Записки научных
семинаров ПОМИ
Том 473, 2018 г.

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OFF-SHELL BETHE STATES AND THE SIX-VERTEX MODEL

Abstract. We study the symmetric six-vertex model on a finite square lattice with the partial domain wall boundary conditions. We use the known connection of the model with the off-shell Bethe states of the Heisenberg XXZ spin chain. We obtain various formulas for the partition function, and also discuss the model in the limit of semi-infinite lattice.

Dedicated to M. A. Semenov-Tain-Shansky on the occasion of his 70-th birthday

§1. Introduction

Possibly the most intriguing results obtained in the study of vertex models with fixed boundary condition are expressions in terms of determinants for their partition functions. Examples include the six-vertex model with domain wall boundary conditions [1–3] as well as with their various modifications closely related to symmetry classes of alternating-sing matrices [4–6]. A nice determinant formula for the partition function is also known for the five-vertex model with the fixed boundary conditions which describe a scalar product of two off-shell Bethe states [7–10].

Under certain restrictions on the vertex weights, some determinant formulas can be derived for the six-vertex model with the so-called partial domain wall boundary condition [11, 12]. These boundary conditions are interesting because they are a mixture of fixed and open boundary conditions. In [11, 12], these formulas were obtained in the limit where the values of a subset of spectral parameters of the six-vertex model with domain wall boundary conditions are sent to infinity; the limit is possible either

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Key words and phrases: partial domain wall boundary conditions, Izergin–Korepin partition function, coordinate Bethe Ansatz, determinant representations.

This work is supported in part by the Russian Foundation for Basic Research, under grant No. 16-01-00296, and by the Program of the Presidium of the Russian Academy of Sciences No. 02 "Nonlinear Dynamics: Fundamental Problems and Applications", under grant PRAS-18-02.

in the case of the rational weights or in the case of trigonometric weights with a non-vanishing asymmetry (external field).

From the point of view of the phase diagram of the six-vertex model these cases fall into the region of the ferroelectric phase ($\Delta \geqslant 1$). At the same time, it is well-known that the most interesting physics (and mathematics) the six-vertex model demonstrates with the weights corresponding to the disordered ($|\Delta| < 1$) and anti-ferroelectric ($\Delta \leqslant -1$) phases. For example, it is well known that in these phases in the case of domain wall boundary conditions the six-vertex model demonstrates phase separation phenomena (see, e.g. [13–16] and references therein). Recent numerical studies show that these phenomena also present in the partial domain-wall case [17]. It is therefore interesting to study in more detail the six-vertex model with partial domain wall boundary conditions with generic weights.

In the present paper we address the problem of calculation of the partition function of this model with arbitrary symmetric weights. We rely on the known connection of the model with the off-shell Bethe states of the Heisenberg XXZ spin chain; specifically, we use their coordinate representation. We obtain various formulas for the partition function and also discuss the model in the limit of semi-infinite lattice. In particular, we show that in this limit the partition function admits the representation in terms of a pfaffian.

§2. Partial domain wall boundary conditions

We consider the six-vertex model in its standard formulation in terms of arrows placed on edges of a square lattice (see, e.g., [18]). The six allowed configurations of arrows around a vertex are shown in Fig. 1, together with their Boltzmann weights. We consider here only the case of symmetric model (zero external field), in which the Boltzmann weights of vertices are invariant under reversal of all arrows. Hence, there are three weight functions: a, b, and c. The parameter

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab} \tag{1}$$

plays an important role in physics of the model. For real positive weights, $\Delta \in \mathbb{R}$.

The partial domain wall boundary conditions mean that the model is considered an $s \times N$ lattice (i.e., the square lattice obtained by intersection of s horizontal and N vertical lines), $s \leqslant N$, and the arrows on the

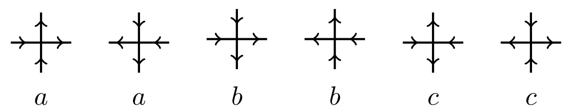

Figure 1. The six vertices and their weights.

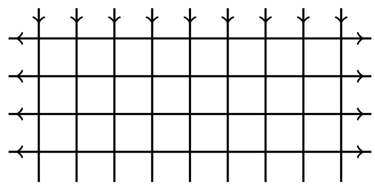

Figure 2. The $N \times s$ lattice with the partial domain-wall boundary conditions; here $N = 9$ and $s = 4$.

external edges are fixed as follows. On the horizontal lines at the left and right boundaries they are outgoing, while on the vertical ones at the top boundary they are incoming. On the vertical lines at the bottom boundary the arrows are not fixed, see Fig. 2, where the empty edges denote the sums over possible orientation of arrows on these edges. In the special case $s = N$ the only possible configuration of arrows at the bottom boundary is with all the arrows being incoming, and hence in total the boundary conditions are exactly the domain wall ones.

We denote the partition function of the model as $Z_{N,s}$. It is defined as the sum over all possible configurations

$$Z_{N,s} = \sum_{\text{arrow configurations}} a^{n_a} b^{n_b} c^{n_c}. \tag{2}$$

Here, n_a, n_b, n_c are the number of the vertices with weights a, b, c, respectively, $n_a + n_b + n_c = sN$.

To study it, we consider here more general model with the weights inhomogeneous along the vertical direction, with the condition that the parameter (1) is independent of the position of the vertex. Denote the weights of the jth horizontal line (counted, say, from the top, $j = 1, \dots, s$), by a_j, b_j, c_j. The inhomogeneity can be described, for example, by the variables

$$t_j = \frac{b_j}{a_j}, \qquad j = 1, \dots, s.$$

As functions of t_j and Δ, the weights read

$$a_j = 1, \qquad b_j = t_j, \qquad c_j = \sqrt{1 - 2\Delta t_j + t_j^2}, \tag{3}$$

where the specific normalization is chosen for a later convenience. The partition function of this model is

$$Z_{N,s}(t_1, \dots, t_s) = \sum_{\text{arrow configurations}} \prod_{j=1}^{s} t_j^{\nu_j} \left(1 - 2\Delta t_j + t_j^2\right)^{\mu_j/2}, \tag{4}$$

where ν_j and μ_j are the numbers of the b- and c-weight vertices in the jth line, respectively. In what follows we mostly work with (4), which, where no confusion may arise, will be denoted simply as $Z_{N,s}$; we will call (2) as $Z_{N,s}$ in the homogeneous limit: $t_j \to t$, $j = 1, \dots, s$.

We end up this section by mentioning a simple property of $Z_{N,s}$. For this purpose it is useful to note that the six-vertex model with the arrow reversal symmetry additionally possess the the so-called crossing symmetry, which means the invariance of the vertices under reflection with respect to the vertical (or horizontal) axis and simultaneous exchange of the weight functions a and b, $a \leftrightarrow b$, see Fig. 1. Hence,

$$Z_{N,s}(t_1, \dots, t_s) = \left(\prod_{j=1}^{s} t_j^N \right) Z_{N,s}(t_1^{-1}, \dots, t_s^{-1}), \tag{5}$$

that can be obtained by noticing the symmetry of the lattice and boundary conditions of Fig. 2 under reflection with respect to the vertical axis.

§3. Relation with the off-shell Bethe states

We first briefly review formulation of the model in terms of the quantum inverse scattering method, and next pass to the formulation in terms of the off-shell Bethe states.

We shall follow paper [19], slightly reverting conventions to fit into the standard interpretation of operators. We define the down and left arrows to be associated with the spin up states $|\uparrow\rangle$, and the top and right arrows with the spin down states $|\downarrow\rangle$. Let us assign the local quantum spaces to the vertical lines of the lattice. Then the boundary conditions on the top boundary correspond to the all-spins-up state

$$|\Uparrow_N\rangle = \otimes_{j=1}^N |\uparrow_j\rangle.$$

Similarly, the open boundary conditions on the bottom boundary correspond to the state

$$|\Updownarrow_N\rangle = \otimes_{j=1}^N (|\uparrow_j\rangle + |\downarrow_j\rangle).$$

Fixing the convention that the operators act from top to bottom, one finds that the partition function (4) can be written as the matrix element

$$Z_{N,s}(t_1,\dots,t_s) = \langle \Updownarrow_N | B(\lambda_s)\cdots B(\lambda_1)|\Uparrow_N\rangle, \tag{6}$$

where the spectral parameters λ_j, $j=1,\dots,s$, are related to the variables t_j by

$$t_j = t(\lambda_j) := \frac{\sin(\lambda_j-\eta)}{\sin(\lambda_j+\eta)}, \qquad \Delta = \cos 2\eta,$$

and the operator $B(\lambda)$ is the top-right element of the quantum monodromy matrix:

$$T(\lambda) = L_N(\lambda)\cdots L_1(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}.$$

Here, $L_n(\lambda)$, $n=1,\dots,N$, are quantum L-operators

$$L_n(\lambda) = \begin{pmatrix} \frac{1+t(\lambda)}{2} + \frac{1-t(\lambda)}{2}\sigma_n^z & c(\lambda)\sigma_n^- \\ c(\lambda)\sigma_n^+ & \frac{1-t(\lambda)}{2} + \frac{1+t(\lambda)}{2}\sigma_n^z \end{pmatrix},$$

where $\sigma_n^{\pm,z}$ are Pauli spin operators of the nth local quantum space, and

$$c(\lambda) = \sin 2\eta / \sin(\lambda+\eta).$$

As it is well known, the commutation relations of the operators-elements of the monodromy matrix—are described by the trigonometric R-matrix, which obeys the Yang–Baxter relation, see, e.g., [20, Chap. VI]. Among these commutation relations is the commutativity property

$$[B(\lambda_j), B(\lambda_k)] = 0.$$

Hence, (6) implies that $Z_{N,s}(t_1,\dots,t_s)$ is totally symmetric with respect permutations of the variables $t_1,\dots,t_s$.

To explore further $Z_{N,s}(t_1,\dots,t_s)$, we shall use another formulation which is closely related to (6). In [19], while studying correlation functions of the six-vertex model with domain wall boundary conditions on an $N\times N$ lattice, it was noticed that somewhat fundamental role in their calculation plays the decomposition on two partition functions $Z^{\mathrm{top}}_{r_1,\dots,r_s}$ and $Z^{\mathrm{bot}}_{r_1,\dots,r_s}$.

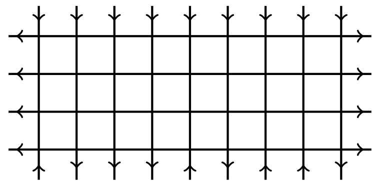

Figure 3. Definition of the partition function $Z^{\rm top}_{r_1,\dots,r_s}$: the up arrows at the bottom boundary are fixed at the positions $1 \leqslant r_1 < \cdots < r_s \leqslant N$, counted from the right. Here, as in Fig. 2, $N = 9$, $s = 4$, and $r_1 = 2, r_2 = 3$, $r_3 = 5$, $r_4 = 9$.

They are defined such that $Z^{\rm top}_{r_1,\dots,r_s}$ (respectively, $Z^{\rm bot}_{r_1,\dots,r_s}$) gives the partition function of the model on the top (bottom) portions of the lattice of the size $s \times N$ ($(N-s)\times N$) with the up arrows located at the positions $r_1, \dots, r_s$ on the bottom (top) boundary, see Fig. 3.

It is clear, that $Z_{N,s}$ can be represented as the sum

$$Z_{N,s} = \sum_{1\leqslant r_1<\ldots<r_s\leqslant N} Z^{\rm top}_{r_1,\dots,r_s}. \tag{7}$$

Similarly to (6), one may write

$$Z^{\rm top}_{r_1,\dots,r_s} = \langle 0|\sigma^+_{r_1}\cdots\sigma^+_{r_s} B(\lambda_s)\cdots B(\lambda_1)|0\rangle.$$

This formula means that $Z^{\rm top}_{r_1,\dots,r_s}$ is a component of the off-shell Bethe s-particle state. Furthermore, using the correspondence between the algebraic and coordinate versions of Bethe Ansatz [21] (see also [20, Chap. VII, App. 2]), one may directly write $Z^{\rm top}_{r_1,\dots,r_s}$ in the coordinate form [19]:

$$\begin{aligned} Z^{\rm top}_{r_1,\dots,r_s} = \prod_{j=1}^{s} c_j \prod_{1\leqslant j<k\leqslant s} \frac{1}{t_k - t_j} \\ \times \sum_{\sigma\in\Omega_s} (-1)^{[\sigma]} \prod_{j=1}^{s} t_{\sigma(j)}^{r_j-1} \prod_{1\leqslant j<k\leqslant s} (1 - 2\Delta t_{\sigma(j)} + t_{\sigma(j)} t_{\sigma(k)}). \end{aligned} \tag{8}$$

Here $c_j \equiv (1-2\Delta t_j + t_j^2)^{1/2}$, see (3). The sum is performed over elements of the symmetric group Ω_s, i.e., permutations $\sigma : 1,\dots,s \mapsto \sigma(1),\dots,\sigma(s)$, and $[\sigma]$ denotes parity of σ.

§4. Some results for the partition function

Let us study $Z_{N,s}$ using (7) and (8). Apparently, from these formulas it follows that $Z_{N,s}(t_1,\dots,t_s)$, modulo factor $c_1\cdots c_s$, see (3), is a symmetric polynomial in its variables. Furthermore, in addition to the relation (5) expressing the crossing symmetry, one can notice the reduction formula

$$Z_{N,s}(t_1,\dots,t_s)\big|_{t_s\to 0}=Z_{N-1,s-1}(t_1,\dots,t_{s-1}),$$

where, due to the symmetry, t_s can be replaced by any of $t_1,\dots,t_{s-1}$.

To get more information about the structure of $Z_{N,s}$, let us consider the summation in (7). Let us denote

$$\Sigma_s=\Sigma_s(t_1,\dots,t_s):=\sum_{1\leqslant r_1<r_2<\ldots<r_s\leqslant N}\prod_{j=1}^{s}t_j^{r_j-1}. \tag{9}$$

We have, for example,

$$\begin{aligned}
\Sigma_1&=\frac{1}{1-t_1}-\frac{t_1^N}{1-t_1},\\
\Sigma_2&=\frac{t_2}{(1-t_2)(1-t_1t_2)}-\frac{t_2^N}{(1-t_1)(1-t_2)}+\frac{t_1^Nt_2^N}{(1-t_1)(1-t_1t_2)},\\
\Sigma_3&=\frac{t_2t_3^2}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)}-\frac{t_2t_3^N}{(1-t_2)(1-t_3)(1-t_1t_2)}\\
&\quad+\frac{t_2^Nt_3^N}{(1-t_1)(1-t_2)(1-t_2t_3)}-\frac{t_1^Nt_2^Nt_3^N}{(1-t_1)(1-t_1t_2)(1-t_1t_2t_3)}.
\end{aligned}$$

Inspecting these expressions it is not difficult to write out the result in the general case.

Lemma 1. *For the quantities Σ_s defined in* (9), *the following formula is valid*:

$$\Sigma_s=\sum_{j=0}^{s}(-1)^j\prod_{k=1}^{s-j}\frac{t_k^{k-1}}{1-\prod\limits_{l=k}^{s-j}t_l}\prod_{k=s-j+1}^{s}\frac{t_k^N}{1-\prod\limits_{l=s-j+1}^{k}t_l}. \tag{10}$$

Clearly, a proof can be given by induction in s; we skip the proof since it is purely technical.

Having in mind the expression (10) for Σ_s, we can write the following representation:

$$
\begin{aligned}
Z_{N,s} = \prod_{j=1}^{s} c_j \prod_{1\leqslant j<k\leqslant s} \frac{1}{t_k - t_j} \sum_{\sigma\in\Omega_s} (-1)^{[\sigma]}\, \Sigma_s(t_{\sigma(1)},\dots,t_{\sigma(s)}) \\
\times \prod_{1\leqslant j<k\leqslant s} (1 - 2\Delta t_{\sigma(j)} + t_{\sigma(j)} t_{\sigma(k)}).
\end{aligned}
\tag{11}
$$

We have, for example,

$$
\begin{aligned}
Z_{N,1} &= \frac{c_1}{1-t_1}\left(1 - t_1^N\right), \\
Z_{N,2} &= \frac{c_1 c_2}{(1-t_1)(1-t_2)(1-t_1t_2)} \\
&\quad \times \Bigg\{ 1 + (1-2\Delta) t_1 t_2 - (1 - 2\Delta + t_1 t_2)\, t_1^N t_2^N \\
&\qquad - (1 - t_1 t_2) \frac{(1+t_1t_2)(t_2^N - t_1^N) - 2\Delta (t_1 t_2^N - t_2 t_1^N)}{t_2 - t_1} \Bigg\}.
\end{aligned}
$$

The expression in the $s = 3$ case is already too bulky to be given here. Nevertheless, studying this and other cases (with the help of symbolic manipulation software) we have observed rather intriguing property of $Z_{N,s}$. Namely, it turns out that there is a nontrivial interplay between the factors standing in the denominator in (10) and the sum over permutations and double product in (11), so that the resulting expression for $Z_{N,s}$ contains only single and double products in the denominator.

To make this property more transparent, let us consider the sum over permutations in (11) but with Σ_s replaced by the expression

$$
\begin{aligned}
\widetilde{\Sigma}_s(t_1,\dots,t_s;x_1,\dots,x_s) = \sum_{j=0}^{s} (-1)^j \prod_{\ell=1}^{s-j} \frac{t_\ell^{\ell-1}}{1 - \prod\limits_{k=\ell}^{s-j} t_k} \\
\times \prod_{\ell=s-j+1}^{s} \frac{x_\ell}{1 - \prod\limits_{k=s-j+1}^{\ell} t_k},
\end{aligned}
\tag{12}
$$

where $x_1,\dots,x_s$ are dummy variables. Our observation is the following.

Conjecture 1. *The sum over permutations with the function $\widetilde{\Sigma}_s$ defined in* (12) *has the form*

$$\sum_{\sigma\in\Omega_s}(-1)^{[\sigma]}\,\widetilde{\Sigma}_s(t_{\sigma(1)},\dots,t_{\sigma(s)};x_{\sigma(1)},\dots,x_{\sigma(s)})$$
$$\times\prod_{1\leqslant j<k\leqslant s}(1-2\Delta t_{\sigma(j)}+t_{\sigma(j)}t_{\sigma(k)})$$
$$=\prod_{j=1}^{s}\frac{1}{1-t_j}\prod_{1\leqslant j<k\leqslant s}\frac{1}{1-t_jt_k}\,\mathcal{A}_s(t_1,\dots,t_s;x_1,\dots,x_s),$$

where $\mathcal{A}_s(t_1,\dots,t_s;x_1,\dots,x_s)$ is a polynomial of all its arguments, totally antisymmetric with respect to permutations of pairs $(t_1,x_1),\dots,(t_s,x_s)$, which possesses the property

$$\mathcal{A}_s(t_1,\dots,t_s;t_1^N,\dots,t_s^N)=\prod_{1\leqslant j<k\leqslant s}(t_k-t_j)\,P_{N,s}(t_1,\dots,t_s).$$

Here, $P_{N,s}(t_1,\dots,t_s)$ is a symmetric polynomial, which has simple zeros in each variable t_j, $j=1,\dots,s$, at the points

$$t_j=t_1^{-1},\dots,t_{j-1}^{-1},1,t_{j+1}^{-1},\dots,t_s^{-1}.$$

Apparently, the polynomial $P_{N,s}$ essentially determines the partition function; we thus have

$$Z_{N,s}=\prod_{j=1}^{s}\frac{c_j}{1-t_j}\prod_{1\leqslant j<k\leqslant s}\frac{1}{1-t_jt_k}\,P_{N,s}(t_1,\dots,t_s). \tag{13}$$

It is very plausibly that this polynomial can be written in some determinant form. In the general situation which we considered so far, existence of such a form remains open, but for some values of the parameters this is indeed the case. In the remaining part of this section we list such cases.

We start with the technically simplest case $\Delta=0$, in which a determinant representation usually exists due to the free-fermion nature of the six-vertex model at this point. In this case,

$$Z_{N,s}^{\Delta=0}=\prod_{j=1}^{s}\frac{c_j}{1-t_j}\prod_{1\leqslant j<k\leqslant s}\frac{1+t_jt_k}{(1-t_jt_k)(t_k-t_j)}\det_{1\leqslant j,k\leqslant s}\left[t_k^{j-1}-t_k^{N+s-j}\right]. \tag{14}$$

This formula can be proven, e.g., using the technique of Schur functions [22]. Indeed, in this case the double product in (11) is symmetric and thus can be moved out of the sum over permutations, so one may use for Σ_s its definition (9).

Much less trivial and very interesting case is $\Delta = 1$, which was studied in [11,12]. In the framework of our approach, the polynomial $P_{N,s}(t_1,\dots,t_s)$ arising in this case factorizes on the double product exactly that standing in the denominator in (13), and a factor which clearly has the form of a determinant, so the result reads

$$Z_{N,s}^{\Delta=1} = \prod_{j=1}^{s} \frac{c_j}{1-t_j} \prod_{1\leqslant j<k\leqslant s} \frac{1}{t_k-t_j} \det_{1\leqslant j,k\leqslant s} \Big[(1-t_k)^{s-j}\big(t_k^{j-1}-t_k^N\big)\Big]. \qquad (15)$$

This formula seems to be new. In fact, it can be proven using the result obtained in [11]; we will give the details of this calculation elsewhere.

The remaining cases are closely related to each other and they are where $s = N-1$ and $s = N$, for arbitrary Δ. In fact these two cases are just the case of domain wall boundary conditions, the former case being a particular generating function for the boundary correlation function of this model [23,24]. The partition of function at $s = N-1$ is related to that at $s = N$ as follows

$$Z_{N,N-1}(t_1,\dots,t_{N-1}) = \frac{Z_{N,N}(t_1,\dots,t_N)}{c_N}\bigg|_{t_N\to 1}. \qquad (16)$$

Thus the determinant formula for $Z_{N,N-1}$ can obtained from that for $Z_{N,N}$. Since the last case is well-known, we do not discuss it here but give the details in appendix.

§5. Homogeneous limit and number of configurations

Let us consider the model in the homogeneous limit, $t_j \to t$, $j = 1,\dots,s$. In [19], it was shown that in this case $Z^{\mathrm{top}}_{r_1,\dots,r_s}$ admits the representation in terms of a multiple contour integral:

$$Z^{\mathrm{top}}_{r_1,\dots,r_s} = \frac{c^s}{(2\pi \mathrm{i})^s} \oint_{C_t} \cdots \oint_{C_t} \prod_{j=1}^{s} \frac{z_j^{r_j-1}}{(z_j-t)^s} \times \prod_{1\leqslant j<k\leqslant s} \big[(z_j-z_k)(1-2\Delta z_j + z_j z_k)\big]\, \mathrm{d}z_1\cdots \mathrm{d}z_s.$$

Here, C_t denotes a simple closed contour enclosing the point $z = t$, and $c \equiv (1-2\Delta t+t^2)^{1/2}$, see (3).

Correspondingly, for the partition function $Z_{N,s}$, we can write

$$Z_{N,s} = \frac{c^s}{(2\pi\mathrm{i})^s} \oint_{C_t} \cdots \oint_{C_t} \Sigma_s(z_1,\dots,z_s) \prod_{j=1}^{s} \frac{1}{(z_j-t)^s} \times \prod_{1\leqslant j<k\leqslant s} [(z_j-z_k)(1-2\Delta z_j+z_jz_k)]\,\mathrm{d}z_1\cdots\mathrm{d}z_s, \tag{17}$$

where Σ_s is given by (10).

The practical meaning of (17) is that it can be used, for example, for computing the number of configurations. Let us denote this number by $K_{N,s}$. It is given by the partition function for $t=c=1$, that is

$$K_{N,s} = Z_{N,s}^{\Delta=1/2}\big|_{t=1}.$$

For generic N and first few values of s, we find:

$$\begin{aligned}
K_{N,1} &= N,\\
K_{N,2} &= \binom{N}{2}\frac{N+4}{3},\\
K_{N,3} &= \binom{N+1}{4}\frac{N^2+14N+54}{15}\\
K_{N,4} &= \binom{N+1}{5}\frac{(N+6)(N+8)\left(N^3+21N^2+128N-30\right)}{2520},\\
K_{N,5} &= \binom{N+2}{7}\frac{N}{907\,200}\big(N^7+67N^6+1897N^5+28\,525N^4\\
&\quad +234\,724N^3+937\,468N^2+786\,498N-3\,753\,180\big).
\end{aligned}$$

These numbers may have a combinatorial meaning. Note that

$$K_{s,s-1} = K_{s,s} = A_s, \tag{18}$$

where A_s is the number of $s\times s$ alternating-sign matrices, $A_1=1$, $A_2=2$, $A_3=7$, $A_4=42$, $A_5=429$, and, in general,

$$A_s = \prod_{j=0}^{s-1}\frac{(3j+1)!}{(s+j)!}.$$

In (18) the first equality is due to (16), while the second one is due the well-known connection of the alternating-sign matrices with the six-vertex model with domain boundary conditions [25].

§6. The case of semi-infinite region

As it is follows from (10), the quantities Σ_s simplify significantly if $t_j^N \to 0$, $j = 1, \dots, s$. This can be achieved in the limit $N \to \infty$ under the condition that, if all t_j's are all real and positive,

$$0 \leqslant t_j < 1, \qquad j = 1, \dots, s. \tag{19}$$

If they need to be complex, then one has to require that $|t_j| < 1$, $j = 1, \dots, s$.

From the point of view of the six-vertex model with partial domain wall boundary conditions, the limit $N \to \infty$ means that the left boundary goes to infinity, so that the lattice is semi-infinite, with s rows. The boundary conditions on the left boundary are now effectively vanishing, that is guaranteed by (19). Note, that the condition (19) breaks the crossing symmetry.

To study the partition function of this model, it is useful to start from the particular cases, and next turn to the general case.

We first consider the case $\Delta = 0$. In the limit $N \to \infty$ the determinant in (14) can be readily evaluated, that yields

$$Z_{\infty,s}^{\Delta=0} = \prod_{j=1}^{s} \frac{c_j}{1 - t_j} \prod_{1 \leqslant j < k \leqslant s} \frac{1 + t_j t_k}{1 - t_j t_k}.$$

This expression remains finite as far as the condition (19) is fulfilled. Using that in the present case $c_j = (1 + t_j^2)^{1/2}$, in the homogeneous limit $t_j \to t$, $j = 1, \dots, s$, we get

$$Z_{\infty,s}^{\Delta=0} = \left(\frac{1 + t^2}{1 - t^2} \right)^{s^2/2} \left(\frac{1 + t}{1 - t} \right)^{s/2}.$$

Since the leading term of $\log Z_{\infty,s}^{\Delta=0}$ is $O(s^2)$ as $s \to \infty$, one may conclude that the model behaves just like it would be defined on an $s \times s$ lattice.

Let us now consider the case $\Delta = 1$. Evaluating the determinant in (15), we obtain

$$Z_{\infty,s}^{\Delta=1} = \prod_{j=1}^{s} \frac{c_j}{1 - t_j} = 1,$$

where we have used that, due to (19), $c_j = 1 - t_j$ in this case. The obtained result has a simple probabilistic meaning, as the normalization condition of a total probability. Indeed, the relation $c_j + t_j = 1$ reflects the stochasticity property of the six-vertex model at $\Delta = 1$ [26].

As for arbitrary Δ, it turns out that $Z_{\infty,s}$ can be given in terms of a pfaffian, due to the following statement[1].

Proposition 1 (L. Cantini [27]). *For $\tau \in \mathbb{C}$, the following identity holds:*

$$\sum_{\sigma\in\Omega_s} (-1)^{[\sigma]} \prod_{j=1}^{s} \frac{1}{x_{\sigma(j)}^{j}\left(1-\prod\limits_{k=1}^{j} x_{\sigma(k)}\right)} \prod_{1\leqslant j<k\leqslant s} (1+\tau x_{\sigma(k)} + x_{\sigma(j)}x_{\sigma(k)})$$

$$= (-1)^{s(s-1)/2} \prod_{j=1}^{s} \frac{1}{x_j^s(1-x_j)} \prod_{1\leqslant j<k\leqslant s} (x_j+x_k+\tau x_j x_k)$$

$$\times \underset{1\leqslant j<k\leqslant g}{\mathrm{Pf}} \left[\frac{(x_k-x_j)\,(1+(1+\tau)x_jx_k)}{(1-x_jx_k)(x_j+x_k+\tau x_jx_k)}\right],$$

where $g=s$, if s is even, and $g=s+1$, with $x_{s+1}\equiv 1$, if s is odd.

Clearly, to apply this result to $Z_{\infty,s}$ one has to set $\tau = -2\Delta$ and $x_j = t_{s-j+1}$, $j=1,\dots,s$. Hence,

$$Z_{\infty,s} = \prod_{j=1}^{s} \frac{c_j}{1-t_j} \prod_{1\leqslant j<k\leqslant s} \frac{t_j+t_k-2\Delta t_jt_k}{t_k-t_j}$$

$$\times \underset{1\leqslant j<k\leqslant g}{\mathrm{Pf}} \left[\frac{(t_k-t_j)\,(1+(1-2\Delta)t_jt_k)}{(1-t_jt_k)(t_j+t_k-2\Delta t_jt_k)}\right]. \qquad (20)$$

Here, g is the same as defined above, and $t_{s+1}\equiv 1$, in the case of s odd.

Apparently, just like for finite N, see (13), the partition function at $N=\infty$ has the structure

$$Z_{\infty,s} = \prod_{j=1}^{s} \frac{c_j}{1-t_j} \prod_{1\leqslant j<k\leqslant s} \frac{1}{1-t_jt_k} P_{\infty,s}(t_1,\dots,t_s),$$

where $P_{\infty,s}(t_1,\dots,t_s) \equiv \lim_{N\to\infty} P_{N,s}(t_1,\dots,t_s)$. The representation (20) essentially determines this symmetric polynomial, but in practical calculations one may prefer more to deal with a determinant rather than with a pfaffian. For example, finding the homogeneous limit of (20) is an awkward task.

Addressing the problem of a determinant form for $P_{\infty,s}(t_1,\dots,t_s)$, we have discovered that at least in the case $\Delta = 1/2$ it indeed admits a desirable solution.

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[1]We are grateful to F. Colomo for informing us about this result by L. Cantini.

Conjecture 2. *At $\Delta = 1/2$ the polynomial $P_{\infty,s}(t_1,\dots,t_s)$ is a polynomial in the variables $u_j = t_j(1-t_j)$, $j=1,\dots,s$, and reads*

$$P^{\Delta=1/2}_{\infty,s}(t_1,\dots,t_s) = \prod_{1\leqslant j<k\leqslant s} \frac{1}{u_k-u_j} \det_{1\leqslant j,k\leqslant s} \left[u_k^{j-1}(1-u_k)^{3[\frac{s-j}{2}]} \right], \quad (21)$$

where $[\frac{s-j}{2}]$ denotes the integer part of $\frac{s-j}{2}$.

We have obtained this result inspecting (for up to $s=7$) the polynomials $P_{\infty,s}(t_1,\dots,t_s)$ arising after evaluation the sum over permutations in (11), taking for Σ_s just single term contributing in the $N\to\infty$ limit (one may prefer instead to use (20) and the specially designed routine for pfaffians [28]). Proof of (21) remains open.

The authors are grateful to F. Colomo for useful discussions.

Appendix §A. The case of domain wall boundary conditions

In the case of $s=N$, the partial domain wall boundary conditions essentially are the domain wall ones. The partition function $Z_{N,N}$ is given by the Izergin-Korepin determinant in which the spectral parameters of one of the two sets (see [2, 3] or [24]) are set to zero. Namely, in our present notation the result has the following form.

Define functions $f_n(t)$, $n=1,2,\dots$, recursively by

$$f_n(t) = \frac{t}{n}\partial_t \left(t + t^{-1} - 2\Delta\right) f_{n-1}(t), \qquad f_0(t) = 1.$$

This definition originates from the formulas

$$f_n(t) = \frac{1}{n!\varphi(\lambda)} \partial_\lambda^n \varphi(\lambda), \qquad \varphi(\lambda) = \frac{1}{\sin(\lambda-\eta)\sin(\lambda+\eta)},$$

where, as we have already used in the main text,

$$t = t(\lambda) = \frac{\sin(\lambda-\eta)}{\sin(\lambda+\eta)}, \qquad \Delta = \cos 2\eta.$$

Then,

$$Z_{N,N} = \prod_{j=1}^{N} c_j \prod_{1\leqslant j<k\leqslant s} \frac{1}{t_k - t_j} \det_{1\leqslant j,k\leqslant N} \left[t_k^{N-1} f_{j-1}(t_k) \right].$$

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Поступило 19 ноября 2018 г.

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