Записки научных
семинаров ПОМИ
Том 476, 2018 г.

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ON SO(3, 3) AS THE PROJECTIVE GROUP OF THE SPACE SO(3)

Abstract. The fractional linear action of SO(3, 3) on the projective space SO(3) is proven to be a (globally defined) projective action.

§1. Introduction

Originally, the choice of topics for the present article was motivated by Segal's Chronometry and Levichev's DLF theory: see [3, Section 7] for details on a (both general as well as DLF specific) notion of parallelization of a vector bundle.

However, it is our opinion that the main statements (Theorems 1 through 4 below) are of general mathematical interest. We believe that the most interesting is our finding that SO(3, 3) can be viewed as the projective group of the (projective) space SO(3).

We notice that there is a 2-cover P of $S^1 \times \mathrm{SO}(3)$ (we denote this group by $D^{1/2}$) by the group $\mathrm{U}(2)$: P sends a matrix z into the pair $(\det z, p(u))$, where p is the standard covering map from SU(2) onto SO(3), see (3.4) below, and u is a matrix in SU(2) such that $z = du$ with $d^2 = \det z$ (u is determined up to a sign). Both P and p are group homomorphisms.

From the theoretical physics viewpoint, the fundamental role of p is well known. In regard to P, its mere existence allows one to present Segal's chronometric theory (see [3] and references therein) starting with the "lowest level" possible (since the center of the group SO(3) is trivial). It is well known that the Lie group $\mathrm{U}(2)$ (with a bi-invariant Lorentzian metric on it) can be viewed as a conformal compactification of the Minkowski spacetime. This provides an approach to how to study physics of Segal's compact cosmos $D = \mathrm{U}(2)$. The Lie group $D^{1/2}$ inherits a bi-invariant metric through the above-introduced homomorphism P. With this metric, we call $D^{1/2}$ the projective world. Thus, $D^{1/2}$ underlies Segal's compact

Key words and phrases: 2-cover of SO(3, 3) by SL(4), fractional linear action of SO(3, 3) on SO(3), bi-invariant metrics, geodesics, 2-cover of $S^1 \times \mathrm{SO}(3)$ by Segal's compact cosmos $\mathrm{U}(2)$.

cosmos $D = \mathrm{U}(2)$. This gives another option of how to build physics in D: to start with that of $D^{1/2}$.

This last topic is to be explored elsewhere while the current presentation takes a turn towards geometry. We introduce the fractional linear action of $\mathrm{SO}(3,3)$ on the projective space $\mathrm{SO}(3)$ and prove that this action is a (globally defined) projective action. (Throughout the article, $\mathrm{SO}(3,3)$ stands for the component of the identity matrix.) One would probably anticipate such an observation: it is known (since long ago) that a projective transformation between two projective lines is a fractional linear one (see [4, p. 22]). The key ingredient of the proof is the commutative diagram, which intertwines the linear action of $\mathrm{SL}(4)$ in $\mathbb{R}^4$ with the $\mathrm{SO}(3,3)$ fractional linear action on $\mathrm{SO}(3)$. In that diagram, the element $\tilde{g}$ of $\mathrm{SO}(3,3)$ corresponds to an element g of the real Lie group $\mathrm{SL}(4)$.

§2. Fractional linear action of SO(3, 3) on SO(3)

In this section, our main goal is to introduce a fractional linear action of $\mathrm{SO}(3,3)$ on $\mathrm{SO}(3)$ and show that this action is globally defined. We denote by $M_{6\times 6}$ the set of all 6×6 real matrices and let $S \in M_{6\times 6}$ be the matrix

$$S = \mathrm{diag}\{1, 1, 1, -1, -1, -1\}.$$

We consider the set W of all 6×6 real matrices $\tilde{g}$ such that

$$\tilde{g}^T S \tilde{g} = S \tag{2.1}$$

This set consists of two components. Following [1], we define the group $\mathrm{SO}(3,3)$ to be the identity-containing component of W. In particular, the negative identity matrix is not an element of thus defined $\mathrm{SO}(3,3)$.

We remark that the authors of [1] use a different matrix S in their analogue of (2.1).

The next step will be to represent the generic element $\tilde{g}$ of $\mathrm{SO}(3,3)$ as a matrix consisting of 3×3 blocks

$$\tilde{g} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{2.2}$$

and find out which conditions are imposed on these blocks by definition (2.1).

Lemma 1. *A matrix* $\tilde{g}$ *of the form* (2.2) *belongs to* $\mathrm{SO}(3,3)$ *if and only if the following conditions hold*

$$A^T A - C^T C = 1, \tag{2.2.1}$$

$$A^T B - C^T D = 0, \tag{2.2.2}$$

$$D^T D - B^T B = 1. \tag{2.2.3}$$

Proof. It is a straightforward calculation based on (2.1). □

Now, let us present the fractional linear action of $\mathrm{SO}(3,3)$ on $\mathrm{SO}(3)$:

$$\tilde{g}z = (Az + B)(Cz + D)^{-1}. \tag{2.3}$$

Theorem 1. *Formula* (2.3) *is globally defined, i. e., it is defined for all elements* z *in* $\mathrm{SO}(3)$.

Proof. We see from (2.3) that, in order to prove the theorem, it suffices to show that the matrix $Cz + D$ in (2.3) is never singular. Let us assume that there exists $z' \in \mathrm{SO}(3)$ with $\det(Cz' + D) = 0$. Then there exists a non-zero vector $\mathbf{v} \in \mathbb{R}^3$ such that $(Cz' + D)\mathbf{v} = \mathbf{0}$. Now, we show that

$$(Az + B)^T (Az + B) - (Cz + D)^T (Cz + D) = 0 \tag{2.4}$$

for all $z \in \mathrm{SO}(3)$. We cross-multiply the terms and reduce (2.4) to

$$z^T(A^T A - C^T C)z + (B^T A - D^T C)z + z^T(A^T B - C^T D) + (B^T B - D^T D) = 0.$$

Using (2.2.1)–(2.2.3), we reduces it to $z^T z = \mathbb{1}$, which proves that (2.4) holds for all $z \in \mathrm{SO}(3)$. If (for z' with $\det(Cz' + D) = 0$) we then multiply both sides of (2.4) by $\mathbf{v}$ from the right and by $\mathbf{v}^T$ from the left, we find that

$$\mathbf{v}^T (Az' + B)^T (Az' + B)\mathbf{v} = 0$$

and $(Az' + B)\mathbf{v} = 0$. Now,

$$B^T (Cz' + D)\mathbf{v} - D^T (Az' + B)\mathbf{v} = 0$$

simplifies to $(B^T B - D^T D)\mathbf{v} = 0$ and $\mathbf{v} = 0$, which contradicts the assumption. Therefore, $Cz + D$ in (2.3) is never singular. □

Theorem 2. *Equation* (2.3) *defines a left action of* $\mathrm{SO}(3,3)$ *on* $\mathrm{SO}(3)$. *In particular,* $\tilde{g}z \in \mathrm{SO}(3)$.

Proof. Assume that $\tilde{g}z$ is defined. In order to show that $\tilde{g}z \in \mathrm{SO}(3)$, we start with $(\tilde{g}z)^T\tilde{g}z = \mathbb{1}$ and reduce it to an obvious identity.

$$\begin{aligned}
&(\tilde{g}z)^T\tilde{g}z = \mathbb{1}\\
&[(Az+B)(Cz+D)^{-1}]^T(Az+B)(Cz+D)^{-1} = \mathbb{1}\\
&[(Cz+D)^{-1}]^T(Az+B)^T(Az+B)(Cz+D)^{-1} = \mathbb{1}\\
&(z^TC^T+D^T)^{-1}(z^TA^T+B^T)(Az+B) = (Cz+D)\\
&(z^TA^T+B^T)(Az+B) = (Cz+D)(z^TC^T+D^T)\\
&z^T(A^TA-C^TC)z = (D^TD-B^TB)\\
&\quad -(B^TA-D^TC)z - z^T(A^TB-C^TD)\\
&z^Tz = \mathbb{1}.
\end{aligned} \tag{2.5}$$

In the last step, we use equations (2.2.1) through (2.2.3). Since z is in SO(3), it follows that (2.5) is true. Therefore, $\tilde{g}z$ is in SO(3). We omit the (straightforward) proof that (2.3) is a left action. Since it is a continuous action of a connected Lie group (and the inputs are in a connected set), $\tilde{g}z$ belongs to SO(3). □

Remark 1. Both Theorems 1 and 2 hold for the case of fractional linear action (2.3) of $\mathrm{SO}(n,n)$ on $\mathrm{SO}(n)$.

§3. Double cover of $\mathrm{SO}(3,3)$ by $\mathrm{SL}(4)$ and the resulting commutative diagram

It is known (see [1, pp. 4, 5]) that there is a homomorphism F (actually, a 2-cover) from SL(4) to $\mathrm{SO}(3,3)$. We will use the notation q for our homomorphism between SL(4) and $\mathrm{SO}(3,3)$ (see Theorem 3 below). Our q will be the following 3-step composition: an element g from SL(4) goes to $Q(g^{-1})^TQ^{-1}$, then F is applied, then the resulting $\overline{g}$ goes to $\tilde{g} = \Omega\overline{g}\Omega^{-1}$. The 4 by 4 matrix Q and the 6 by 6 matrix Ω will be specified below.

Now that we are equipped with the $\mathrm{SO}(3,3)$ action on SO(3), we can introduce the commutative diagram. In Section 1, we have recalled a homomorphism p from SU(2) to SO(3): this p is specified as (3.4) below. Before we introduce the commutative diagram, we agree on the following notation: let u be an element of (viewed as the unit sphere in $\mathbb{R}^4$) SU(2), $\tilde{u} = gu$ is its image under the SL(4) projective action on SU(2), z is an element of SO(3), $\tilde{z}$ is its image under $\tilde{g}$ that implements the $\mathrm{SO}(3,3)$ fractional action (in other words, $\tilde{g} = q(g)$).

Remark 2. The SL(4) projective action on SU(2) originates from the SL(4) linear action in $\mathbb{R}^4$ together with SU(2) identification with the space of rays emanating from the origin in $\mathbb{R}^4$, [2, p. 1368].

Theorem 3. *The* (*described above and specified below*) *homomorphism* q *of* SL(4) *onto* SO(3, 3) *makes the following diagram*

$$\begin{array}{ccc} u & \xrightarrow{g} & \widetilde{u} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ z & \xrightarrow[\widetilde{g}]{} & \widetilde{z} \end{array} \tag{3.1}$$

commutative.

Proof. It is helpful to consider a standard basis $\{\mathbf{L}_{ij}\}$ (with $\mathbf{L}_{ij} = -\mathbf{L}_{ji}$) in so(3, 3). Here is the commutation table:

$$[\mathbf{L}_{im}, \mathbf{L}_{mk}] = -e_m \mathbf{L}_{ik}, \tag{3.2}$$

$(e_{-2}, e_{-1}, e_0, e_1, e_2, e_3)$ stands for $(1, 1, 1, -1, -1, -1)$.

These fifteen 6 by 6 matrices $\mathbf{L}_{ij}$ will be specified below. We choose a $6D$ *maximal compact subalgebra* k spanned by the matrices $\mathbf{L}_{23}$, $\mathbf{L}_{12}$, $\mathbf{L}_{13}$, $\mathbf{L}_{-2-1}$, $\mathbf{L}_{-20}$, $\mathbf{L}_{-10}$. Clearly, k is a direct sum of two copies of so(3). This k acts in the $9D$ subspace spanned by the remaining nine (*non-compact*) generators $\mathbf{L}_{ij}$. To each matrix $\mathbf{L}_{ij}$ from so(3, 3), we will assign a matrix from sl(4). In other words, one can think of (3.2) as of the commutation table for the matrix Lie algebra sl(4). Such an assignment originates from the (described above) homomorphism q, which is specified by the following choice of Q and Ω:

$$Q = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \Omega = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Observe that $Q^{-1} = Q^T$ and $\Omega^{-1} = \Omega^T$.

In the remaining part of the proof, we denote $\cos(t)$ by c and $\sin(t)$ by s.

Compact generators. We will prove (3.1) for one-parameter subgroups generated by compact elements $\mathbf{L}_{23}$, $\mathbf{L}_{12}$, $\mathbf{L}_{-2-1}$, $\mathbf{L}_{-10}$, in that order (to avoid fractions, we will use vectors proportional to these four). Clearly, these four vectors generate k.

Let

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

be in SU(2), which means that

$$u_1^2 + u_2^2 + u_3^2 + u_4^2 = 1. \tag{3.3}$$

The matrix $z = p(u)$ in SO(3) is defined as follows ([6, p. 161]):

$$z = \begin{bmatrix} u_1^2 - u_2^2 - u_3^2 + u_4^2 & -2(u_1u_2 + u_3u_4) & 2(u_1u_3 - u_2u_4) \\ 2(-u_1u_2 + u_3u_4) & -u_1^2 + u_2^2 - u_3^2 + u_4^2 & -2(u_1u_4 + u_2u_3) \\ 2(u_1u_3 + u_2u_4) & 2(u_1u_4 - u_2u_3) & -u_1^2 - u_2^2 + u_3^2 + u_4^2 \end{bmatrix}. \tag{3.4}$$

1. Generator $2\mathrm{L}_{32}$:

$$\widetilde{g}_1 = \begin{bmatrix} \mathbb{1} & & \mathbb{0} & \\ & 1 & 0 & 0 \\ \mathbb{0} & 0 & \cos(2t) & -\sin(2t) \\ & 0 & \sin(2t) & \cos(2t) \end{bmatrix},$$

$$g_1 = \begin{bmatrix} \cos(t) & 0 & 0 & -\sin(t) \\ 0 & \cos(t) & \sin(t) & 0 \\ 0 & -\sin(t) & \cos(t) & 0 \\ \sin(t) & 0 & 0 & \cos(t) \end{bmatrix}.$$

We compute $\widetilde{z} = \widetilde{g}_1 z$:
$[\widetilde{g}_1 z]_{1,1} = u_1^2 - u_2^2 - u_3^2 + u_4^2$;
$[\widetilde{g}_1 z]_{1,2} = -2(u_1u_2 + u_3u_4)\cos(2t) - 2(u_1u_2 - u_3u_4)\sin(2t)$;
$[\widetilde{g}_1 z]_{1,3} = 2(u_1u_3 - u_2u_4)\cos(2t) - 2(u_1u_2 + u_3u_4)\sin(2t)$;
$[\widetilde{g}_1 z]_{2,1} = 2(-u_1u_2 + u_3u_4)$;
$[\widetilde{g}_1 z]_{2,2} = (-u_1^2 + u_2^2 - u_3^2 + u_4^2)\cos(2t) + 2(u_1u_4 + u_2u_3)\sin(2t)$;
$[\widetilde{g}_1 z]_{2,3} = -2(u_1u_4 + u_2u_3)\cos(2t) + (-u_1^2 + u_2^2 - u_3^2 + u_4^2)\sin(2t)$;
$[\widetilde{g}_1 z]_{3,1} = 2(u_1u_3 + u_2u_4)$;
$[\widetilde{g}_1 z]_{3,2} = 2(u_1u_4 - u_2u_3)\cos(2t) + (u_1^2 + u_2^2 - u_3^2 - u_4^2)\sin(2t)$;
$[\widetilde{g}_1 z]_{3,3} = (-u_1^2 - u_2^2 + u_3^2 + u_4^2)\cos(2t) + 2(u_1u_4 - u_2u_3)\sin(2t)$.

$$\widetilde{u}_1 = g_1 u = \begin{pmatrix} u_1\cos(t) - u_4\sin(t) \\ u_2\cos(t) + u_3\sin(t) \\ -u_2\sin(t) + u_3\cos(t) \\ u_1\sin(t) + u_4\cos(t) \end{pmatrix}.$$

We compute $p(g_1 u)$:

$$\begin{aligned}
[\widetilde{z}]_{1,1} &= \widetilde{u}_1^2 - \widetilde{u}_2^2 - \widetilde{u}_3^2 + \widetilde{u}_4^2 \\
&= u_1^2c^2 - \cancel{2u_1u_4sc} + u_4^2s^2 - u_2^2c^2 - \cancel{2u_2u_3sc} - u_3^2s^2 \\
&\quad - u_2^2s^2 + \cancel{2u_2u_3sc} - u_3^2c^2 + u_1^2s^2 + \cancel{2u_1u_4sc} + u_4^2c^2 \\
&= u_1^2 - u_2^2 - u_3^2 + u_4^2;
\end{aligned}$$

$$\begin{aligned}
[\widetilde{z}]_{1,2} &= -2(\widetilde{u}_1\widetilde{u}_2 + \widetilde{u}_3\widetilde{u}_4) \\
&= -2(u_1u_2c^2 - u_2u_4sc + u_1u_3sc - u_3u_4s^2 \\
&\quad - u_1u_2s^2 + u_1u_3sc - u_2u_4sc + u_3u_4c^2) \\
&= -2(u_1u_2 + u_3u_4)\cos(2t) - 2(u_1u_2 - u_3u_4)\sin(2t);
\end{aligned}$$

$$\begin{aligned}
[\widetilde{z}]_{1,3} &= 2(\widetilde{u}_1\widetilde{u}_3 - \widetilde{u}_2\widetilde{u}_4) \\
&= 2(-u_1u_2sc + u_2u_4s^2 + u_1u_3c^2 - u_3u_4sc \\
&\quad - u_1u_2sc - u_1u_3s^2 - u_2u_4c^2 - u_3u_4sc) \\
&= 2(u_1u_3 - u_2u_4)\cos(2t) - 2(u_1u_2 + u_3u_4)\sin(2t);
\end{aligned}$$

$$\begin{aligned}
[\widetilde{z}]_{2,1} &= 2(-\widetilde{u}_1\widetilde{u}_2 + \widetilde{u}_3\widetilde{u}_4) \\
&= 2(-u_1u_2s^2 + \cancel{u_1u_3sc} - \cancel{u_2u_4sc} + u_3u_4c^2 \\
&\quad - u_1u_2c^2 + \cancel{u_2u_4sc} - \cancel{u_1u_3sc} + u_3u_4s^2) \\
&= 2(-u_1u_2 + u_3u_4);
\end{aligned}$$

$$\begin{aligned}
[\widetilde{z}]_{2,2} &= -\widetilde{u}_1^2 + \widetilde{u}_2^2 - \widetilde{u}_3^2 + \widetilde{u}_4^2 \\
&= -u_1^2c^2 + 2u_1u_4sc - u_4^2s^2 + u_2^2c^2 + 2u_2u_3sc + u_3^2s^2 \\
&\quad - u_2^2s^2 + 2u_2u_3sc - u_3^2c^2 + u_1^2s^2 + 2u_1u_4sc + u_4^2c^2 \\
&= (-u_1^2 + u_2^2 - u_3^2 + u_4^2)\cos(2t) + 2(u_1u_4 + u_2u_3)\sin(2t);
\end{aligned}$$

$$
\begin{aligned}
[\widetilde{z}]_{2,3} &= -2(\widetilde{u}_1\widetilde{u}_4 + \widetilde{u}_2\widetilde{u}_3) \\
&= -2(u_1^2 sc - u_1u_4s^2 + u_1u_4c^2 - u_4^2 sc \\
&\quad - u_2^2 sc - u_2u_3s^2 + u_2u_3c^2 + u_3^2 sc) \\
&= -2(u_1u_4 + u_2u_3)\cos(2t) + (-u_1^2 + u_2^2 - u_3^2 + u_4^2)\sin(2t);
\end{aligned}
$$

$$
\begin{aligned}
[\widetilde{z}]_{3,1} &= 2(\widetilde{u}_1\widetilde{u}_3 + \widetilde{u}_2\widetilde{u}_4) \\
&= 2(-\cancel{u_1u_2sc} + u_2u_4s^2 + u_1u_3c^2 - \cancel{u_3u_4sc} \\
&\quad + \cancel{u_1u_2sc} + u_1u_3s^2 + u_2u_4c^2 + \cancel{u_3u_4sc}) \\
&= 2(u_1u_3 + u_2u_4);
\end{aligned}
$$

$$
\begin{aligned}
[\widetilde{z}]_{3,2} &= (\widetilde{u}_1\widetilde{u}_4 - \widetilde{u}_2\widetilde{u}_3) \\
&= 2(u_1^2 sc - u_1u_4s^2 + u_1u_4c^2 - u_4^2 sc \\
&\quad + u_2^2 sc + u_2u_3s^2 - u_2u_3c^2 - u_3^2 sc) \\
&= 2(u_1u_4 - u_2u_3)\cos(2t) + (u_1^2 + u_2^2 - u_3^2 - u_4^2)\sin(2t);
\end{aligned}
$$

$$
\begin{aligned}
[\widetilde{z}]_{3,3} &= -\widetilde{u}_1^2 - \widetilde{u}_2^2 + \widetilde{u}_3^2 + \widetilde{u}_4^2 \\
&= -u_1^2c^2 + 2u_1u_4sc - u_4^2s^2 - u_2^2c^2 - 2u_2u_3sc - u_3^2s^2 \\
&\quad + u_2^2s^2 - 2u_2u_3sc + u_3^2c^2 + u_1^2s^2 + 2u_1u_4sc + u_4^2c^2 \\
&= (-u_1^2 - u_2^2 + u_3^2 + u_4^2)\cos(2t) + 2(u_1u_4 - u_2u_3)\sin(2t).
\end{aligned}
$$

Thus, we proved (3.1) for the first compact generator $\widetilde{g}_1$. For the remaining three compact generators, we only give corresponding group elements (both in SO(3, 3) and SL(4)) as matrices:

2. Generator $2\mathrm{L}_{21}$:

$$
\widetilde{g}_2 = \begin{bmatrix} \mathbb{1} & & \mathbb{0} & \\ & \cos(2t) & -\sin(2t) & 0 \\ \mathbb{0} & \sin(2t) & \cos(2t) & 0 \\ & 0 & 0 & 1 \end{bmatrix},
$$

$$g_1 = \begin{bmatrix} \cos(t) & \sin(t) & 0 & 0 \\ -\sin(t) & \cos(t) & 0 & 0 \\ 0 & 0 & \cos(t) & -\sin(t) \\ 0 & 0 & \sin(t) & \cos(t) \end{bmatrix};$$

3. Generator $2\mathrm{L}_{-1-2}$:

$$\widetilde{g}_2 = \begin{bmatrix} \cos(2t) & -\sin(2t) & 0 & \\ \sin(2t) & \cos(2t) & 0 & \mathbb{0} \\ 0 & 0 & 1 & \\ & \mathbb{0} & & \mathbb{1} \end{bmatrix},$$

$$g_1 = \begin{bmatrix} \cos(t) & \sin(t) & 0 & 0 \\ -\sin(t) & \cos(t) & 0 & 0 \\ 0 & 0 & \cos(t) & \sin(t) \\ 0 & 0 & -\sin(t) & \cos(t) \end{bmatrix};$$

4. Generator $2\mathrm{L}_{0-1}$:

$$\widetilde{g}_2 = \begin{bmatrix} 1 & 0 & 0 & \\ 0 & \cos(2t) & -\sin(2t) & \mathbb{0} \\ 0 & \sin(2t) & \cos(2t) & \\ & \mathbb{0} & & \mathbb{1} \end{bmatrix},$$

$$g_1 = \begin{bmatrix} \cos(t) & 0 & 0 & \sin(t) \\ 0 & \cos(t) & \sin(t) & 0 \\ 0 & -\sin(t) & \cos(t) & 0 \\ -\sin(t) & 0 & 0 & \cos(t) \end{bmatrix};$$

Non-compact generators. Under the linear action of $g \in \mathrm{SL}(4)$ corresponding to a non-compact generator of SO(3, 3), the resulting image $\overline{u}$ would no longer be in SU(2). Therefore, $\overline{u}$ needs to be normalized. So, for (3.1) to hold, we take

$$\widetilde{u} = \frac{\overline{u}}{\|\overline{u}\|} = a\overline{u}.$$

We take

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

from SU(2), $z = p(u)$ is defined by (3.4).

Let us show that (3.1) holds for the generator $2\mathrm{L}_{-23}$:

$$g = \begin{bmatrix} \cosh(t) & 0 & -\sinh(t) & 0 \\ 0 & \cosh(t) & 0 & \sinh(t) \\ -\sinh(t) & 0 & \cosh(t) & 0 \\ 0 & \sinh(t) & 0 & \cosh(t) \end{bmatrix},$$

$$\widetilde{g} = \begin{bmatrix} \cosh(2t) & 0 & 0 & 0 & 0 & -\sinh(2t) \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -\sinh(2t) & 0 & 0 & 0 & 0 & \cosh(2t) \end{bmatrix}.$$

First of all, we have

$$\widetilde{u} = gu = \begin{pmatrix} u_1\cosh(t) - u_3\sinh(t) \\ u_2\cosh(t) + u_4\sinh(t) \\ -u_1\sinh(t) + u_3\cosh(t) \\ u_2\sinh(t) + u_4\cosh(t) \end{pmatrix},$$

so that

$$p(\widetilde{u}) = p(a\overline{u}) = a^2 p(\overline{u}).$$

Therefore, in order to prove (3.1), it suffices to show that

$$Az + B = a^2\overline{z}(Cz + D). \tag{3.5}$$

Here, we have $\overline{z} = p(\overline{u})$: we apply (3.4) even though $\overline{u}$ is not in SU(2). We denote $\cosh(t)$ by c and $\sinh(t)$ by s. Then we have

$$\begin{aligned} a^2 &= u_1^2c^2 - 2u_1u_3sc + u_3^2s^2 + u_2^2c^2 + 2u_2u_4cs + u_4^2s^2 \\ &\quad + u_1^2s^2 - 2u_1u_3sc + u_3^2c^2 + u_2^2s^2 + 2u_2u_4sc + u_4^2c^2 \\ &= (u_1^2 + u_2^2 + u_3^2 + u_4^2)\cosh(2t) + 2(-u_1u_3 + u_2u_4)\sinh(2t) \\ &= \cosh(2t) - z_{13}\sinh(2t). \end{aligned}$$

Now, Eq. (3.5) becomes

$$\begin{aligned} \begin{bmatrix} cz_{11} & cz_{12} & cz_{13} - s \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix} &= (\cosh(2t) - z_{13}\sinh(2t))^{-1} \\ &\quad \times \begin{bmatrix} \overline{z}_{11} - sz_{11}\overline{z}_{13} & \overline{z}_{12} - sz_{12}\overline{z}_{13} & \overline{z}_{13}(c - sz_{13}) \\ \overline{z}_{21} - sz_{11}\overline{z}_{23} & \overline{z}_{22} - sz_{12}\overline{z}_{23} & \overline{z}_{23}(c - sz_{13}) \\ \overline{z}_{31} - sz_{11}\overline{z}_{33} & \overline{z}_{32} - sz_{12}\overline{z}_{33} & \overline{z}_{33}(c - sz_{13}) \end{bmatrix}. \end{aligned} \tag{3.6}$$

We compute $\overline{z} = p(\overline{u})$:

$$
\begin{aligned}
[\overline{z}]_{1,1} &= \overline{u}_1^2 - \overline{u}_2^2 - \overline{u}_3^2 + \overline{u}_4^2 \\
&= u_1^2c^2 - \cancel{2u_1u_3sc} + u_3^2s^2 - u_2^2c^2 - \cancel{2u_2u_4sc} - u_4^2s^2 \\
&\quad - u_1^2s^2 + \cancel{2u_1u_3sc} - u_3^2c^2 + u_2^2s^2 + \cancel{2u_2u_4sc} + u_4^2c^2 \\
&= u_1^2 - u_2^2 - u_3^2 + u_4^2 = z_{11};
\end{aligned}
$$

$$
\begin{aligned}
[\overline{z}]_{1,2} &= -2(\overline{u}_1\overline{u}_2 + \overline{u}_3\overline{u}_4) \\
&= -2(u_1u_2c^2 - \cancel{u_2u_3sc} + \cancel{u_1u_4sc} - u_3u_4s^2 - \\
&\quad - u_1u_2s^2 + u_3u_4c^2 + \cancel{u_2u_3sc} - \cancel{u_1u_4sc}) \\
&= -2(u_1u_2 + u_3u_4) = z_{12};
\end{aligned}
$$

$$
\begin{aligned}
[\overline{z}]_{1,3} &= 2(\overline{u}_1\overline{u}_3 - \overline{u}_2\overline{u}_4) \\
&= 2(-u_1^2sc + u_1u_3s^2 + u_1u_3c^2 - u_3^2sc \\
&\quad - u_2^2sc - u_2u_4s^2 - u_2u_4c^2 - u_4^2sc) \\
&= 2(u_1u_3 - u_2u_4)\cosh(2t) - (u_1^2 + u_2^2 + u_3^2 + u_4^2)\sinh(2t) \\
&= z_{13}\cosh(2t) - \sinh(2t);
\end{aligned}
$$

$$
\begin{aligned}
[\overline{z}]_{2,1} &= 2(-\overline{u}_1\overline{u}_2 + \overline{u}_3\overline{u}_4) \\
&= 2(-u_1u_2s^2 + u_2u_3sc - u_1u_4sc + u_3u_4c^2 \\
&\quad - u_1u_2c^2 + u_2u_3sc - u_1u_4sc + u_3u_4s^2) \\
&= 2(-u_1u_2 + u_3u_4)\cosh(2t) + 2(-u_1u_4 + u_2u_3) \\
&= z_{21}\cosh(2t) - z_{32}\sinh(2t);
\end{aligned}
$$

$$
\begin{aligned}
[\overline{z}]_{2,2} &= -\overline{u}_1^2 + \overline{u}_2^2 - \overline{u}_3^2 + \overline{u}_4^2 \\
&= -u_1^2c^2 + 2u_1u_3sc - u_3^2s^2 + u_2^2c^2 + 2u_2u_4sc + u_4^2s^2 \\
&\quad - u_1^2s^2 + 2u_1u_3sc - u_3^2c^2 + u_2^2s^2 + 2u_2u_4sc + u_4^2c^2 \\
&= (-u_1^2 + u_2^2 - u_3^2 + u_4^2)\cosh(2t) + 2(u_1u_3 + u_2u_4)\sinh(2t) \\
&= z_{22}\cosh(2t) - z_{31}\sinh(2t);
\end{aligned}
$$

$$\begin{aligned}
[\overline{z}]_{2,3} &= -2(\overline{u}_1\overline{u}_4 + \overline{u}_2\overline{u}_3)\\
&= -2(\cancel{u_1u_2sc} - u_2u_3s^2 + u_1u_4c^2 - \cancel{u_3u_4sc}\\
&\quad - \cancel{u_1u_2sc} - u_1u_4s^2 + u_2u_3c^2 + \cancel{u_3u_4sc})\\
&= -2(u_1u_4 + u_2u_3) = z_{23};
\end{aligned}$$

$$\begin{aligned}
[\overline{z}]_{3,1} &= 2(\overline{u}_1\overline{u}_3 + \overline{u}_2\overline{u}_4)\\
&= 2(-u_1^2sc + u_1u_3s^2 + u_1u_3c^2 - u_3^2sc\\
&\quad + u_2^2sc + u_2u_4s^2 + u_2u_4c^2 + u_4^2sc)\\
&= 2(u_1u_3 + u_2u_4)\cosh(2t) + (-u_1^2 + u_2^2 - u_3^2 + u_4^2)\sinh(2t)\\
&= z_{31}\cosh(2t) + z_{22}\sinh(2t);
\end{aligned}$$

$$\begin{aligned}
[\overline{z}]_{3,2} &= (\overline{u}_1\overline{u}_4 - \overline{u}_2\overline{u}_3)\\
&= 2(u_1u_2sc - u_2u_3s^2 + u_1u_4c^2 - u_3u_4sc\\
&\quad + u_1u_2sc + u_1u_4s^2 - u_2u_3c^2 - u_3u_4sc)\\
&= 2(u_1u_4 - u_2u_3)\cosh(2t) + 2(u_1u_2 - u_3u_4)\sinh(2t)\\
&= z_{32}\cosh(2t) - z_{21}\sinh(2t);
\end{aligned}$$

$$\begin{aligned}
[\overline{z}]_{3,3} &= -\overline{u}_1^2 - \overline{u}_2^2 + \overline{u}_3^2 + \overline{u}_4^2\\
&= -u_1^2c^2 + \cancel{2u_1u_3sc} - u_3^2s^2 - u_2^2c^2 - \cancel{2u_2u_4sc} - u_4^2s^2\\
&\quad + u_1^2s^2 - \cancel{2u_1u_3sc} + u_3^2c^2 + u_2^2s^2 + \cancel{2u_2u_4sc} + u_4^2c^2\\
&= (-u_1^2 - u_2^2 + u_3^2 + u_4^2) = z_{33}.
\end{aligned}$$

Now, we need to verify (3.6) for each entry.

$$(3.6)_{1,1} : z_{11}\cosh(2t) = (\cosh(2t) - z_{13}\sinh(2t))^{-1} \times (\overline{z}_{11} - z_{11}\overline{z}_{13}\sinh(2t))$$

$$\begin{aligned}
\cosh(2t) = (\cosh(2t) - z_{13}\sinh(2t))^{-1}(1 - \sinh(2t)\times\cosh(2t) + \sinh^2(2t))\\
= \cosh(2t)
\end{aligned}$$

$$(3.6)_{1,2} : z_{12}\cosh(2t) = (\cosh(2t) - z_{13}\sinh(2t))^{-1} \times (\overline{z}_{12} - z_{12}\overline{z}_{13}\sinh(2t))$$

$$\cosh(2t) = (\cosh(2t) - z_{13}\sinh(2t))^{-1}(1 - \sinh(2t) \times \cosh(2t) + \sinh^2(2t)) \\ = \cosh(2t)$$

$$(3.6)_{1,3} : z_{13}\cosh(2t) - \sinh(2t) = (\cosh(2t) - z_{13}\sinh(2t))^{-1}\overline{z}_{13}(\cosh(2t) \\ - z_{13}\sinh(2t))$$

$$1 = (\cosh(2t) - z_{13}\sinh(2t))^{-1}(\cosh(2t) - z_{13}\sinh(2t)) = 1$$

$$(3.6)_{2,1} : z_{21} = (\cosh(2t) - z_{13}\sinh(2t))^{-1} \times (\overline{z}_{21} - z_{11}\overline{z}_{23}\sinh(2t))$$
$$z_{13}z_{21} = z_{32} + z_{11}z_{23}$$
$$u_1u_4 - u_2u_3 = (u_1u_4 - u_2u_3)(u_1^2 + u_2^2 + u_3^2 + u_4^2) = u_1u_4 - u_2u_3$$

$$(3.6)_{2,2} : z_{22} = (\cosh(2t) - z_{13}\sinh(2t))^{-1} \times (\overline{z}_{22} - z_{12}\overline{z}_{23}\sinh(2t))$$
$$z_{13}z_{22} = z_{31} + z_{12}z_{23}$$
$$u_1u_3 + u_2u_4 = (u_1u_3 + u_2u_4)(u_1^2 + u_2^2 + u_3^2 + u_4^2) = u_1u_3 + u_2u_4$$

$$(3.6)_{2,3} : z_{23} = (\cosh(2t) - z_{13}\sinh(2t))^{-1} \times \overline{z}_{23}(\cosh(2t) - z_{13}\sinh(2t))$$
$$1 = (\cosh(2t) - z_{13}\sinh(2t))^{-1}(\cosh(2t) - z_{13}\sinh(2t)) = 1$$

$$(3.6)_{3,1} : z_{31} = (\cosh(2t) - z_{13}\sinh(2t))^{-1} \times (\overline{z}_{31} - z_{11}\overline{z}_{33}\sinh(2t))$$
$$\begin{aligned} z_{11}z_{33} &= z_{22} + z_{13}z_{31} - u_1^2 + u_2^2 - u_3^2 + u_4^2 \\ &= (-u_1^2 + u_2^2 - u_3^2 + u_4^2)(u_1^2 + u_2^2 + u_3^2 + u_4^2) \\ &= -u_1^2 + u_2^2 - u_3^2 + u_4^2 \end{aligned}$$

$$(3.6)_{3,2} : z_{32} = (\cosh(2t) - z_{13}\sinh(2t))^{-1} \times (\overline{z}_{32} - z_{12}\overline{z}_{33}\sinh(2t))$$
$$z_{13}z_{32} = z_{21} + z_{12}z_{33}$$
$$\begin{aligned} - u_1u_2 + u_3u_4 &= (-u_1u_2 + u_3u_4)(u_1^2 + u_2^2 + u_3^2 + u_4^2) \\ &= -u_1u_2 + u_3u_4 \end{aligned}$$

$$(3.6)_{3,3} : z_{33} = (\cosh(2t) - z_{13}\sinh(2t))^{-1} \times \overline{z}_{33}(\cosh(2t) - z_{13}\sinh(2t))$$
$$1 = (\cosh(2t) - z_{13}\sinh(2t))^{-1}(\cosh(2t) - z_{13}\sinh(2t)) = 1$$

This finishes our verification of (3.6).

Let us use the expression "(3.1) holds for g" if

$$p(g(z)) = (qg)(pz) \tag{3.1g}$$

holds for *that* element g of SL(4) (whereas z goes through all of SU(2)); here, q is the described above homomorphism. At this point in time, we know that (3.1) holds for any element g of the five 1-parameter subgroups in SL(4). Four of these subgroups correspond to compact generators and one subgroup corresponds to the non-compact generator. Similarly, "(3.1h) is satisfied" means that (3.1) holds for an element h of SL(4).

Lemma 2. *If* (3.1) *holds for both* g *and* h*, then it holds for their product* gh*.*

Proof. We use f to denote gh. Then, for the left-hand side of (3.1f), we have

$$p(f(z)) = p(g(h(z)) = (qg)(p(h(z)).$$

In the above, we use that SL(4) acts on SU(2) and that (3.1g) holds. For the right-hand side of (3.1f), we have

$$(qf)(pz) = \{(qg)(qh)\}\{(pz)\} = (qg)\{(qh)(pz)\} = (qg)\{p(h(z))\}.$$

Here, we use that q is a homomorphism, that SO(3, 3) acts on SO(3), and that (3.1h) holds. □

Lemma 2 guarantees that the diagram (3.1) is commutative: it is well known that any element of SL(4) can be represented as the product of a finite number of elements from the above-mentioned five 1-parameter subgroups. The same holds for any element of $SO(3,3)$. Theorem 3 is proved. □

Theorem 4. *The* SO(3, 3) *action* (2.3) *is projective.*

Proof. We recall that the 2-cover p can be viewed as an isometry from SU(2) onto SO(3) when both spaces are equipped with bi-invariant Riemannian metrics. Let $\widetilde{l}$ be a (one-dimensional) coset in SO(3) and $\widetilde{g}$ be an element of SO(3, 3). We need to prove that $\widetilde{g}(\widetilde{l})$ is a coset. To do so, we cover $\widetilde{g}$ by g from SL(4). Also, we cover $\widetilde{l}$ by a coset l from SU(2). Since SL(4) action on SU(2) is projective, it follows that $g(l)$ is a coset. Due to the diagram (3.1), $g(l)$ covers $\widetilde{g}(\widetilde{l})$. Since p is an isometry, this last curve has to be a geodesic, hence a coset. Theorem 4 is proved. □

Acknowledgment. The second author is grateful to Steve Rosenberg for a helpful discussion.

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Поступило 3 ноября 2018 г.

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