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LEGENDRIAN CURVES IN $\mathbb{C}P^3$: CUBICS AND CURVES ON A QUADRIC SURFACE

Abstract. We prove that the number of Legendrian rational cubics in $\mathbb{C}P^3$ through three generic points and a line is three. Also, we classify all Legendrian curves on a quadric surface. Additionally, several computations are verified using Macaulay2 computer algebra system.

§1. Introduction

Inspired by Gromov–Witten invariants, one can try to count holomorphic curves under some additional restrictions. For example, I. Vainsencher asked to count Legendrian curves passing through a prescribed number of generic points or lines. His student, Éden Amorim [2] used localizations to count rational Legendrian curves through $2d+1$ generic lines in $\mathbb{C}P^3$. Then G. Mikhalkin proposed to me this problem as a potential topic for my thesis. However, not much has been accomplished. In this paper, we show that the family of Legendrian cubics passing through three generic points in $\mathbb{C}P^3$ forms a line in the space of coefficients and classify all algebraic Legendrian curves on a quadric surface. Some computations are performed in Macaulay2, [12].

The recent study of the complex Legendrian curves is motivated by minimal surfaces in the four-dimensional sphere. The map

$$(z_1, z_2, z_3, z_4) \mapsto (z_1 + jz_2, z_3 + jz_4)$$

from $\mathbb{C}^4$ to $\mathbb{H}^2$ yields the so-called twistor (or Penrose) map

$$\phi\colon \mathbb{C}P^3 \to \mathbb{H}P^1 = S^4,$$

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and Bryant has shown [5] that the images of the Legendrian curves in $\mathbb{C}P^3$ under ϕ are superminimal surfaces in S^4. Furthermore, each minimal immersion $S^2 \to S^4$ can be obtained as $\phi(C)$, where C is a rational Legendrian curve in $\mathbb{C}P^3$. Then, each Riemann surface M can be mapped to a Legendrian curve in $\mathbb{C}P^3$ using two meromorphic functions (f, g) on M. This leads to the fact that for each Riemann surface M^2 there exists a conformal minimal immersion $M^2 \to S^4$. Nowadays, such immersions are mostly constructed by this approach. See a recent survey [1] about minimal surfaces.

The area of the image of a harmonic map $f\colon S^2 \to S^4$ is equal to $4\pi d$ if $f(S^2)$ comes as the projection of a Legendrian rational curve in $\mathbb{C}P^3$ of degree d. The dimension of the space $\mathfrak{M}_{d,0}$ of Legendrian maps $\mathbb{C}P^1 \to \mathbb{C}P^3$ of degree d is proven to be $2d+4$, see [13,17,24,25]; see [15] for the Legendrian maps $\mathbb{C}P^1 \to \mathbb{C}P^{2n+1}$. This is done via studying the pairs (f, g) of meromorphic functions of degree d with the same ramification divisor. Up to degree six, the space $\mathfrak{M}_{d,0}$ is a smooth complex manifold, see [4].

If $d \geqslant g + 3$, then the part of the space $\mathfrak{M}_{d,g}$ consisting of smooth contact curves in $\mathbb{C}P^3$ of degree d and genus g, is smooth, [19,26]. Besides, a complete intersection cannot be a contact curve [6]. This complicates the study of the contact curves of higher genus, which was approached in [8,9]. The dimension of $\mathfrak{M}_{d,g}$ is $2d - g + 4$ for $d \geqslant \max(2g, g + 2)$, [18]; the dimension of each irreducible component of $\mathfrak{M}_{d,g}$ is between $2d-4g+4$ and $2d-g+4$, where the upper bound is always attained by totally geodesic immersions (whose images belong to a line) and the lower bound is achieved on $\mathfrak{M}_{6,1}$ and $\mathfrak{M}_{8g+1+3k,g}$, [8]. See [9] for further details about other possible pairs (d, g) with non-trivial contact curve. All this means that, for $g \geqslant 1$, we need to take the degree d of the curve at least 6, which is now beyond our abilities to compute with formulae even using computer.

For a general overview of complex contact varieties and deformations of contact curves, see [7], [26, 27]. Real algebraic contact structures are numerous, the questions about polynomial distributions go back to [14,22], see Example 2.4.

For the works of the same spirit, we mention the study of Legendrian curves of minimal degree through two points with prescribed tangency [11] and contact curves in $\mathrm{PSL}(2, \mathbb{C})$ [20].

§2. The contact structure on $\mathbb{C}P^3$

Definition 2.1. *A section ω of the projectivization $P(\Omega^1(\mathbb{C}P^3))$ of the cotangent bundle of $\mathbb{C}P^3$ is said to be a contact holomorphic form on $\mathbb{C}P^3$ if $\omega \wedge d\omega$ is nowhere zero.*

Formally, there are charts A_i, holomorphic 1-forms ω_i on A_i, a set f_{ij} of transition functions on $A_i \cap A_j$, $f_{ij}\omega_i = \omega_j$, such that $\bigcup A_i = \mathbb{C}P^3$ and $\omega_i \wedge d\omega_i \neq 0$ on A_i.

Note that if ω is locally a contact form and f is a function, then $f\omega$ is also a contact form since

$$f\omega \wedge d(f\omega) = f^2\omega \wedge d\omega. \tag{1}$$

Example 2.2. The form $\omega = ydx - xdy + wdz - zdw$ is contact.

Indeed, consider the restriction of ω to the chart $w = 1$. We have

$$\omega|_{w=1} = dz + ydx - xdy,$$

$$\omega|_{w=1} \wedge d\omega|_{w=1} = -2dx \wedge dy \wedge dz \neq 0,$$

similar formulae hold in other charts.

Theorem 2.3 ([16]). *Each contact holomorphic form ω on $\mathbb{C}P^3$ is of the type*

$$(py - qz + aw)dx + (-px + rz + bw)dy + (qx - ry + cw)dz \\ + (-ax - by - cz)dw, \tag{2}$$

where a, b, c, p, q, r are constants and $pc + qb + ra \neq 0$. Furthermore, all such forms are equivalent under the $\mathrm{GL}(4, \mathbb{C})$ *action.*

Proof. We only sketch a proof from [16]. Let α be a holomorphic contact form in $\mathbb{C}P^3$. Note that the set f_{ij} of transition functions determines a linear bundle whose first Chern class we denote by $c_1(\alpha)$. The form $\alpha \wedge d\alpha$ gives a section of the canonical bundle. Considering transition function (1), we conclude that $c_1(\mathbb{C}P^3) = 2c_1(\alpha)$. This means that if $Pdx + Qdy + Rdz$ is a contact form in the chart $w = 1$, then it extends to all of $\mathbb{C}P^3$ only if the transition functions to other charts have w in the denominator in degree at most two. Therefore, P, Q, and R are polynomials of degree one. The explicit form of all such polynomials follows by a direct computation. $\square$

Quite the contrary, there are many algebraic contact structures on $\mathbb{R}P^3$.

Example 2.4. The following forms are contact forms on $\mathbb{R}P^3$:

$$\begin{aligned}\omega_1' &= (yz^2 + yw^2)dx + (-xz^2 - xw^2)dy \\ &\quad + (x^2w + y^2w + w)dz + (-x^2z - y^2z - z)dw, \\ \omega_2' &= (x^2y + y^3 + yz^2 + yw^2)dx - (x^3 + xy^2 + xz^2 + xw^2)dy \\ &\quad + (x^2w + y^2w + z^2w + w^3)dz - (x^2z + y^2z + z^3 + zw^2)dw.\end{aligned}$$

Note also that a small perturbation of the coefficients of a real contact form does not affect the fact that $\omega' \wedge d\omega'$ never vanishes.

It seems not easy to enumerate real algebraic curves that are contact with respect to these contact structures.

Proposition 2.5. *Any irreducible algebraic curve C in $\mathbb{C}P^3$ that is not a collection of lines is Legendrian with at most one holomorphic contact structure.*

Indeed, when we intersect the distribution given by (2) with the distribution given by $\omega = yxd - xdy + wdz - zdw$, we obtain a vector field v almost everywhere (except for a finite collection of points as the Macaulay2 code below shows). On the other hand, we know that there is a line, tangent to the obtained distribution, through each point in $\mathbb{C}P^3$. Therefore, the integral curves for v are lines almost everywhere. Hence, the only locus where a curve tangent to both distributions can live, is the set where these two contact forms coincide, i. e., a finite collection of points.

The following code in Macaulay2, [12], obtains the ideal J of the variety of the points where two contact structures coincide. Comments start with "`--`". What follows after "`=`" is the output of the corresponding command. We use these conventions throughout this paper.

```
use QQ[p,q,r,a,b,c,x,y,z,w]
a1=p*y-q*z+a*w
a2=-p*x+r*z+b*w
a3=q*x-r*y+c*w
a4=-a*x-b*y-c*z

I=ideal(a1*x-a2*y, a2*w-a3*x,a3*z-a4*w)
C=minimalPrimes I
J=C_8  -- all the other ideals C_0, C_1, ...
       -- give lines if we fix a,b,c,p,q,r
dim J  -- =7
```

```
-- 7-6=1 because we have 6 parameters p,q,r,a,b,c
-- so it is just several lines through the origin,
-- that is, a collection of points after the homogenisation.
```

The global Reeb vector field for the contact structure

$$\omega = ydx - xdy + wdz - zdw$$

is given by

$$y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} - z\frac{\partial}{\partial w}.$$

Its trajectories (which are also the fibers of the Penrose map $\mathbb{C}P^3 \to S^4$) are given by

$$\varphi(t) = \left(A\frac{(t^2-1)}{(t^2+1)}, 2A\frac{t}{(t^2+1)}, B\frac{((t+k)^2-1)}{((t+k)^2+1)}, 2B\frac{(t+k)}{((t+k)^2+1)}\right) \quad (3)$$

and

$$\left(\frac{t^2-1}{t^2+1}\right)' = \frac{4t}{t^2+1}, \quad \left(\frac{2t}{t^2+1}\right)' = \frac{t^2-1}{t^2+1}.$$

So, the Reeb vector field just rotates in xy plane and zw plane on the same angle. For each fixed angle, this gives a linear transformation.

§3. Contact form automorphisms

It is known that the group of automorphisms of $\mathbb{C}P^3$ that preserve the form

$$\omega = ydx - xdy + wdz - zdw$$

is the symplectic group $\mathrm{Sp}(4,\mathbb{C})$. Indeed, we have 6 conditions on the coefficients of a matrix $A \in \mathrm{GL}(4,\mathbb{C})$, since A preserves ω, and the condition $\det A \neq 0$, but one can check (by Macaulay2, for example) that the set of such $A \in \mathbb{C}^{16}$ is a quasiprojective variety of dimension 10. The dimension count gives $\dim \mathrm{Sp}(4,\mathbb{C}) = 10$ and $\dim \mathrm{PGL}(4,\mathbb{C}) = 15$, which agrees with the fact the set of all contact structures in (2) is five-dimensional.

Proposition 3.1. *We list the set of generators of this group* $\mathrm{Sp}(4,\mathbb{C})$.

- 1) $x \to x + \lambda y$, $\begin{pmatrix} 1 & \lambda & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$,

- 2) $x \to y,\ \ y \to -x,\ \ \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$
- 3) $x \to z,\ \ y \to w,\ \ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$
- 4) $x \to x + \lambda w,\ \ z \to z + \lambda y,\ \ \begin{pmatrix} 1 & 0 & 0 & \lambda \\ 0 & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$
- 5) $x \to \lambda x,\ \ y \to y/\lambda,\ \ \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & 1/\lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$

Proposition 3.2. *The restriction of a contact structure of the type* (2) *on the plane $z = w = 0$ is $p(ydx - xdy) = 0$ by an easy computation.*

Therefore, the vector field generated by the contact form, at a point (x, y) equals the vector (x, y), so that the only integral curves are the lines passing through the origin. Since all of the planes are equivalent under the action of $\mathrm{GL}(4, \mathbb{C})$, it follows that all of the planar contact curves are collections of lines.

Let us choose an arbitrary plane L.

Proposition 3.3. *Each contact curve in L is a collection of lines through a point $p \in L$. Moreover, L is the contact plane at p, i. e., L is the zero set of ω computed at p.*

Proposition 3.4. *All of the elements of* $\mathrm{Sp}(4, \mathbb{C})$ *that preserve the points*

$$(0, 0, 0, 1),\ (1, 1, 1, 1),\ (-1, 1, -1, 1)$$

are of the form

$$\begin{gathered} \mathrm{Stab}^3_\mu\colon x \to x,\ \ y \to y + \mu(z - x),\ \ z \to z, \\ w \to w - \mu(z - x),\ \ \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\mu & 1 & \mu & 0 \\ 0 & 0 & 1 & 0 \\ \mu & 0 & \mu & 1 \end{pmatrix}. \end{gathered} \tag{4}$$

Proof. Direct computation. □

It is easy to send any point of $\mathbb{C}P^3$ to $(0,0,0,1)$ by an element of $\mathrm{Sp}(4)$. Consequently, the points of $\mathbb{C}P^3$ can be divided into two classes: those lying on the plane L through $(0,0,0,1)$ such that $\omega((0,0,0,1))|_L = 0$ and all the others. The subgroup of $\mathrm{Sp}(4)$ stabilizing $(0,0,0,1)$ acts on both these classes transitively. Now, consider a point p that is not on the contact planes through $(0,0,0,1)$ and $(1,1,1,1)$. A direct computation shows that the subgroup of $\mathrm{Sp}(4)$ stabilizing $(0,0,0,1)$ and $(1,1,1,1)$ contains an element that sends p to $(-1,1,-1,1)$. Thus, we have the following lemma.

Lemma 3.5. *The group* $\mathrm{Sp}(4)$ *is generically 3-transitive, i. e., any three points* $p_1, p_2, p_3 \in \mathbb{C}P^3$ *in general position can be sent to any three points* $q_1, q_2, q_3 \in \mathbb{C}P^3$ *in general position by an element* $a \in \mathrm{Sp}(4,\mathbb{C})$*. In general, the set*

$$\{a \in \mathrm{Sp}(4,\mathbb{C}) \mid a(p_i) = q_i, i = 1,2,3\}$$

is of dimension one.

§4. Curves on a hypersurface of degree two

Consider a contact form ω of the type (2). We will find the restriction of ω on the surface X given by

$$\begin{aligned} &\{xy - zw = 0\} = \mathrm{Im}\,(f\colon \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^3),\\ &f\colon (\mu:\mu'),(\nu:\nu') \to (\mu\nu', \mu'\nu, \mu\nu, \mu'\nu'). \end{aligned} \tag{5}$$

Note that any irreducible hypersurface X' of degree 2 in $\mathbb{C}P^3$ is projectively equivalent to X. Therefore, this method describes all of the Legendrian curves on all of the non-degenerate hypersurfaces X' of degree 2.

Computing in the affine chart $(\mu,\nu) \to (\mu,\nu,\mu\nu,1) \in X$, we obtain

$$f_*\colon \frac{\partial}{\partial\mu} \to \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \frac{\partial}{\partial\nu} \to \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}.$$

The fact that $\omega(f_*(M\frac{\partial}{\partial\mu} + N\frac{\partial}{\partial\nu}) = 0$ at $(\mu,\nu,\mu\nu,1)$ is equivalent to

$$(p\nu - q\mu\nu + a)M + (-p\mu + r\mu\nu + b)N + (q\mu - r\nu + c)(M\nu + N\mu) = 0,$$

i.e.,

$$M(p\nu + a - r\nu^2 + c\nu) + N(-p\mu + b + q\mu^2 + c\mu) = 0.$$

If a curve is locally of type $(\mu(t), \nu(t))$, then its tangent vector is given by the formula $\mu' \frac{\partial}{\partial \mu} + \nu' \frac{\partial}{\partial \nu}$. But this one, after reparametrization, rewrites as

$$\frac{d\mu}{dt} = (c-p)\mu + b + q\mu^2, \quad \frac{d\nu}{dt} = -((p+c)\nu + a - r\nu^2). \tag{6}$$

We are looking for the algebraic leaves of this foliation. See [10, 23] for details about the space of foliations with algebraic leafs, [3] for the classification of the quadratic systems with the first integral.

Example 4.1. Consider the curve $(t, t^2, t^3, 1)$, which lies on the hypersurface $\{xy - zw = 0\}$. It is Legendrian with respect to the form

$$3dx - 3dy + wdz - zdw = 0,$$

so we put

$$p = 3, \quad c = 1, \quad q = a = r = b = 0$$

and (6) becomes

$$(\mu', \nu') = (-2\mu, -4\nu) = (\mu, 2\nu),$$

whence we have $\mu = e^t$, $\nu = e^{2t}$, which is the same as $(\mu, \nu) = (t, t^2)$. Subsequently,

$$(\mu, \nu, \mu\nu, 1) = (t, t^2, t^3, 1).$$

Depending on the coefficients, each equation $\frac{dx}{dt} = c_0 + c_1 x + c_2 x^2$ after a linear change of the coordinates (over the complex numbers) becomes one in the following list:

- $\frac{dx}{dt} = c$,
- $\frac{dx}{dt} = cx$,
- $\frac{dx}{dt} = cx^2$,
- $\frac{dx}{dt} = c(x^2 - 1)$.

Example 4.2. If $\frac{d\mu}{dt} = c_0(\mu^2 - 1)$, $\frac{d\nu}{dt} = c_1(\nu^2 - 1)$, then $\frac{d\mu}{\mu^2 - 1} = c_3 \frac{d\nu}{\nu^2 - 1}$. This implies that

$$\log\left(\frac{\mu - 1}{\mu + 1}\right) = c_4 \log\left(\frac{\nu - 1}{\nu + 1}\right) + c_5,$$

and finally $c_6(\frac{\nu-1}{\nu+1}) = c_7(\frac{\mu-1}{\mu+1})^{d_1}$, which is algebraic if $d_1 \in \mathbb{Q}$.

To the contrary, the case where

$$\frac{d\mu}{dt} = c_0(\mu^2 - 1), \quad \frac{d\nu}{dt} = c_1\nu^2$$

always gives a non-algebraic curve if $c_0c_1 \neq 0$ because this gives an equation of the type $\frac{\mu-1}{\mu+1} = e^{1/\nu}$. So, by a direct computation, we prove the following theorem.

Theorem 4.3. *After a linear change of coordinates*

$$\tilde{\mu} = c_0 + c_1\mu, \quad \tilde{\nu} = c_2 + c_3\nu,$$

any Legendrian curve on the quadric $xy - zw = 0$ with parametrization (5) *can be written in one of the following standard forms:*

- $c_0(\frac{\nu-1}{\nu+1})^{d_1} = c_1(\frac{\mu-1}{\mu+1})^{d_2}$,
- $c_0\nu^{d_1} = c_1\mu^{d_2}$,
- $c_0(\frac{\nu-1}{\nu+1})^{d_1} = c_1\mu^{d_2}$,
- $c_0(\frac{\mu-1}{\mu+1})^{d_1} = c_1\nu^{d_2}$,
- $c_0\mu\nu + c_1\mu + c_2\nu = 0$,
- $c_0\mu + c_1\nu + c_2 = 0$,
- $c_0\mu\nu + c_1\mu + c_2 = 0$,
- $c_0\mu\nu + c_1\nu + c_2 = 0$,
- $\mu = c_0$,
- $\nu = c_0$,

where $c_i \in \mathbb{C}, d_i \in \mathbb{N}_0$ are some constants.

Remark 4.4. Given this classification, one might count the Legendrian curves of given degree and genus lying in a quadric. For example, all rational quartics lie on a quadric surface.

§5. Legendrian curves of degrees one and two

Definition 5.1. *A map $f\colon M \to \mathbb{C}P^3$ is totally geodesic if $f(M)$ is a Legendrian line.*

Let us study the rational Legendrian curves of degrees one and two. In the case $\deg x, y, z = 1$ or 2, it happens that such curves are parametrized by $(f, p + qf, r + pf)$, where f is a polynomial of degree 1 or 2.

Consider a general line

$$l = (a_0 + b_0s, a_1 + b_1s, a_2 + b_2s, a_3 + b_3s)$$

in $\mathbb{C}P^3$. Putting it into the contact form we conclude that the line l is Legendrian if and only if

$$a_1b_0 - a_0b_1 + a_3b_2 - a_2b_3 = 0.$$

This means that for a point A we have a one-dimensional family of Legendrian lines through A. This family is just the contact plane through A. Therefore, the number of Legendrian lines through one point and one line equals one.

Let us observe one important property of Legendrian lines. One can think of a line l in $\mathbb{C}P^3$ as of four section x, y, z, w of $\mathcal{O}(1)$ on $\mathbb{C}P^1$. Let X, Y, Z, W be the roots of x, y, z, w, $x = a_0+b_0 s$, $X = -\frac{a_0}{b_0}$, $y = a_1+b_1 s$, $Y = -\frac{a_1}{b_1}$, etc.

Proposition 5.2. *The following three conditions are equivalent:*

- *the line l is Legendrian,*
- $y(X)/z(X) = w(Z)/x(Z)$,
- $x(Y)/w(Y) = z(W)/y(W)$.

Proof. Look at the table with values of x, y, z, w in X, Y, Z, W.

$$\begin{pmatrix} X = & (0, & \frac{a_1b_0-b_1a_0}{b_0}, & \frac{a_2b_0-b_2a_0}{b_0}, & \frac{a_3b_0-b_3a_0}{b_0}) \\ Y = & (\frac{a_0b_1-a_1b_0}{b_1}, & 0, & \frac{a_2b_1-a_1b_2}{b_1}, & \frac{a_3b_1-a_1b_3}{b_1}) \\ Z = & (\frac{a_0b_2-a_2b_0}{b_2}, & \frac{a_1b_2-a_2b_1}{b_2}, & 0, & \frac{a_3b_2-a_2b_3}{b_2}) \\ W = & (\frac{a_0b_3-a_3b_0}{b_3}, & \frac{a_1b_3-a_3b_1}{b_3}, & \frac{a_2b_3-a_3b_2}{b_3}, & 0) \end{pmatrix}. \qquad \square$$

Remark 5.3. Is it possible to generalize this proposition for the curves of higher degree?

§6. Legendrian cubics

Let us find all of the Legendrian cubics passing through three generic points in $\mathbb{C}P^3$. We parametrize our curve and suppose that it passes through chosen points at $t = -1, 0, 1$. This imposes constraints on the coefficients of this parametrization and we will find that the corresponding subvariety of the space of coefficients of cubics through three generic points is a line. This subvariety happens to be of dimension one (as expected) and of degree one (it was not expected). First, we do it using Macaulay2; then, we do it "by hands".

```
clearAll
--coefficients of the parametrization of the cubic
mainvar=(a0,a1,a2,a3,b0,b1,b2,b3,c0,c1,c2,c3,d0,d1,d2,d3)
R=QQ[mainvar]; P=R[s];
--polynomials for each coordinate
x=a0+a1*s+a2*s*s+a3*s*s*s; y=b0+b1*s+b2*s*s+b3*s*s*s;
```

```
z=c0+c1*s+c2*s*s+c3*s*s*s; t=d0+d1*s+d2*s*s+d3*s*s*s;

ourconditions=y*diff(s,x)-x*diff(s,y)+t*diff(s,z)-z*diff(s,t)
--in M, we have our relation for variables
--since in the variable ourconditions
--(as a polynomial in z) all the coef. should be zeroes
(C,M) = coefficients ourconditions
(A,B,C)=(0,1,-1)

xA=sub(x,{s=>A}); xB=sub(x,{s=>B}); xC=sub(x,{s=>C});

yA=sub(y,{s=>A}); yB=sub(y,{s=>B}); yC=sub(y,{s=>C});

zA=sub(z,{s=>A}); zB=sub(z,{s=>B}); zC=sub(z,{s=>C});

tA=sub(t,{s=>A}); tB=sub(t,{s=>B}); tC=sub(t,{s=>C});

--choose random points
(p11,p12,p13,p14)=(29,-6,13,11)
(p21,p22,p23,p24)=(-3,-17,7,-5)
(p31,p32,p33,p34)=(16,-5,6,23)
--conditions that our curve passes through chosen points
(i1,i2,i3)=(p14*xA-p11*tA,p14*yA-p12*tA,p14*zA-p13*tA)
(j1,j2,j3)=(p24*xB-p21*tB,p24*yB-p22*tB,p24*zB-p23*tB)
(k1,k2,k3)=(p34*xC-p31*tC,p34*yC-p32*tC,p34*zC-p33*tC)

use R; N= M_0; l=i->lift(i,R);

J  = ideal(i1,i2,i3,l(N_0),l(N_1),l(N_2),l(N_3),l(N_4))
S = minimalPrimes J
J0 = S_0; J1 = S_1; J2 = S_2;
--S_3 does not exist

di=i->dim variety i; use P;
Null = ideal(x,y,z,t) --if Null is a subset of our ideal,
-- it means that x, y, z, t are all zeroes at some point,
-- so we are not interested in such coefficients a0, a1, ...
```

```
di J0 --=7
di J1 --=8 that raises our suspicions
           -- that it contains Null...
di J2 --=7

--ideal(s-A) means evaluation at A
isSubset(Null, promote(J0,P)+ideal(s-A))  --=false,
isSubset(Null, promote(J1,P)+ideal(s-A))
                --=true, eliminate from our consideration!
isSubset(Null, promote(J2,P)+ideal(s-A))  --=false

use R; S0 = minimalPrimes (J0+ideal(j1,j2,j3));
J00=S0_0; J01=S0_1;  --S0_2 do not exist

use P
isSubset(Null, promote(J00,P)+ideal(s-B)) --=false
isSubset(Null, promote(J01,P)+ideal(s-B)) --=true, eliminate!

use R; S01 = minimalPrimes (J00+ideal(k1,k2,k3))
J000=S01_0; J001=S01_1;

use P; isSubset(Null, promote(J000,P)+ideal(s-C))  --=false
isSubset(Null, promote(J001,P)+ideal(s-C)) --=true, eliminate!

di J000  --=1
degree J000  --=1, so it is linear!
---------
S2 = minimalPrimes (J2 + ideal(j1,j2,j3))
J20=S2_0  --S2_1 does not exist

isSubset(Null, promote(J20,P)+ideal(s-B)) --=true, eliminate!
```

Any rational non-planar cubic is equivalent to $(t, t^2, t^3, 1)$. We can choose a contact form ω_1 such that $(t, t^2, t^3, 1)$ is Legendrian with respect to it.

Lemma 6.1. *The cubic $(t, t^2, t^3, 1)$ is Legendrian with respect to only one contact structure*

$$\omega_1 = 3ydx - 3xdy + wdz - zdw.$$

Proof. Direct calculation, using (2). $\square$

We fix the contact form w_1, then by a contactomorphism we can bring any three generic points to the points

$$(0,0,0,1), \quad (1,1,1,1), \quad (-1,1,-1,1).$$

The main result of this section is the following theorem (above we have just predicted that the family of such curves is a line in the space of the coefficients).

Theorem 6.2. *All of the rational cubics passing through*

$$(0,0,0,1), \quad (1,1,1,1), \quad (-1,1,-1,1)$$

and tangent to

$$\omega_1 = 3ydx - 3xdy + wdz - zdw$$

are of the form

$$l(t,\mu) = (t, t^2 + \mu(t - t^3), t^3, 1 - 3\mu(t - t^3)). \tag{7}$$

The result of the theorem is not surprising. This is the orbit of the action of Stab^3_μ (see Eq. (4)) on $(t, t^2, t^3, 1)$. Therefore, the only problem is to show that there are no other solutions.

Corollary 6.3. *For each holomorphic contact form on $\mathbb{C}P^3$, the number of rational contact cubics through three generic points and a line in general position is equal to three.*

Proof. We intersect family (7) with a generic line L of the type

$$(t', p_1 + q_1 t', p_2 + q_2 t', p_3 + q_3 t').$$

Because of the genericity, L does not pass through

$$(0,0,0,1) = l(0,\mu).$$

Therefore, we may assume that $t \neq 0$ at any intersection of L and $l(t,\mu)$. Consequently, at a point of intersection, we have $t' = ct$ for some c. Then it follows that

$$\begin{aligned} p_1 + q_1 t' &= c(t^2 + \mu(t - t^3)), \\ p_2 + q_2 t' &= ct^3, \\ p_3 + q_3 t' &= c(1 - 3\mu(t - t^3)). \end{aligned} \tag{8}$$

We have

$$3(p_1 + q_1 t') + p_3 + q_3 t' = c(3t^2 + 1).$$

Therefore, substituting $t' = ct$, we obtain

$$c = \frac{3p_1 + p_3}{3t^2 - 3q_1 t - q_3 t + 1}.$$

Then, using the first equality in (8), we get

$$\mu = \frac{p_1 + q_1 ct - ct^2}{c(t - t^3)}.$$

Then, since $c(t^3 - q_2 t) = p_2$, we have

$$t^3 - q_2 t = \frac{p_2}{3p_1 + p_3}(3t^2 - 3q_1 t - q_3 t + 1).$$

Choosing p_2, q_2 appropriately, we see that the last equation usually has three roots. □

Corollary 6.4. *For the contact form*

$$\omega = ydx - xdy + wdz - zdw,$$

the parametrization of the family of Legendrian rational cubics through the points

$$(0,0,0,1), \quad (1,1,1,1), \quad (-1,1,-1,1)$$

is

$$(3t - t^3, 2t^2 + 2\mu(t - t^3), 2t^3, 1 + t^2 - 2\mu(t - t^3)). \tag{9}$$

The surface swept by all these cubics is given by $F = 0$, where

$$F(x,y,z,w) = 2x^3 + 21x^2z - 27y^2z - 54yzw - 27zw^2 + 60xz^2 + 25z^3.$$

Such a surface intersects a generic line in three points. This gives another proof of Corollary 6.3.

§7. Proof of Theorem 6.2

Each rational cubic curve has a parametrization of the form

$$\begin{aligned}
&(a_0 + a_1 t + a_2 t^2 + a_3 t^3,\\
&b_0 + b_1 t + b_2 t^2 + b_3 t^3,\\
&c_0 + c_1 t + c_2 t^2 + c_3 t^3,\\
&d_0 + d_1 t + d_2 t^2 + d_3 t^3).
\end{aligned}$$

We supposed that our cubic passes through the points

$$(0,0,0,1), \quad (1,1,1,1), \quad (-1,1,-1,1)$$

at $t = 0, 1, -1$, respectively. Substituting $t = 0$ in the parametrization, we obtain

$$a_0 = b_0 = c_0 = 0, \quad d_0 = 1.$$

Substitutions $t = \pm 1$ give us

$$\begin{aligned} a_1 + a_2 + a_3 = b_1 + b_2 + b_3 = c_1 + c_2 + c_3 = 1 + d_1 + d_2 + d_3, \\ a_1 - a_2 + a_3 = -b_1 + b_2 - b_3 = c_1 - c_2 + c_3 = 1 - d_1 + d_2 - d_3. \end{aligned}$$

Therefore,

$$a_2 = b_1 + b_3 = c_2 = d_1 + d_3, \quad a_1 + a_3 = b_2 = c_1 + c_3 = 1 + d_2.$$

Substituting indeterminates with bigger indices as functions of the indeterminates with smaller indices, we see that our curve is parametrized by

$$\begin{gathered} (a_1 t + a_2 t^2 + (b_2 - a_1)t^3, \\ b_1 t + b_2 t^2 + (a_2 - b_1)t^3, \\ c_1 t + a_2 t^2 + (b_2 - c_1)t^3, \\ 1 + d_1 t + (b_2 - 1)t^2 + (a_2 - d_1)t^3). \end{gathered}$$

Evaluating the form $3ydx - 3xdy + wdz - zdw$ on the curve, we obtain

$$\begin{aligned} &3(b_1 t + b_2 t^2 + (a_2 - b_1)t^3)(a_1 + 2a_2 t + 3(b_2 - a_1)t^2) \\ &- 3(a_1 t + a_2 t^2 + (b_2 - a_1)t^3)(b_1 + 2b_2 t + 3(a_2 - b_1)t^2) \\ &+ (1 + d_1 t + (b_2 - 1)t^2 + (a_2 - d_1)t^3)(c_1 + 2a_2 t + 3(b_2 - c_1)t^2) \\ &- (c_1 t + a_2 t^2 + (b_2 - c_1)t^3)(d_1 + 2(b_2 - 1)t + 3(a_2 - d_1)t^2) = 0. \end{aligned}$$

The coefficient before t^0 should be equal to 0, so $c_1 = 0$. The parametrization rewrites as

$$\begin{aligned} &3(b_1 t + b_2 t^2 + (a_2 - b_1)t^3)(a_1 + 2a_2 t + 3(b_2 - a_1)t^2) \\ &- 3(a_1 t + a_2 t^2 + (b_2 - a_1)t^3)(b_1 + 2b_2 t + 3(a_2 - b_1)t^2) \\ &+ (1 + d_1 t + (b_2 - 1)t^2 + (a_2 - d_1)t^3)(2a_2 t + 3b_2 t^2) \\ &- (a_2 t^2 + b_2 t^3)(d_1 + 2(b_2 - 1)t + 3(a_2 - d_1)t^2) = 0. \end{aligned}$$

The coefficient before t^1 equals $2a_2$, so $a_2 = 0$.

$$\begin{aligned}
&3(b_1t+b_2t^2-b_1t^3)(a_1+3(b_2-a_1)t^2)\\
&\qquad -3(a_1t+(b_2-a_1)t^3)(b_1+2b_2t-3b_1t^2)\\
&\qquad +(1+d_1t+(b_2-1)t^2-d_1t^3)(3b_2t^2)\\
&\qquad -(b_2t^3)(d_1+2(b_2-1)t-3d_1t^2)\\
&=3(b_1t+b_2t^2-b_1t^3)(a_1+3(b_2-a_1)t^2)\\
&\qquad -3(a_1t+(b_2-a_1)t^3)(b_1+2b_2t-3b_1t^2)\\
&\qquad +b_2t^2(3+3d_1t+3(b_2-1)t^2-3d_1t^3-d_1t-2(b_2-1)t^2+3d_1t^3)\\
&=3(b_1t-b_1t^3)(a_1+3(b_2-a_1)t^2)-3(a_1t+(b_2-a_1)t^3)(b_1-3b_1t^2)\\
&\qquad +b_2t^2(3(b_2-a_1)t^2-3a_1+3+2d_1t+(b_2-1)t^2)\\
&=b_1t^3(-3a_1-9(b_2-a_1)t^2+9(b_2-a_1)+9a_1-3(b_2-a_1)+9(b_2-a_1)t^2)\\
&\qquad +b_1t(3a_1-3a_1)+b_2t^2(3(b_2-a_1)t^2-3a_1+3+2d_1t+(b_2-1)t^2)\\
&=6b_1b_2t^3+b_2t^2(3(b_2-a_1)t^2-3a_1+3+2d_1t+(b_2-1)t^2)\\
&=b_2t^2(6b_1t+3(b_2-a_1)t^2-3a_1+3+2d_1t+(b_2-1)t^2)\\
&=b_2t^2(t(6b_1+2d_1)+t^2(4b_2-3a_1-1)-3a_1+3)=0.
\end{aligned}$$

Therefore, either $b_2=0$ or $a_1=1, b_2=1, d_1=-3b_1$.

In the first case, the curve is going to be as follows:

$$(a_1t-a_1t^3, b_1t-b_1t^3, 0, 1+d_1t-t^2-d_1t^3)=(a_1t, b_1t, 0, 1+d_1t).$$

This is not really a cubic, but in the second case we have

$$(t, b_1t+t^2-b_1t^3, t^3, 1-3b_1t+3b_1t^3)=(t, t^2+\mu(t-t^3), t^3, 1-3\mu(t-t^3)).$$

As it was predicted by Macaulay2, we have obtained a linear family of cubics.

Remark 7.1. One can look at what happens in the limiting case $\mu=\infty$. The family of curves converges (if we look at the parametrizations) to a point $(0,-1/3,0)$. On the other hand, their tangent vectors at $t=0,1,-1$ converge to

$$(0,1,0),\quad (-3,-4,-3),\quad (3,-4,3),$$

respectively. Then, contact lines from

$$(0,0,0,1),\quad (1,1,1,1),\quad (-1,1,-1,1)$$

with these tangent vectors all intersect in $(0, -1/3, 0)$. So, the family $l(t, \mu)$ converges to these three lines as $\mu \to \infty$. These three lines with the embedded point $(0, -1/3, 0)$ is a point on the boundary of the Hilbert scheme of rational cubics in $\mathbb{C}P^3$ (see [21] for more details about the compactification of the space of rational cubics).

Remark 7.2. Is it true that the number of higher degree rational Legendrian curves passing through given points can be computed by means of degeneration? A hypothesis: there always exist at least d Legendrian rational curves of degree d passing through d generic points and a line. A heuristic argument is as follows. We take the one-dimensional family (because Stab^3_μ acts on these curves) of the degree d Legendrian curves through d points that all belong to a given plane L, and write the equation of the surface that they sweep. Then, we intersect this surface with L. We obtain a collection of d lines in the intersection. Therefore, the degree of the surface is at least d. Consequently, there are at least d Legendrian curves through d generic points and one generic line. Also, this approach by perturbation of degenerate families might work for any genus, as long as the set of the curves is not empty.

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