}(w,t,x,\mu)+\tfrac{1}{2}\int_0^t \mathscr{D}(w,\tau,x,\mu)d\tau < \tfrac{1}{8c^2}\,, \text{ for all } t\in[0,t^*),\, \mu>0. \tag{3.13}$$

Since $w=u-v$ is right continuous in L^2-norm in $t=0$, for all $\mu>0$ the same continuity property holds for $\mathscr{E}(w,t,x,\mu)$. Therefore there exists a $\delta=\delta(\mu)>0$ such that

$$\mathscr{E}(w,t,x,\mu) < \frac{1}{8c^2}\,, \text{ for all } t\in[0,\delta). \tag{3.14}$$

Hence the validity of estimates (3.8) and (3.11)–(3.12) yields for any $t \in [0,\delta)$

$$\mathscr{E}(w,t,x,\mu)+\int_0^t \mathscr{D}(w,\tau,x,\mu)d\tau < \frac{1}{8c^2} + c\int_0^t \mathscr{E}^{\frac{1}{2}}(w,\tau,x,\mu)\mathscr{D}(w,\tau,x,\mu)d\tau,$$

that, thanks to (3.14), gives (3.13) on $[0,\delta)$.

Let us show that estimate (3.14) holds for $t \in [0,t^*)$. For all $\mu > 0$, the function

$$f(t,\mu) := \mathscr{E}(w,0,x,\mu)+ c\int_0^t \mathscr{E}^{\frac{1}{2}}(w,\tau,x,\mu)\mathscr{D}(w,\tau,x,\mu)d\tau + H(v,t,x,\mu)$$

is uniformly continuous on $[0,t^*]$. Hence there exists $\eta = \eta(\mu) > 0$ such that

$$|t_1 - t_2| < \eta \Rightarrow |f(t_1) - f(t_2)| < \frac{1}{8c}{}_2 - \mathscr{E}(w,0,x,\mu) - H(t^*(x)).$$

We state that estimate (3.14) and, consequently, estimate (3.13), also holds for $t \in [\delta,\delta+\eta)$. Assuming the contrary, there exists $\overline{t} \in [\delta,\delta+\eta)$ such that

$$\mathscr{E}(w,\overline{t},x,\mu) > \frac{1}{8c^2}\,. \tag{3.15}$$

On the other hand, the validity of (3.8) yields

$$\mathscr{E}(w,\overline{t},x,\mu)+\int_0^{\overline{t}} \mathscr{D}(w,\tau,x,\mu)d\tau \leqslant (f(\overline{t}) - f(\delta)) + f(\delta)$$

$$< \frac{1}{8c^2} + c\int_0^{\delta} \mathscr{E}^{\frac{1}{2}}(w,\tau,x,\mu)\mathscr{D}(w,\tau,x,\mu)d\tau.$$

Estimate (3.14) allows to deduce that

$$c\int_0^{\delta} \mathscr{E}^{\frac{1}{2}}(w,\tau,x,\mu)\mathscr{D}(w,\tau,x,\mu)d\tau < \frac{1}{\sqrt{8}}\int_0^{\overline{t}} \mathscr{D}(w,\tau,x,\mu)d\tau.$$

Hence the last two estimates imply

$$\mathscr{E}(w,\overline{t},x,\mu) < \frac{1}{8c^2},$$

which is in contradiction with (3.15). Since the arguments are independent of δ, the result holds for any $t \in [0, t^*(x))$, which proves (3.13)

Step 5. Since $u = w + v$, via estimate (2.8) and via estimate (3.13) we deduce, with obvious meaning of N and $t^*(x)$ independent of μ, the following inequality

$$\int\limits_{\mathbb{R}^3} \frac{|u(t,y)|^2}{(|x-y|^2+\mu^2)^{\frac{1}{2}}} + (1-\tfrac{1}{\sqrt{8}}) \int\limits_0^t \int\limits_{\mathbb{R}^3} \frac{|\nabla u(t,y)|^2}{(|x-y|^2+\mu^2)^{\frac{1}{2}}} dyd\tau \leqslant N\,, \quad \text{for all } t \in [0, t^*(x))\,.$$

The thesis is an easy consequence of estimate (3.2) and the following remark: the families of functions

$$\Big\{ \int\limits_0^t \int\limits_{\mathbb{R}^3} \frac{|\nabla u(t,y)|^2}{(|x-y|^2+\mu^2)^{\frac{1}{2}}} dyd\tau \Big\} \text{ and } \Big\{ \int\limits_{\mathbb{R}^3} \frac{|u(t,y)|^2}{(|x-y|^2+\mu^2)^{\frac{1}{2}}} dy \Big\}$$

are monotone in $\mu > 0$. Hence, by virtue of the Beppo Levi's theorem, in the limit as $\mu \to 0$, we deduce (3.1). □

Corollary 3.1. *Let* (u, π_u) *be a suitable weak solution. Let* $\sigma \geqslant 0$ *such that* (1.2) *is verified. Then there exists a set* $\mathbb{E} \subseteq \mathbb{R}^3$, *with* $\mathbb{R}^3 - \mathbb{E}$ *having zero Lebesgue measure, enjoying the property: for all* $x \in \mathbb{E}(\sigma)$ *there exists a* $t^*(x) > 0$ *such that*

$$\mathscr{E}(u,t,x) + (1-\tfrac{1}{\sqrt{8}}) \int\limits_\sigma^t \mathscr{D}(u,\tau,x)d\tau \leqslant N < \infty, \text{ for all } t \in [\sigma, \sigma + t^*(x))\,. \tag{3.16}$$

Proof. For all $\sigma \geqslant 0$ for which u verifies (1.2), via Lemma 2.6, there exists a set $\mathbb{E}$ such that for $x \in \mathbb{E}$ and $\varepsilon > 0$ there exists a function $\overline{u}(\sigma) \in J^{1,2}(\mathbb{R}^3)$ that allows us to verify (1.13) of Theorem 1.5 with $u(\sigma) - \overline{u}(\sigma)$. As the assumptions of Lemma 3.1 are satisfied, the result follows. □

§4. Proof of Theorems 1.5–1.6 and Corollaries 1.2–1.3.

To prove Theorem 1.5 we employ the result of Proposition 1.1. To this aim, in the following Lemma 4.1 we prove that, for a suitable $r > 0$, estimate (3.1) of Lemma 3.1 implies condition (1.8) of Proposition 1.1.

Lemma 4.1. *Let the assumption of Lemma* 3.1 *be satisfied. Then, there exists* $\delta > 0$ *such that*

$$M(t,x,r) \leqslant \varepsilon_1\,, \text{ for all } r \in (0, [(1-\delta)t]^{\frac12}) \text{ and } t \in (0, t^*(x)). \tag{4.1}$$

with $t^*(x)$ *given in Lemma* 3.1.

Proof. By virtue of our assumption, and by virtue of representation formula (2.2) and Lemma 2.2, a.e. in $t \in (0, t^*(x))$, we get that

$$\|\pi_u(t)|x-y|^{-\frac43}\|_{\frac32} \leqslant c\||u(t)||x-y|^{-\frac23}\|_3^2\,. \tag{4.2}$$

Applying Hölder's inequality, from (4.2) and from Lemma 2.1, for all $t \in (0, t^*(x))$ and $t - r^2 > 0$, we have

$$\begin{aligned} & r^{-2}\int\limits_{t-r^2}^{t}\int\limits_{|x-y|<r}\Big[|u|^3+|v||\pi_u|\Big]dyd\tau \\ & \leqslant c\int\limits_{t-r^2}^{t}\Big[\|\frac{u(\tau)}{|x-y|^{\frac23}}\|_3^3+\|\frac{u(\tau)}{|x-y|^{\frac23}}\|_3\|\frac{\pi_u(\tau)}{|x-y|^{\frac43}}\|_{\frac32}\Big]d\tau \\ & \leqslant c\int\limits_{t-r^2}^{t}\|\frac{u(\tau)}{|x-y|^{\frac12}}\|_2\|\frac{\nabla u(\tau)}{|x-y|^{\frac12}}\|_2^2 d\tau \\ & = c\int\limits_{t-r^2}^{t}\mathscr{E}(\tau,x)^{\frac12}\mathscr{D}(\tau,x)d\tau =: N_1. \end{aligned} \tag{4.3}$$

Considering the second term on the right-hand side of $M(t,x,r)$ in (1.7), applying twice Hölder's inequality, (4.2), or all $t \in (0, t^*(x))$ and $t-r^2 > 0$, we get

$$\begin{aligned} & r^{-\frac{13}{4}}\int\limits_{t-r^2}^{t}\Big[\int\limits_{|x-y|<r}|\pi_u(\tau,y)|dy\Big]^{\frac54}d\tau \leqslant cr^{-\frac13}\int\limits_{t-r^2}^{t}\Big[\|\frac{\pi_u(\tau)}{|x-y|^{\frac43}}\|_{\frac32}\Big]^{\frac54}d\tau \\ & \leqslant cr^{-\frac13}\int\limits_{t-r^2}^{t}\|\frac{u(\tau)}{|x-y|^{\frac12}}\|_2^{\frac56}\|\frac{\nabla u(\tau)}{|x-y|^{\frac12}}\|_2^{\frac53}d\tau \leqslant c\Big[\int\limits_{t-r^2}^{t}\mathscr{E}^{\frac12}(\tau,x)\mathscr{D}(\tau,x)d\tau\Big]^{\frac56} =: N_2. \end{aligned} \tag{4.4}$$

Hence (4.3) and (4.4) imply that

$$M(t,x,r) \leqslant N_1 + N_2.$$

Employing estimate (3.1), we get

$$N_1 + N_2 \leqslant cN^{\frac{1}{2}} \int\limits_{t-r^2}^{t} \mathscr{D}(\tau,x)d\tau + \Big[cN^{\frac{1}{2}} \int\limits_{t-r^2}^{t} \mathscr{D}(\tau,x)d\tau\Big]^{\frac{5}{6}},$$

$$\text{for all} \quad t \in (0, t^*(x)) \text{ and } \quad t - r^2 > 0.$$

On the other hand the function

$$\int\limits_{t}^{t^*(x)} \mathscr{D}(\tau)d\tau \text{ is uniformly continuous on } [0, t^*(x)].$$

Hence there exists a $\delta \in (0,1)$ such that

$$\Big[cN^{\frac{1}{2}} \int\limits_{(1-\delta)t}^{t} \mathscr{D}(\tau,x)d\tau\Big]^{\frac{5}{6}} + cN^{\frac{1}{2}} \int\limits_{(1-\delta)t}^{t} \mathscr{D}(\tau,x)d\tau < \varepsilon_1 \, \forall t \in (0, t^*(x)).$$

Hence the lemma is proved. □

Now we are in a position to prove the results of Theorem 1.5 and Theorem 1.6.

Proof of Theorem 1.5. By virtue of Lemma 3.1, for any x satisfying the assumptions, estimate (3.1) holds on some interval $[0, t^*(x))$. Set $t(x) := \frac{6}{7}t^*(x)$, by virtue of Lemma 4.1, there exists a $\delta > 0$ such that $M(\frac{7}{6}s, x, r) \leqslant \varepsilon_1$, for all $r \in (0, [(1-\delta)\frac{7}{6}s]^{\frac{1}{2}})$, $s \in (0, t(x))$. This, via Proposition 1.1, implies the local regularity (1.14), provided that $\delta \in (0, \frac{1}{7})$[1]. Finally, in order to prove (1.15) it is enough to observe that the point (s,x) belongs to $Q_{(\frac{s}{4})^{\frac{1}{2}}}(\frac{7}{6}s, x)$ and, if (s,x) is a Lebesgue point, then, via estimate (1.9), we can state (1.15). The theorem is completely proved. □

Proof of Corollary 1.2. By virtue of Corollary 3.1, there exists a set $\mathbb{E}(\sigma)$ such that for all $x \in \mathbb{E}(\sigma)$ estimate (3.1) holds on some interval $[\sigma, \sigma + t^*(x))$. Then one can conclude as in the proof of Theorem 1.5. □

[1]This condition ensures that we can choose $r = \sqrt{s}$, being $(1-\delta)\frac{7}{6}s > s$.

Proof of Theorem 1.6. Under the assumption of the theorem, Lemma 3.1 holds for any x in Ω, with $t^*(x)$ uniform in Ω. The last claim is a consequence of the fact that in the definition of ψ the smooth function v_0 is independent of $x \in \Omega$. Hence under our assumption (1.21) we have that both (3.11) and (3.12) are uniform with respect to x. Setting $T_0 := t^*$, we write (3.1) for $t \in [0, T_0)$ for all $x \in \Omega$. As a consequence, all the arguments employed for the proof of Theorem 1.5 work independently of $x \in \Omega$. The theorem is proved. □

Proof of Corollary 1.3. Fixed the ball $B(R)$ and given $\varepsilon > 0$, we can employ Lemma 2.6 which furnishes property (2.11). Hence the assumption of Theorem 1.6 holds for any x in Ω_ε. □

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