Записки научных
семинаров ПОМИ
Том 473, 2018 г.

D. Keating, N. Reshetikhin, A. Sridhar

CONFORMAL LIMIT FOR DIMER MODELS ON THE HEXAGONAL LATTICE

Abstract. In this note we derive the asymptotical behavior of local correlation functions in dimer models on a domain of the hexagonal lattice in the continuum limit, when the size of the domain goes to infinity and parameters of the model scale appropriately.

Dedicated to the 70-th birthday of M. Semenov-Tian-Shansky

§1. Introduction

In this note we study the asymptotics of local correlation functions for dimer models on special domains of the hexagonal lattices. The main result is a formula for the asymptotics of the inverse to the Kasteleyn operator computed in two different ways: from the integral formula and from the definition. This note is a research report. Missing details will be completed in an extended version which will be also posted on the ArXiv.

Asymptotical formulae for local correlation functions of height functions in dimer models were computed in a number of papers for various regions and lattices, see for example [1–3].

Here we emphasize the relation to Dirac fermions, rather than to a Gaussian field as it was done, for example, in [1–3]. Dirac fermions can be written in terms of Gaussian field due to the Bose–Fermi correspondence in space one dimension, but the resulting expression is non-local. However, in many ways it is preferable to think of Dirac fermions as more fundamental objects.

Here is the plan of the paper. The first section is the introduction. In the second section we recall basic facts about dimer models on the hexagonal lattice. We compute the asymptotic of correlation functions for special domains using the integral representation in the third section. In section 5 we compute the same asymptotic using the definition of the inverse to the

Key words and phrases: Dimer models, Dirac fermions, Kasteleyn operator, Burgers equation, conformal correlation functions.

This work was partially supported by the NSF grant DMS-1601947. N.R. also acknowledges the support from the RFBR grant No. 18-01-00916.

Kasteleyn operator in terms of the difference equation. In the fifth section we state the asymptotical behavior of Kastelyn fermions in the continuum limit. The details will be given in an extended version of the paper.

§2. Dimers on the hexagonal lattice and the Kasteleyn operator

2.1. Dimer models on the hexagonal lattice. Let H be the hexagonal lattice with the bipartite structure shown on Fig. 1 and $\Gamma \subset H$ be a finite subgraph which is a connected, simply-connected domain in H without 1-valet vertices. In other words Γ is a connected, simply-connected domain assembled from elementary hexagons.

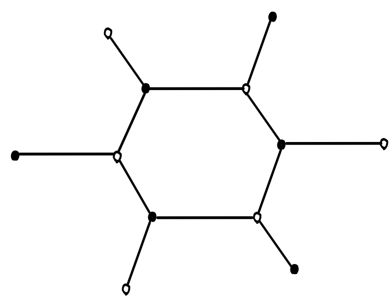

Figure 1. Hexagonal lattice with bipartite structure.

A dimer configuration on D is a perfect matching on vertices connected by edges. In other words, it is a partition of edges into two groups, occupied by a dimer and not occupied, such that each vertex should be occupied by a dimer and two dimers never share a common vertex. The Boltzmann weight of a dimer configuration is

$$w(D) = \prod_{e \in D} w(e),$$

where the product is taken over edges occupied in the dimer configuration D, and $w(e) > 0$ are weights of edges which should be fixed in order to define the model.

Boltzmann weights define the probability distribution on dimer configurations on Γ with

$$\mathrm{Prob}(D) = \frac{w(D)}{Z},$$

where Z is the partition function

$$Z = \sum_{D \subset \Gamma} w(D).$$

The characteristic function of an edge e on the space of dimer configurations is the function σ_e which has value on D of 1 when the e is occupied and 0 when e is not occupied. Local correlation functions for dimer models are expectation values of products of characteristic functions

$$E(e_1, \dots, e_n) = \sum_{D \subset \Gamma} \mathrm{Prob}(D) \prod_{i=1}^{n} \sigma_{e_i}.$$

It is clear that the dimer probability distribution and therefore local correlation functions are invariant with respect to transformations $w(e) \mapsto s(e_+)w(e)s(e_-)$ where s is any function on vertices with positive values and $e_\pm$ are endpoints of e.

For the hexagonal lattice (our terminology will match Fig. 1) this means that we can choose weights of tilted NW-SE edges and of the horizontal edges to be 1. And we will denote remaining weights of SW-NE edges by $x(e)$.

2.2. The Kasteleyn operator. As it was discovered in the 1960's the partition function and correlation functions of dimer models can be computed in terms of determinants. For details see original references [4,5] and an expository part of [6].

To define such determinantal solution we should choose a special orientation of edges, a Kasteleyn orientation. On the hexagonal lattice it can be chosen as it is shown on Fig. 2. In order to have determinants, not Pfaffians, one should choose an identification of black and white vertices. We assume that they are identified by horizontal edges.

Choose an embedding of the hexagonal lattice in a square grid as is shown on Fig. 2. We will denote coordinates of centers of horizontal edges as (t, h). Here $h \in \frac{1}{2}\mathbb{Z}$ and $t \in \mathbb{Z}$. Let $\mathcal{D} \subset \mathbb{Z} \times \frac{1}{2}\mathbb{Z}$ be a domain in the hexagonal lattice embedded imbedded in the square grid.

The Kasteleyn operator is a linear operator (a difference operator) acting on vertices of the graph. After the identification of black and white

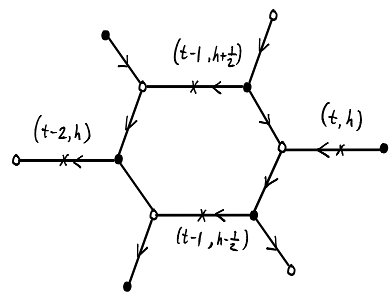

Figure 2. Hexagonal lattice with the Kasteleyn orientation which we use and with coordinates of horizontal edges which are identified with adjacent vertices.

vertices by horizontal dimers it becomes a difference operator on a domain in a square grid with coordinates $(t, h) \in \mathcal{D}$ acting as

$$(Kf)(h,t) = f(t,h) - f\Big(t-1, h+\frac{1}{2}\Big) + x\Big(t-\frac{1}{2}, h\Big) f\Big(t-1, h-\frac{1}{2}\Big). \quad (1)$$

It is convenient to think about such functions as functions on an extended domain $\tilde{\mathcal{D}}$ where we add edges with 1-valent vertices to boundary vertices and define $f(v) = 0$ for each 1-valet vertex v. According to the Kasteleyn theorem, the partition function Z is the absolute value of the determinant of K and local correlation functions can be computed in terms the inverse to K.

Let $R(t, h|t', h')$ be kernel of the inverse to the Kasteleyn operator on $\mathcal{D} \subset \mathbb{Z} \times \frac{1}{2}\mathbb{Z}$. That is if

$$Kf = g$$

then

$$f(t,h) = \sum_{(t',h') \in \mathcal{D}} R(t,h|t',h') g(t',h').$$

We have

$$R(t,h|t',h') - R\Big(t-1, h+\frac{1}{2}|t',h'\Big) + x\Big(t-\frac{1}{2}, h\Big) R\Big(t-1, h-\frac{1}{2}|t',h'\Big) = \delta(t,t')\delta(h,h'). \quad (2)$$

with boundary conditions $R(t,h|t',h') = 0$ when (h,t) correspond to a 1-valent vertex. When the domain $\mathcal{D}$ is noncompact one should impose boundary conditions when $(t,h) \to \infty$ but we will not go into details of this here.

Consider horizontal edges with coordinates $x_k = (t_k, h_k)$. Then for the local correlation function we have the following formula

$$E(x_1, \ldots, x_n) = \det(R(t_k, h_k|t_l, h_l))_{k,l=1}^n. \tag{3}$$

Note that Kasteleyn operators can be defined for non-compact domains as well, but that should be supplemented by appropriate boundary conditions.

2.3. Kasteleyn formions. The Kasteleyn solution of dimer models (the determinant formulae above) can be written in terms of Grassman integral. Let $V_{\mathcal{D}}$ be the real vector space where the basis is enumerated by vertices in the region $\mathcal{D}$. Choose an element $I \in \wedge^N V_{\mathcal{D}}$. It defines the Grassman integral over $\wedge^\bullet V_{\mathcal{D}}$ as

$$\int f = f_I$$

where $f \in \wedge^\bullet V_{\mathcal{D}}$ and f_I is its component in the basis $I \in \wedge^N V_{\mathcal{D}}$. Let $\psi(t,h)$ be elements of $\wedge^\bullet V_{\mathcal{D}}$ corresponding to the basis vectors in $V_{\mathcal{D}}$. Typically I is chosen as a monomial in ψ (longest ordered product with no repetitions). There are two choices of such integral I and $-I$.

Elements ψ are generators of the Grassman algebra $\wedge^\bullet V_{\mathcal{D}}$. In physics they are called fermions since

$$\psi(t',h')\psi(t,h) = -\psi(t,h)\psi(t',h').$$

In terms of generators we will write

$$\int f = \int f d\psi.$$

Similarly the Grassman integral can be defined for the dual vector space $V_{\mathcal{D}}^*$. We will denote corresponding fermions as $\psi^*(t,h)$.

The Grassman algebra $\wedge^\bullet(V_{\mathcal{D}} \oplus V_{\mathcal{D}}^*)$ is naturally isomorphic to $\wedge^\bullet V_{\mathcal{D}} \otimes \wedge^\bullet V_{\mathcal{D}}^*$. The integral on this algebra can be identifies with the tensor product of integrals. We will write

$$\int F = \int F d\psi^* d\psi$$

for such integral where F is a polynomial in anticommuting variables ψ, ψ^*, generators of $\wedge^\bullet(V_{\mathcal{D}} \oplus V_{\mathcal{D}}^*)$.

Define

$$A = \sum_{(t,h)\in\mathcal{D}} \psi^*(t,h)(K\psi)(t,h),$$

where $K\psi$ is defined as in (1).

The determinant formulae for the partition function and for correlation functions can be written in terms of fermions as

$$Z = \Big| \int e^A d\psi^* d\psi \Big|$$

and

$$E(x_1, \ldots, x_n) = \frac{\int e^A \psi(t_1,h_1)\psi^*(t_1,h_1)\ldots\psi(t_n,h_n)\psi^*(t_n,h_n)d\psi^* d\psi}{\int e^A d\psi^* d\psi}.$$

Note that neither the formula for the partition function, nor the formula for local correlation functions depends on the choice of monomials defining the integrals.

Also, note that the inverse to the Kasteleyn matrix can be written as

$$R(t,h|t',h') = \frac{\int e^A \psi(t,h)\psi^*(t',h')d\psi^* d\psi}{\int e^A d\psi^* d\psi}.$$

We will call ψ, ψ^* Kasteleyn fermions.

§3. Continuum limit from the integrals representation

3.1. Continuum limit. Denote by $\varphi_\epsilon : \mathbb{Z} \times \frac{1}{2}\mathbb{Z} \to \mathbb{R}^2$ the embedding of the square grid into $\mathbb{R}^2$ such that $(t,h) \mapsto (\epsilon t, \epsilon h)$. We are interested in the asymptotic of local correlation functions in the limit $\epsilon \to 0$ when the lattice domain D expands such that the image $\varphi_\epsilon(\mathcal{D})$ fills an $\mathbb{R}^2$ domain $\mathbb{D}$. Because of the determinantal formulae (3) it is enough to find the asymptotic of the kernel $R((t_1,h_1),(t_2,h_2))$ of the inverse Kasteleyn matrix.

We assume that as $\epsilon \to 0$ and the lattice region is expanding accordingly to fill the Euclidean domain $\mathbb{D}$, the coordinates t_i and h_i behaving as $t_i = \tau_i/\epsilon, h_i = \chi_i/\epsilon$ where $(\tau_i, \chi_i) \in \mathbb{D}$.

3.2. The integral formula for inverse to the Kasteleyn operator. For special lattice domains $\mathcal{D} \in \mathbb{Z} \times \frac{1}{2}\mathbb{Z}$ the kernel of $R = K^{-1}$ has a convenient integral representation. For a semiinfinite domain shown on Fig. 3 such representation was found in [7]. Boundary conditions at infinity are determined by asymptotical configuration of dimers as it is shown on Fig. 3.

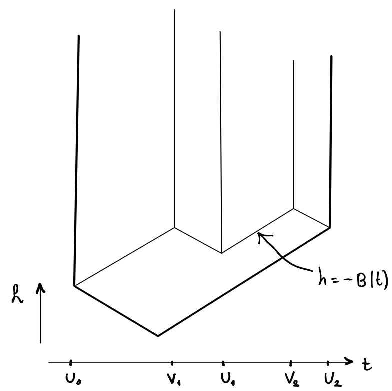

Figure 3. The lattice domain $\mathcal{D}$ with asymptotical boundary configuration of dimers. The function $B(t)$ is defined in (5). For details see [7].

Assume that the edge weights $x(t-\frac{1}{2}, h)$ in (1) are $x(m,h) = q^m$ when $V_i < m < U_i$ and $x(m,h) = q^{-m}$ when $U_i < m < V_{i+1}$. Define $\mathcal{D}_+$ to be the set of m such that $V_i < m < U_i$ for some i, and $\mathcal{D}_-$ to be the set of m such that $U_i < m < V_{i+1}$ for some i. Then formulae from [7] give the following integral representation of the inverse Kasteleyn operator:

$$R((t_1,h_1),(t_2,h_2)) = \left(\frac{1}{2\pi i}\right)^2 \int\limits_{C_z}\int\limits_{C_w} \frac{\Phi_-(z,t_1)\Phi_+(w,t_2)}{\Phi_+(z,t_1)\Phi_-(w,t_2)} \times z^{-h_1-B(t_1)} w^{h_2+B(t_2)} \frac{\sqrt{zw}}{z-w}\frac{dz}{z}\frac{dw}{w} \tag{4}$$

where

$$\Phi_+(z,t) = \prod_{\substack{m>t,\\ m\in\mathcal{D}_+}} (1-zq^m), \quad \Phi_-(z,t) = \prod_{\substack{m<t,\\ m\in\mathcal{D}_-}} (1-z^{-1}q^{-m})$$

and

$$B(t) = \frac{1}{2}\sum_{i=1}^{N}|t - V_i| - \frac{1}{2}\sum_{i=1}^{N-1}|t - U_i| \tag{5}$$

for $m \in \mathbb{Z} + \frac{1}{2}$ and $t \in \mathbb{Z}$. We assume $\sum_{i=1}^{N} V_i = \sum_{i=1}^{N-1} U_i$ and $U_0 + U_N = 0$.

>From our set up we see that for the case when $t \in \mathcal{D}_+$, $V_i < t < U_i$, we have:

$$\Phi_+(z,t) = \prod_{m=t+\frac{1}{2}}^{U_i-\frac{1}{2}}(1 - zq^m)\prod_{m=V_{i+1}+\frac{1}{2}}^{U_{i+1}-\frac{1}{2}}(1 - zq^m)\prod_{m=V_{i+2}+\frac{1}{2}}^{U_{i+2}-\frac{1}{2}}\cdots$$

$$\Phi_-(z,t) = \prod_{m=U_{i-1}+\frac{1}{2}}^{V_i-\frac{1}{2}}(1 - z^{-1}q^{-m})\prod_{m=U_{i-2}+\frac{1}{2}}^{V_{i-1}-\frac{1}{2}}\cdots$$

and for the case when $t \in \mathcal{D}_-$, $U_i < t < V_{i+1}$:

$$\Phi_+(z,t) = \prod_{m=V_{i+1}+\frac{1}{2}}^{U_{i+1}-\frac{1}{2}}(1 - zq^m)\prod_{m=V_{i+2}+\frac{1}{2}}^{U_{i+2}-\frac{1}{2}}\cdots$$

$$\Phi_-(z,t) = \prod_{m=U_i+\frac{1}{2}}^{t-\frac{1}{2}}(1 - z^{-1}q^{-m})\prod_{m=U_{i-1}+\frac{1}{2}}^{V_i-\frac{1}{2}}(1 - z^{-1}q^{-m})\prod_{m=U_{i-2}+\frac{1}{2}}^{V_{i-1}-\frac{1}{2}}\cdots.$$

3.3. Continuum limit. Now assume that $q = \exp(-\epsilon)$, $\epsilon \to 0$ and that $u_i = U_i\epsilon$, $v_i = V_i\epsilon$, $\tau_a = t_a\epsilon$, $\chi_a = h_a\epsilon$ are kept finite in this limit.

3.3.1. *Lemma on q-dilogarithms.* The following lemma is known. We present it anyway for completeness.

Lemma 1.

$$\prod_{m=t_1+\frac{1}{2}}^{t_2-\frac{1}{2}}(1 - zq^m) = e^{\frac{1}{\epsilon}\int_{ze^{-\tau_2}}^{ze^{-\tau_1}}\frac{\ln(1-t)}{t}dt}(1 + O(\epsilon)).$$

Proof. Recall the q-Pochhammer symbol (q-dilogarithm) defined by $(z;q)_\infty = \prod_{k=0}^{\infty}(1 - zq^k)$. Suppose $(z;q)_\infty$ can be expanded as:

$$(z;q)_\infty = e^{\frac{S(z)}{\epsilon}}f(z)(1 + O(\epsilon))$$

as $\epsilon \to 0$. Then we have:

$$(zq;q)_\infty = \frac{1}{1-z}(z;q)_\infty = e^{\frac{S(z)}{\epsilon}}\frac{f(z)}{1-z}(1+O(\epsilon))$$

as well as:

$$(zq;q)_\infty = e^{\frac{S(zq)}{\epsilon}}f(zq)(1+O(\epsilon)).$$

We now write $q = e^{-\epsilon}$ and expand the above in orders of ϵ:

$$\begin{aligned} S\Big(z - z\epsilon + z\frac{\epsilon^2}{2}\Big) &= S(z) + z\Big(-\epsilon + \frac{\epsilon^2}{2}\Big)S'(z) + z^2\frac{\epsilon^2}{2}S''(z) + \dots \\ &= S(z) - \epsilon z S'(z) + \frac{\epsilon^2}{2}(zS'(z) + z^2 S''(z)) + O(\epsilon^3). \end{aligned}$$

Equating the two expressions for $(zq;q)_\infty$ we have:

$$e^{-zS'(z)}\left(1 + \frac{\epsilon}{2}\left(zS'(z) + z^2S''(z)\right) + \dots\right)(f(z) - \epsilon z f'(z) + \dots) = \frac{1}{1-z}f(z).$$

Now let's look at terms order-by-order. For the 0-order terms we have:

$$S'(z) = \frac{\ln(1-z)}{z}.$$

If S is chosen such that $S(0) = 0$ then

$$S(z) = \int_0^z \frac{\ln(1-t)}{t}dt.$$

For the ϵ-order terms we have:

$$\frac{1}{2}(zS'(z) + z^2S''(z))f(z) - zf'(z) = \frac{1}{2}z(zS'(z))'f(z) - zf'(z) = 0.$$

Using what we know about $S(z)$ this becomes :

$$f'(z) = -\frac{1}{2}\frac{f(z)}{1-z}$$

giving

$$f(z) = \sqrt{z-1}.$$

Putting this all together we have:

$$(z,q)_\infty = \exp\left(\frac{1}{\epsilon}\int_0^z \frac{\ln(1-t)}{t}dt\right)\sqrt{z-1}(1+O(\epsilon)).$$

Now we write our finite product as a ratio of infinite products and use the above result:

$$\prod_{m=t_1+\frac{1}{2}}^{t_2-\frac{1}{2}}(1-zq^m)=\frac{(zq^{t_1+\frac{1}{2}};q)_\infty}{(zq^{t_2+\frac{1}{2}};q)_\infty}$$

$$=\exp\left(\frac{1}{\epsilon}\int\limits_{zq^{t_2+\frac{1}{2}}}^{zq^{t_1+\frac{1}{2}}}\frac{\ln(1-t)}{t}dt\right)\sqrt{\frac{zq^{-\tau_1}-1}{zq^{-\tau_2}-1}}(1+O(\epsilon))$$

$$=\exp\left(\frac{1}{\epsilon}\int\limits_{z_2(1-\frac{\epsilon}{2})}^{z_1(1-\frac{\epsilon}{2})}\frac{\ln(1-t)}{t}dt\right)\sqrt{\frac{zq^{-\tau_1}-1}{zq^{-\tau_2}-1}}(1+O(\epsilon))$$

$$=\exp\left(\frac{1}{\epsilon}\int\limits_{z_2}^{z_1}\frac{\ln(1-t)}{t}dt-\frac{1}{2}(\ln(1-ze^{-\tau_1})+\ln(1-ze^{-\tau_2}))\right)$$

$$\times\sqrt{\frac{zq^{-\tau_1}-1}{zq^{-\tau_2}-1}}(1+O(\epsilon))=\exp\left(\frac{1}{\epsilon}\int\limits_{ze^{-\tau_2}}^{ze^{-\tau_1}}\frac{\ln(1-t)}{t}dt\right)(1+O(\epsilon)).$$

□

Similarly we have:

$$\prod_{t_1+\frac{1}{2}}^{t_2-\frac{1}{2}}(1-z^{-1}q^{-m})=(-z)^{-(t_2-t_1)}\prod_{m=t_1+\frac{1}{2}}^{t_2-\frac{1}{2}}q^{-m}\prod_{t_1+\frac{1}{2}}^{t_2-\frac{1}{2}}(1-zq^m).$$

So as $\epsilon\to 0$ we have:

$$\prod_{t_1+\frac{1}{2}}^{t_2-\frac{1}{2}}(1-z^{-1}q^{-m})$$

$$=(-1)^{t_2-t_1}z^{-\frac{\tau_2-\tau_1}{\epsilon}}e^{\frac{\tau_2^2-\tau_1^2}{2\epsilon}}\exp\left(\frac{1}{\epsilon}\int\limits_{ze^{-\tau_2}}^{ze^{-\tau_1}}\frac{\ln(1-t)}{t}dt\right)(1+O(\epsilon)).$$

Note that this asymptotic expansion is a meromorphic function of z on the complex plane with branch cuts along $[e^{\tau_1},e^{\tau_2}]$.

3.3.2. *Functions $\Phi_\pm$ in the continuum limit.* Now we can use computations from the previous section to find the asymptotic of $\Phi_\pm(z,t)$.

Indeed for $t \in \mathcal{D}_+$, i.e $V_i < t < U_i$ we have[1]

$$\Phi_+(z,t) = \prod_{m=t+\frac{1}{2}}^{U_i-\frac{1}{2}} (1-zq^m) \prod_{m=V_{i+1}+\frac{1}{2}}^{U_{i+1}-\frac{1}{2}} (1-zq^m) \prod_{m=V_{i+2}+\frac{1}{2}}^{U_{i+2}-\frac{1}{2}} \dots$$
$$= \exp\left(\frac{1}{\epsilon}\int_{ze^{-u_i}}^{ze^{-\tau}} \frac{\ln(1-t)}{t}dt\right) \exp\left(\frac{1}{\epsilon}\int_{ze^{-u_{i+1}}}^{ze^{-v_{i+1}}} \frac{\ln(1-t)}{t}dt\right)$$
$$\exp\left(\frac{1}{\epsilon}\int_{ze^{-u_{i+2}}}^{ze^{-v_{i+2}}} \frac{\ln(1-t)}{t}dt\right)\dots$$
$$= \exp\left(\frac{1}{\epsilon}\int_{ze^{-u_i}}^{ze^{-\tau}} + \frac{1}{\epsilon}\int_{ze^{-u_{i+1}}}^{ze^{-v_{i+1}}} + \frac{1}{\epsilon}\int_{ze^{-u_{i+2}}}^{ze^{-v_{i+2}}} \frac{\ln(1-t)}{t}dt + \dots\right)$$

$$\Phi_-(z,t) = \prod_{m=U_{i-1}+\frac{1}{2}}^{V_i-\frac{1}{2}} (1-z^{-1}q^{-m}) \prod_{m=U_{i-2}+\frac{1}{2}}^{V_{i-1}-\frac{1}{2}} \dots$$
$$= (-z)^{-\frac{v_i-u_{i-1}}{\epsilon}} e^{\frac{v_i^2-u_{i-1}^2}{2\epsilon}} \exp\left(\frac{1}{\epsilon}\int_{ze^{-v_i}}^{ze^{-u_{i-1}}} \frac{\ln(1-t)}{t}dt\right)$$
$$(-z)^{\frac{v_{i-1}-u_{i-2}}{\epsilon}} e^{\frac{v_{i-1}^2-u_{i-2}^2}{2\epsilon}} \exp\left(\frac{1}{\epsilon}\int_{ze^{-v_{i-1}}}^{ze^{-u_{i-2}}} \frac{\ln(1-t)}{t}dt\right)\dots$$
$$= (-z)^{-\frac{1}{\epsilon}\sum_{j\le i}(v_j-u_{j-1})} e^{\frac{1}{2\epsilon}\sum_{j\le i} v_j^2-u_{j-1}^2}$$
$$\exp\left(\frac{1}{\epsilon}\int_{ze^{-v_i}}^{ze^{-u_{i-1}}} + \frac{1}{\epsilon}\int_{ze^{-v_{i-1}}}^{ze^{-u_{i-2}}} \frac{\ln(1-t)}{t}dt + \dots\right)$$

[1] In the expressions below we will omit the integrand if it is clear which function is integrated.

Similarly, for $t \in \mathcal{D}_-$, i.e. $U_i < t < V_{i+1}$ we obtain:

$$\Phi_+(z,t) = \exp\left(\frac{1}{\epsilon}\int\limits_{ze^{-u_{i+1}}}^{ze^{-v_{i+1}}} + \frac{1}{\epsilon}\int\limits_{ze^{-u_{i+2}}}^{ze^{-v_{i+2}}} \frac{\ln(1-t)}{t}dt + \dots\right)$$

$$\Phi_-(z,t) = (-z)^{-\frac{\tau-u_i}{\epsilon}-\frac{1}{\epsilon}\sum\limits_{j<i}(v_j-u_{j-1})} e^{\frac{1}{2\epsilon}(\tau^2-u_i^2+\sum\limits_{j<i}v_j^2-u_{j-1}^2)}$$
$$\exp\left(\frac{1}{\epsilon}\int\limits_{ze^{-\tau}}^{ze^{-u_i}} + \frac{1}{\epsilon}\int\limits_{ze^{-v_i}}^{ze^{-u_{i-1}}} \frac{\ln(1-t)}{t}dt + \dots\right).$$

Recall:

$$B(t) = \frac{1}{2}\sum_{i=1}^{N}|t-V_i| - \frac{1}{2}\sum_{i=1}^{N-1}|t-U_i|.$$

Now define:

$$L(t) = \begin{cases} \sum\limits_{j\le i} V_j - U_{j-1}, & \text{for } t \in \mathcal{D}_+, V_i < t < U_i \\ t - U_i + \sum\limits_{j<i} V_j - U_{j-1}, & \text{for } t \in \mathcal{D}_-, U_i < t < V_{i+1}. \end{cases}$$

For $t \in \mathcal{D}_+$, $V_i < t < U_i$:

$$B(t) + L(t) = \frac{1}{2}\sum_{j=1}^{N}|t-V_j| - \frac{1}{2}\sum_{j=1}^{N-1}|t-U_j| + \sum_{j\le i}(V_j - U_{j-1})$$
$$= \frac{1}{2}\left(\sum_{j=1}^{i} t - \sum_{j=i+1}^{N} t + \sum_{j=i}^{N-1} t - \sum_{j=1}^{i-1} t - \sum_{j=1}^{i} V_j + \sum_{j=i+1}^{N} V_j - \sum_{j=i}^{N-1} U_j + \sum_{j=1}^{i-1} U_j\right)$$
$$+ \sum_{j=1}^{i} V_j - \sum_{j=1}^{i-1} U_j + U_0 = \frac{t}{2} + \frac{1}{2}\left(\sum_{j=1}^{N} V_j - \sum_{j=1}^{N-1} U_j\right) - U_0 = \frac{t}{2} - U_0,$$

where we use that $\sum\limits_{j=1}^{N} V_j = \sum\limits_{j=1}^{N-1} U_j$. A similar calculation can be done for $t \in \mathcal{D}_-$, $U_i < t < V_{i+1}$.

Now for the ratio of the Φ's:

$$\frac{\Phi_-(z,t)}{\Phi_+(z,t)} z^{-h-B(t)} = C_\tau \exp\left(\frac{S(z)}{\epsilon}\right)(1 + O(\epsilon))$$

where C_τ is a constant in z. The function $S(z)$ is :

$$S(z) = \sum_{i=0}^{N} Li_2(ze^{-u_i}) - \sum_{i=1}^{N} Li_2(ze^{-v_i}) - Li_2(ze^{-\tau}) - \Big(\frac{\tau}{2} - u_0 + \chi\Big) \ln z$$

and $Li_2(z) = \int\limits_0^z \frac{\ln(1-x)}{x} dx$ is the dilogarithm.

Combining result from above we have the following asymptotical integral representation for R.

$$R((t_1,h_1),(t_2,h_2)) = \frac{C_{\tau_1}}{C_{\tau_2}} \left(\frac{1}{2\pi i}\right)^2 \int\limits_{C_z}\int\limits_{C_w} e^{\frac{S(z,\tau_1,\chi_1)-S(w,\tau_2,\chi_2)}{\epsilon}} \frac{\sqrt{zw}}{z-w}\frac{dz}{z}\frac{dw}{w}(1+O(\epsilon)) \tag{6}$$

where the function $S(z)$ as above. The integration contours are shown on Fig. 4.

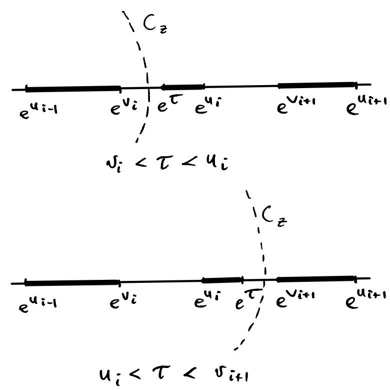

Figure 4. The integration contours in (6) are circles with $|z| < |w|$ when $\tau_1 < \tau_2$ and $|w| > |z|$ when $\tau_1 > \tau_2$ centered at the origin. They the contour C_z intersect the positive part of the real line as it is shown above with $\tau = \tau_1$. The contour C_w intersect the positive part of the real line similarly with $\tau = \tau_2$.

3.3.3. *The asymptotic of the integral (6).* We will be computing the asymptotic using the method of steepest descent, so first we should study critical points of the function $S(z)$.

Lemma 2. *The following identity holds*

$$z_0^2 S''(z_0) = \left(\frac{\partial z_0}{\partial \chi}\right)^{-1}$$

where z_0 is a critical point of $S(z)$.

Proof. For the first derivative of S in z we have:

$$\begin{aligned} z\frac{\partial S(z)}{\partial z} &= \sum_{i=0}^{N} \ln(1-ze^{-u_i}) - \sum_{i=1}^{N} \ln(1-ze^{-v_i}) - \ln(1-ze^{-\tau}) \\ &- \left(\frac{\tau}{2} - u_0 + \chi\right) = \ln\left(\frac{\prod_{i=0}^{N}(1-ze^{-u_i})}{\prod_{i=1}^{N}(1-ze^{-v_i})}\frac{1}{(1-ze^{-\tau})}\right) \\ &- \left(\frac{\tau}{2} - u_0 + \chi\right) = \ln\left(\frac{f(z)}{(1-ze^{-\tau})}\right) - \left(\frac{\tau}{2} + \chi\right) \end{aligned}$$

where $f(z) = \frac{\prod_{i=0}^{N}(1-ze^{-u_i})}{\prod_{i=1}^{N}(1-ze^{-v_i})} e^{u_0}$.

From this we see that if z_0 is a critical point of S, i.e. $S'(z_0) = 0$ we have:

$$e^{\chi+\frac{\tau}{2}} = \frac{f(z_0)}{1-z_0e^{-\tau}}.$$

This defines z_0 as an implicit function of χ and τ. Taking a derivative we have:

$$1 = \frac{\partial z_0}{\partial \chi}\left(\ln\left(\frac{f(z_0)}{1-z_0e^{-\tau}}\right)\right)'.$$

For the second derivative of $S(z)$ we have

$$\left(z\frac{\partial}{\partial z}\right)^2 S(z) = z\left(\ln\left(\frac{f(z_0)}{1-z_0e^{-\tau}}\right)\right)'.$$

Taking into account The equation for the derivative of the critical point in χ and that

$$\left(z\frac{\partial}{\partial z}\right)^2 S(z) = zS'(z) + z^2 S''(z)$$

we obtain the value of the second derivative of $S(z)$ at the critical point z_0 and the desired identity. □

Before we will compute the asymptotic of (6), we need one more lemma.

Lemma 3. *The following identities hold:*

$$e^{-S_\tau} = \frac{\sqrt{z_0}}{1 - z_0 e^{-\tau}} \quad e^{-\frac{1}{2}S_\chi} = \sqrt{z_0}.$$

Indeed, we have the following identities which imply the lemma.

$$\frac{d}{d\tau}S(z_0) = \frac{\partial z_0}{\partial \tau}S'(z_0) + \frac{\partial S}{\partial \tau}|_{z_0} = -\frac{1}{2}\ln(z_0) + \ln(1 - z_0 e^{-\tau})$$
$$\frac{d}{d\chi}S(z_0) = \frac{\partial z_0}{\partial \chi}S'(z_0) + \frac{\partial S}{\partial \chi}|_{z_0} = -\ln(z_0).$$

Theorem 1. *The integral (6) has the following asymptotic when $\epsilon \to 0$ and all parameters are scaling as before*

$$\begin{aligned} R(t,h|t',h') = \epsilon \frac{C_{\tau_1}}{C_{\tau_2}} \Bigg(& e^{\frac{S(z_0)-S(w_0)}{\epsilon}} \frac{\sqrt{\frac{\partial z_0}{\partial \chi_1}\frac{\partial w_0}{\partial \chi_2}}}{z_0 - w_0} + e^{\frac{S(\bar z_0)-S(w_0)}{\epsilon}} \frac{\sqrt{\frac{\partial \bar z_0}{\partial \chi_1}\frac{\partial w_0}{\partial \chi_2}}}{\bar z_0 - w_0} \\ & + e^{\frac{S(z_0)-S(\bar w_0)}{\epsilon}} \frac{\sqrt{\frac{\partial z_0}{\partial \chi_1}\frac{\partial \bar w_0}{\partial \chi_2}}}{z_0 - \bar w_0} + e^{\frac{S(\bar z_0)-S(\bar w_0)}{\epsilon}} \frac{\sqrt{\frac{\partial \bar z_0}{\partial \chi_1}\frac{\partial \bar w_0}{\partial \chi_2}}}{\bar z_0 - \bar w_0} \Bigg)(1 + O(\epsilon)). \end{aligned} \tag{7}$$

Proof. As it is shown in [7] for (τ, χ) inside the discriminant curve there are two complex conjugate critical points of $S(z)$. The discriminant curve also know as the arctic circle is

$$S'(z) = S''(z) = 0.$$

Deforming integration contours to the contours which pass critical points in the steepest descent direction and computing corresponding Gaussian integrals we arrive at (7). □

§4. Asymptotical solutions to Kasteleyn difference equation

4.1. Formal asymptotical solutions to the Kasteleyn equations. Here we will study the difference equation

$$f(t,h) - f\Big(t-1, h+\frac{1}{2}\Big) + x\Big(t - \frac{1}{2}, h\Big) f\Big(t-1, h-\frac{1}{2}\Big) = 0 \tag{8}$$

in the continuum limit when $\epsilon \to 0$ and $\tau = \epsilon t$, $\chi = \epsilon h$ are fixed. It is convenient to change of coordinates:

$$\xi_+ = \chi + \frac{\tau}{2}, \quad \xi_- = \chi - \frac{\tau}{2}$$
$$\partial_+ = \partial_\tau + \frac{1}{2}\partial_\chi, \quad \partial_- = -\partial_\tau + \frac{1}{2}\partial_\chi.$$

Let us look for asymptotic solutions to the difference equation (8) of the form $f(t,h) = e^{\frac{1}{\epsilon}S(\xi_+,\xi_-)}\phi(\xi_+,\xi_-)$. The equation (8) gives:

$$e^{\frac{1}{\epsilon}S(\xi_+,\xi_-)}\phi(\xi_+,\xi_-) - e^{\frac{1}{\epsilon}S(\xi_+,\xi_-+\epsilon)}\phi(\xi_+,\xi_-+\epsilon)$$
$$+ v\Big(\xi_+ - \xi_- - \frac{\epsilon}{2}\Big)e^{\frac{1}{\epsilon}S(\xi_+-\epsilon,\xi_-)}\phi(\xi_+-\epsilon,\xi_-) = 0.$$

Taking the limit $\epsilon \to 0$, we get the following nonlinear differential equation for S at 0-th order in ϵ:

$$1 - e^{\partial_- S} + ve^{-\partial_+ S} = 0. \tag{9}$$

The first order terms 1-order terms give linear differential equation for ϕ:

$$-\frac{1}{2}\partial_-^2 S e^{\partial_- S} - \frac{\partial_-\phi}{\phi}e^{\partial_- S} + \frac{1}{2}v\partial_+^2 S e^{-\partial_+ S} - v\frac{\partial_+\phi}{\phi}e^{-\partial_+ S} - \frac{1}{2}v' e^{-\partial_+ S} = 0. \tag{10}$$

4.1.1. *The function S.* Taking into account equation (9) we can write:

$$e^{\partial_- S} = \frac{1}{1 - z_0 v}, \quad e^{-\partial_+ S} = \frac{z_0}{1 - z_0 v}$$

for some function $z_0(\xi_+,\xi_-)$.

Lemma 4. *The function $z_0(\xi_+,\xi_-)$ satisfies differential equation*

$$\partial_- z_0(\xi_+,\xi_-) + z_0(\xi_+,\xi_-)v(\xi_+ - \xi_-)\partial_+ z_0(\xi_+,\xi_-) = 0. \tag{11}$$

Proof. Indeed, differentiating (9) we obtain:

$$\partial_+(\partial_- S) = -\partial_+ \ln(1 - z_0 v) = \frac{\partial_+(z_0 v)}{1 - z_0 v}$$
$$\partial_-(\partial_+ S) = \partial_-(\ln(1 - z_0 v) - \ln(z_0)) = -\frac{\partial_-(z_0 v)}{(1 - z_0 v)z_0 v} + \frac{\partial_- v}{v}.$$

These two identities imply

$$\partial_-(z_0 v) + (z_0 v)\partial_+(z_0 v) = (1 - z_0 v)z_0\partial_- v.$$

Because $v = v(\xi_+ - \xi_-)$ we can rewrite this as:

$$(\partial_- z_0)v - z_0 v' + z_0 v(\partial_+ z_0)v + z_0^2 v v' = -z_0 v' + z_0^2 v v'$$

this gives the desired identity. □

Note that when v is constant the equation for z_0 is exactly the complex Burgers equation from [8].

4.1.2. *The function ϕ.* Here we will describe general solution to the differential equation for ϕ.

Theorem 2. *Let $z_0(\xi_+, \xi_-)$ be as above, then the function*

$$\phi = \psi \frac{\sqrt{(\partial_+ + \partial_-)z_0}}{z_0 - w_0}, \tag{12}$$

where w_0 does not depend on $\xi_\pm$, is a solution to (10) if and only if ψ satisfies the equation

$$(\partial_- + z_0 v \partial_+)\psi = 0.$$

Proof. First, let us now look at the the terms in equation (10) not containing ϕ. We have:

$$\begin{aligned}
&\frac{1}{2}\frac{\partial_- \ln(1 - z_0 v)}{1 - z_0 v} + \frac{1}{2} v \partial_+ (\ln(z_0) - \ln(1 - z_0 v)) \frac{z_0}{1 - z_0 v} - \frac{1}{2} v' \frac{z_0}{1 - z_0 v} \\
&= -\frac{1}{2}\frac{\partial_-(z_0 v)}{(1 - z_0 v)^2} - \frac{1}{2} v \left(\frac{\partial_+ z_0}{z_0} + \frac{\partial_+(z_0 v)}{1 - z_0 v} \right) \frac{z_0}{1 - z_0 v} - \frac{1}{2} v' \frac{z_0}{1 - z_0 v} \\
&= -\frac{1}{2}\frac{\partial_-(z_0 v) + v z_0 \partial_+(z_0 v)}{(1 - z_0 v)^2} - \frac{1}{2} v \frac{\partial_+ z_0}{1 - z_0 v} - \frac{1}{2} v' \frac{z_0}{1 - z_0 v} \\
&= -\frac{1}{2}\frac{v(\partial_-(z_0) + v z_0 \partial_+(z_0)) + z_0(\partial_- v + v z_0 \partial_+ v)}{(1 - z_0 v)^2} - \frac{1}{2} v \frac{\partial_+ z_0}{1 - z_0 v} - \frac{1}{2}\frac{v' z_0}{1 - z_0 v} \\
&= \frac{1}{2}\frac{z_0 v'}{1 - z_0 v} - \frac{1}{2}\frac{v \partial_+ z_0}{1 - z_0 v} - \frac{1}{2}\frac{v' z_0}{1 - z_0 v} \\
&= -\frac{1}{2}\frac{v \partial_+ z_0}{1 - z_0 v}
\end{aligned}$$

where in the fourth line we use the lemma from above.

Now the terms containing ϕ after the substitution (12) can be transformed as:

$$\begin{aligned}
&\frac{1}{1-z_0v}\frac{\partial_-\phi}{\phi}+\frac{vz_0}{1-z_0v}\frac{\partial_+\phi}{\phi}\\
&=\frac{\partial_-\psi+z_0v\partial_+\psi}{\psi(1-z_0v)}+\frac{1}{2}\frac{(\partial_-+vz_0\partial_+)(\partial_+z_0+\partial_-z_0)}{(\partial_+z_0+\partial_-z_0)(1-z_0v)}-\frac{(\partial_-+z_0v\partial_+)z_0}{(1-z_0v)(z_0-\tilde{z}_0)}\\
&=\frac{\partial_-\psi+z_0v\partial_+\psi}{\psi(1-z_0v)}+\frac{1}{2}\frac{(\partial_-+vz_0\partial_+)(\partial_+z_0+\partial_-z_0)}{(\partial_+z_0+\partial_-z_0)(1-z_0v)}
\end{aligned}$$

where we again use the lemma from above. The denominator of the second term we can write as

$$\begin{aligned}
&(\partial_-+z_0v\partial_+)(\partial_+z_0+\partial_-z_0)\\
&\quad=(\partial_++\partial_-)(\partial_-+z_0v\partial_+)z_0+(\partial_++\partial_-)(z_0v)\partial_+z_0\\
&\quad=(\partial_++\partial_-)(z_0v)\partial_+z_0\\
&\quad=v(\partial_++\partial_-)(z_0)\partial_+z_0.
\end{aligned}$$

Here we use the lemma and the fact that $(\partial_++\partial_-)v=0$. Combining expressions above, we have the following expression for terms in (10)

$$\frac{1}{1-z_0v}\frac{\partial_-\phi}{\phi}+\frac{vz_0}{1-z_0v}\frac{\partial_+\phi}{\phi}=\frac{\partial_-\psi+z_0v\partial_+\psi}{\psi(1-z_0v)}+\frac{1}{2}\frac{v\partial_+z_0}{1-z_0v}.$$

Putting everything together, the equation for ϕ becomes:

$$-\frac{1}{2}\frac{v\partial_+z_0}{1-z_0v}+\frac{\partial_-\psi+z_0v\partial_+\psi}{\psi(1-z_0v)}+\frac{1}{2}\frac{v\partial_+z_0}{1-z_0v}=\frac{\partial_-\psi+z_0v\partial_+\psi}{\psi(1-z_0v)}.$$

The theorem follows. □

4.2. The asymptotical behavior of the inverse to the Kasteleyn operator in the continuum limit. Now let us find the asymptotic of the inverse to the Kasteleyn operator from the difference equation. Note that critical points of function S from the asymptotic of integral representation satisfy the equation (11).

Let $z_0(\tau,\chi)$ be the relevant solution to (11), denote $z_0=z_0(\tau,\chi)$ and $w_0=z_0(\tau',\chi')$. Combining the previous results of this section we arrive to the following asymptotic of $R(t,h|t'h')$:

$$R(t,h|t',h') = \epsilon\frac{C_{\tau_1}}{C_{\tau_2}}\left(e^{\frac{S(z_0)-S(w_0)}{\epsilon}}\frac{\sqrt{\frac{\partial z_0}{\partial\chi_1}\frac{\partial w_0}{\partial\chi_2}}}{z_0-w_0} + e^{\frac{S(\bar z_0)-S(w_0)}{\epsilon}}\frac{\sqrt{\frac{\partial \bar z_0}{\partial\chi_1}\frac{\partial w_0}{\partial\chi_2}}}{\bar z_0-w_0}\right.$$
$$\left. + e^{\frac{S(z_0)-S(\bar w_0)}{\epsilon}}\frac{\sqrt{\frac{\partial z_0}{\partial\chi_1}\frac{\partial \bar w_0}{\partial\chi_2}}}{z_0-\bar w_0} + e^{\frac{S(\bar z_0)-S(\bar w_0)}{\epsilon}}\frac{\sqrt{\frac{\partial \bar z_0}{\partial\chi_1}\frac{\partial \bar w_0}{\partial\chi_2}}}{\bar z_0-\bar w_0}\right)(1+O(\epsilon)). \tag{13}$$

This agrees with (7) when $v(x) = e^{-x}$. We will give detailed proof in an extended version of this paper.

§5. Conformal correlation functions

Note that the asymptotical formula for the inverse to the Kasteleyn operator can be interpreted in terms of Kasteleyn fermions in the following way. In the appropriate sense one can say that as $\epsilon \to 0$

$$\psi(t,h) = \sqrt{\epsilon}C_\tau\left(a(z_0(\tau,\chi))e^{\frac{S(z_0(\tau,\chi))}{\epsilon}} + a(\overline{z_0(\tau,\chi)})e^{\frac{\overline{S(z_0(\tau,\chi))}}{\epsilon}}\right)(1+O(\epsilon))$$

$$\psi^*(t,h) = \sqrt{\epsilon}C_\tau^{-1}\left(b(z_0(\tau,\chi))e^{-\frac{S(z_0(\tau,\chi))}{\epsilon}} + b(\overline{z_0(\tau,\chi)})e^{-\frac{\overline{S(z_0(\tau,\chi))}}{\epsilon}}\right)(1+O(\epsilon))$$

where $a(z)$ and $b(z)$ are components of the Dirac fermionic field with correlation functions

$$\langle a(z)b(w)\rangle = \frac{1}{z-w}.$$

We will explain the exact meaning of the convergence and the definition of the Dirac fermionic field in an extended version of this paper. The square roots in the formulae (13) appear from the spinor nature of conformal fields a and b.

The height function of the dimer model is a quadratic combination in a and b, see the extended version of the paper.

References

1. R. Kenyon, *Height Fluctuations in the Honeycomb Dimer Model.* —Commun. Math. Phys., **281** (2008), 675.
2. A. Bufetov, A. Knizel, *Asymptotics of random domino tilings of rectangular Aztec diamonds.* Ann. de l'Inst. Henri Poincare, 2016
3. L. Petrov, *Asymptotics of uniformly random lozenge tilings of polygons. Gaussian free field.* — Ann. Probab., **43**, No. 1 (2014), 1–43. arXiv:1206.5123

4. W. Kasteleyn, *Graph theory and crystal physics.* In: Graph Theory and Theoretical Physics, Academic Press, London, pp. 43–110.
5. M. Fisher, *Statistical mechanics of dimers on a plane lattice.* —Phys. Review, **124**, No. 6 (1961), 1664.
6. D. Cimasoni, N. Reshetikhin, *Dimers on surface graphs and spin structures.* — Commun. Math. Phys., **275**, No. 1 (2007), 187–208; arXiv:0704.0273.
7. A. Okounkov, N. Reshetikhin, *Random skew plane partitions and the Pearcey process.* — Comm. Math. Phys., **269**, No. 3, February 2007, arXiv math. CO/0503508.
8. R. Kenyon, A. Okounkov, *Limit shapes and the complex Burgers equation*, arXiv math-ph/0507007.

Поступило 22 ноября 2018 г.

D.K.: Department of Mathematics,
University of California,
Berkeley, CA 94720, USA
E-mail: dkeating@berkeley.edu

N.R.: Department of Mathematics,
University of California,
Berkeley, CA 94720, USA;
St. Petersburg University,
Russia;
KdV Institute for Mathematics,
University of Amsterdam,
Science Park 904,
1098 XH Amsterdam, The Netherlands.
E-mail: reshetik@math.berkeley.edu

A.S.: Google LLC
E-mail: asridhar@berkeley.edu