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THE PARTITION FUNCTION OF THE FOUR-VERTEX MODEL IN A SPECIAL EXTERNAL FIELD

Аннотация. The exactly solvable four-vertex model on a square grid with the fixed boundary conditions in a presence of a special external field is considered. Namely, we study a system in a linear field acting on the central column of a grid. The partition function of the model is calculated by Quantum Inverse Scattering Method. The answer is written in the determinantal form.

Dedicated to M. A. Semenov-Tian-Shansky on the occasion of his 70th birthday

§1. Four-vertex model

The four-vertex model is a particular case of the six-vertex model [1] in which two vertices are frozen out. This model was considered in [2,3]. The Quantum Inverse Scattering Method [4,5] was applied to the solution of the four-vertex model on a finite lattice with different boundary conditions in [6–8]. In these papers the connection of the model with the theory of random walks and plane partitions [9,10] was also discussed.

Consider a square grid of $2N$ vertical lines and $M+1$ horizontal ones. A four-vertex model is described by four different arrows arrangements pointing in and out of each vertex on a grid (Fig. 1). Representing the arrows pointed up and to the right by the lines we obtain an alternate description of the vertices in terms of lines flowing through the vertices. Since a lattice edge can exist in two states, line or no line, there exists a one-to-one correspondence between the arrow configuration on the lattice and the graphs of lines on the lattice – nests of lattice paths. A statistical weight corresponds to each type of the vertices and there are three vertex weights $\omega_a = \omega_2$, $\omega_b = \omega_4$ and $\omega_c = \omega_5 = \omega_6$. For the general inhomogeneous case the weights are site dependent.

Key words and phrases: four-vertex model, integrable models, Quantum Inverse Scattering Method.

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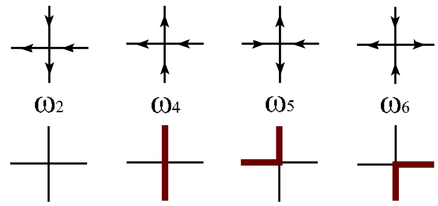

Рис. 1. The four allowed types of vertices in terms of arrows and lines.

The allowed configurations of arrows depend on the imposed boundary conditions which are specified by the direction of the arrows on the boundary of the grid. Under the fixed boundary conditions we shall understand the following arrangement of boundary arrows: the arrows on the top and bottom of the N vertical lines (counting from the left) are pointing inwards, and the arrows on the top and bottom of the last N ones are pointing outwards. All arrows on the left and right boundaries of the grid are pointing to the left.

To enumerate all possible configurations of the vertices it is more convenient to use the description of the model in terms of flowing lines and to represent the allowed configurations as nests of lattice paths. The path is running from one of the N down left vertices to the top N right ones and always moves east or north. The paths cannot touch each other and arbitrary number of consequent steps are allowed in vertical direction while only one step at a time is allowed in the horizontal one. A typical nest of lattice paths is represented in (Fig. 2). The length of the path is $N + M$. The number of the (c) vertices in the admissible path is $2N$ since only one step is allowed in horizontal direction, so the number of (b) vertices is $M - N + 1$. It means that the number of (c) and (b) vertices in the nest is equal to $l^c = 2N^2$ and $l^b = l^a = N(M - N + 1)$ respectively.

The nest of lattice paths uniquely defines the configuration of vertex weights on a lattice. Let $Z^\nu(\boldsymbol{\omega}_a, \boldsymbol{\omega}_b, \boldsymbol{\omega}_c)$ be the configuration that corresponds to the nest ν, then the partition function of the model is equal to

$$Z(\boldsymbol{\omega}_a, \boldsymbol{\omega}_b, \boldsymbol{\omega}_c) = \sum_\nu Z^\nu(\boldsymbol{\omega}_a, \boldsymbol{\omega}_b, \boldsymbol{\omega}_c). \tag{1}$$

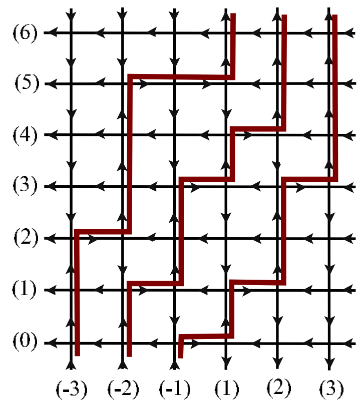

Рис. 2. A typical nest of admissible lattice paths with the fixed boundary conditions.

The summation here is taken over all admissible nests of lattice paths. In this paper we shall consider a case when the vertex weight depends on the position of the vertical line which it only belongs to.

In this letter we shall calculate the partition function of the model in the presence of the linearly growing external field hj ($j = 0, 1, \ldots, M;\ h \geqslant 0$) acting on arrows turned to the right in the central column of the lattice. The set of positions μ_i ($i = 1, 2, \ldots, N$) of N arrows turned to the right in the central column of the lattice form a strict partition $\boldsymbol{\mu}$, that is $M \geqslant \mu_1 > \mu_2 > \ldots > \mu_N \geqslant 0$ with the parts satisfying the condition $\mu_i > \mu_{i+1} + 1$. If $Z^{\nu}_{\boldsymbol{\mu}}(\boldsymbol{\omega}_a, \boldsymbol{\omega}_b, \boldsymbol{\omega}_c)$ is the configuration that corresponds to the nest ν with the fixed set $\boldsymbol{\mu}$, then the partition function of the model under consideration is equal to

$$G(\boldsymbol{\omega}_a, \boldsymbol{\omega}_b, \boldsymbol{\omega}_c | h) = \sum_{\boldsymbol{\mu}} e^{-h|\boldsymbol{\mu}|} \sum_{\nu} Z^{\nu}_{\boldsymbol{\mu}}(\boldsymbol{\omega}_a, \boldsymbol{\omega}_b, \boldsymbol{\omega}_c), \tag{2}$$

where the summation is over all strict partitions $\boldsymbol{\mu}$, and $|\boldsymbol{\mu}| = \sum\limits_{i=1}^{M} \mu_i$.

§2. Partition function

The partition function (2) will be calculated by the Quantum Inverse Scattering Method. The L-operator of the four vertex model is equal to

[6, 7]:

$$L(n|u) = \begin{pmatrix} -ue_n & \sigma_n^- \\ \sigma_n^+ & u^{-1}e_n \end{pmatrix}, \tag{3}$$

where the parameter $u \in \mathbb{C}$, $\sigma^{z,\pm}$ are the Pauli matrices, and $e = \frac{1}{2}(\sigma^z + 1)$ is the projector on the state with the spin up. The matrix with subindex n acts nontrivially only in the n-th space: $s_n = I \otimes \cdots \otimes I \otimes s \otimes I \otimes \cdots \otimes I$.

The operator valued matrix (3) is associated with the following $R(u, v)$ matrix:

$$R(u,v) = \begin{pmatrix} f(v,u) & 0 & 0 & 0 \\ 0 & g(v,u) & 1 & 0 \\ 0 & 0 & g(v,u) & 0 \\ 0 & 0 & 0 & f(v,u) \end{pmatrix}, \tag{4}$$

with the entries equal to

$$f(v,u) = \frac{u^2}{u^2 - v^2}, \qquad g(v,u) = \frac{uv}{u^2 - v^2}. \tag{5}$$

The monodromy matrix is the product of L-operators

$$T(u) = L(M|u)L(M-1|u)\dots L(0|u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{6}$$

The transfer matrix $\tau(u)$ is the trace of the monodromy matrix

$$\tau(u) = trT(u) = A(u) + D(u). \tag{7}$$

The L-operator (3) satisfies relations

$$e^{h\sigma_n^z} L_n(u) e^{-h\sigma_n^z} = e^{-h\sigma^z} L_n(u) e^{h\sigma^z}, \tag{8}$$

and

$$e^{h\sigma^z} L_n(u) = e^{\frac{h}{2}\sigma^z} L_n(e^h u) e^{-\frac{h}{2}\sigma^z}. \tag{9}$$

From these equations and from the definition of the monodromy matrix (6) it follows that

$$e^{h\sum\limits_{i=0}^{M}\sigma_i^z} T(u) e^{-h\sum\limits_{i=0}^{M}\sigma_i^z} = e^{-\frac{\zeta}{2}\sigma^z} T(u) e^{\frac{\zeta}{2}\sigma^z}, \tag{10}$$

and

$$e^{h\sum\limits_{j=1}^{M} j\sigma_j^z} T(u) = e^{-h(M+\frac{1}{2})\sigma^z} T(e^h u) e^{\frac{h}{2}\sigma^z} e^{h\sum\limits_{j=1}^{M} j\sigma_j^z}. \tag{11}$$

The relations

$$e^{h\sum\limits_{j=1}^{M} j\sigma_j^z} B(u) = e^{-h(M+1)} B(e^h u) e^{h\sum\limits_{j=1}^{M} j\sigma_j^z}, \tag{12}$$

and

$$e^{h\sum\limits_{j=1}^{M} j\sigma_j^z} A(u) = e^{-hM} A(e^h u) e^{h\sum\limits_{j=1}^{M} j\sigma_j^z},$$
$$e^{h\sum\limits_{j=1}^{M} j\sigma_j^z} C(u) = e^{h(M+1)} C(e^h u) e^{h\sum\limits_{j=1}^{M} j\sigma_j^z},$$
$$e^{h\sum\limits_{j=1}^{M} j\sigma_j^z} D(u) = e^{hM} D(e^h u) e^{h\sum\limits_{j=1}^{M} j\sigma_j^z}$$

are the consequence of the equation (11).

Let us consider the scalar product

$$W(\mathbf{v};\mathbf{u}) = \langle \Leftarrow |C(v_1)\dots C(v_N)B(u_1)\dots B(u_N)| \Leftarrow\rangle, \tag{13}$$

where $\mathbf{v} \equiv (v_1,\dots,v_N)$ and $\mathbf{u} \equiv (u_1,\dots,u_N)$ are the sets of N independent parameters. For arbitrary N and M this scalar product is evaluated by means of the commutation relations and may be represented in the determinantal form [6, 7]:

$$W(\mathbf{y}^{-1};\mathbf{x}) = (-1)^{(M-1)N} \prod_{j=1}^{N} (x_j y_j)^{-(M-2N+2)} \frac{\det H(\mathbf{x}^2,\mathbf{y}^2)}{\Delta_N(\mathbf{x}^2)\Delta_N(\mathbf{y}^2)}. \tag{14}$$

The matrix $H(\mathbf{x}^2,\mathbf{y}^2) \equiv \big(H_{ij}(\mathbf{x}^2,\mathbf{y}^2)\big)_{1\leqslant i,j\leqslant N}$ is given by the entries

$$H(\mathbf{x}^2,\mathbf{y}^2) = \frac{1-\left(x_i^2 y_j^2\right)^{M-N+1}}{1-\left(x_i^2 y_j^2\right)}, \tag{15}$$

and $\Delta_N(\mathbf{x})$ is the Vandermonde determinant

$$\Delta_N(\mathbf{x}) = \prod_{1\leqslant i<k\leqslant N} (x_k - x_i). \tag{16}$$

The partition function (1) is expressed through the scalar product (13)

$$Z(\boldsymbol{\omega}_a,\boldsymbol{\omega}_b,\boldsymbol{\omega}_c) = (-1)^{MN} W(\mathbf{u};\mathbf{v}), \tag{17}$$

if we put

$$\begin{aligned}
(\omega_a)_j &= v_{-j}^{-1}, \quad (\omega_b)_j = v_{-j}; \quad -N \leqslant j \leqslant -1, \\
(\omega_a)_j &= u_j^{-1}, \quad (\omega_b)_j = u_j; \qquad 1 \leqslant j \leqslant N, \\
(\omega_c)_j &= 1.
\end{aligned} \tag{18}$$

In the spin language the partition function (2) is expressed in terms of the scalar product (13) in the following way:

$$
\begin{aligned}
&G(\boldsymbol{\omega}_a, \boldsymbol{\omega}_b, \boldsymbol{\omega}_c|h) \\
&= (-1)^{MN} \langle \Leftarrow | C(v_1) \dots C(v_N) e^{-\frac{h}{2} \sum\limits_{j=1}^{M} j(1-\sigma_j^z)} B(u_1) \dots B(u_N) | \Leftarrow \rangle. \qquad (19)
\end{aligned}
$$

The applications of the Eq. (12) gives

$$
\begin{aligned}
&e^{-\frac{h}{2} \sum\limits_{j=1}^{M} j(1-\sigma_j^z)} B(u_1) B(u_2) \dots B(u_N) | \Leftarrow \rangle \qquad (20) \\
&= e^{-\frac{h}{2}(M+1)N} B(e^{\frac{h}{2}} u_1) B(e^{\frac{h}{2}} u_2) \dots B(e^{\frac{h}{2}} u_N) e^{-\frac{h}{2} \sum\limits_{j=1}^{M} j(1-\sigma_j^z)} | \Leftarrow \rangle \\
&= e^{-\frac{h}{2}(M+1)N} B(e^{\frac{h}{2}} u_1) B(e^{\frac{h}{2}} u_2) \dots B(e^{\frac{h}{2}} u_N) | \Leftarrow \rangle.
\end{aligned}
$$

Taking into account equations (13), (14) and (15) we find that

$$
\begin{aligned}
&\langle \Leftarrow | C(v_1) \dots C(v_N) e^{-\frac{h}{2} \sum\limits_{j=1}^{M} j(1-\sigma_j^z)} B(u_1) \dots B(u_N) | \Leftarrow \rangle \\
&= (-1)^{(M-1)N} e^{-\frac{h}{2}(M+1)N} \prod_{j=1}^{N} \Big(\frac{e^h u_j}{v_j} \Big)^{-(M-2N+2)} \qquad (21) \\
&\times \frac{\det H(e^h \mathbf{u}^2, \mathbf{v}^{-2})}{\Delta_N(e^h \mathbf{u}^2) \Delta_N(\mathbf{v}^{-2})}.
\end{aligned}
$$

Substituting relations (18) into (21) we obtain the answer for the partition function (2), (19) in the determinantal form:

$$
\begin{aligned}
G(\boldsymbol{\omega}_a, \boldsymbol{\omega}_b, 1|h) &= (-1)^N e^{-\frac{h}{2}N} \prod_{j=1}^{N} \left[\frac{(\omega_b)_j (\omega_a)_{-j}}{(\omega_a)_j (\omega_b)_{-j}} \right]^{\frac{1}{2}(M-2N+2)} \\
&\times \prod_{1 \leqslant i < k \leqslant N} \big(e^h (\omega_b)_k^2 - e^h (\omega_b)_i^2 \big)^{-1} \qquad (22) \\
&\times \big((\omega_a)_{-k}^2 - (\omega_a)_{-i}^2 \big)^{-1} \det \mathcal{H}(\omega_a, \omega_b, h).
\end{aligned}
$$

The entries of matrix $\mathcal{H}(\omega_a, \omega_b, h)$ are

$$
\mathcal{H}(\omega_a, \omega_b, h)_{ij} = \frac{e^{-h(M-N+1)} - \left[\frac{(\omega_b)_i (\omega_a)_{-j}}{(\omega_a)_i (\omega_b)_{-j}} \right]^{(M-N+1)}}{1 - e^h \frac{(\omega_b)_i (\omega_a)_{-j}}{(\omega_a)_i (\omega_b)_{-j}}}. \qquad (23)
$$

When the height of a chain M tends to infinity the determinant in (23) turns into the Cauchy determinant and may be calculated. In this limit the normalized partition function is equal to

$$\mathcal{G}(\boldsymbol{\omega}_a, \boldsymbol{\omega}_b, 1|h) = \lim_{M\to\infty} \prod_{j=1}^{N} \left[\frac{(\omega_b)_j (\omega_a)_{-j}}{(\omega_a)_j (\omega_b)_{-j}} \right]^{-\frac{1}{2}M} G(\boldsymbol{\omega}_a, \boldsymbol{\omega}_b, 1|h)$$

$$= e^{-hN} \prod_{i,j=1}^{N} \frac{1}{e^{h \frac{(\omega_b)_i (\omega_a)_{-j}}{(\omega_a)_i (\omega_b)_{-j}}} - 1}. \quad (24)$$

The discussed partition function is related to the norm-trace generating function of plane partitions [11, 12]. The method discussed in this paper allows to calculate the partition function of four-vertex model in a linear field acting on several columns of the grid.

Список литературы

1. R. G. Baxter, *Exactly Solved Models in Statistical Mechanics*, San Diego, Academic press, 1982.
2. W. Li, H. Park, M. Widom, *Finite-size scaling amplitudes in a random tiling model*, J. Phys. A **23**, L573 (1990).
3. W. Li, H. Park, *Logarithmic singularity in the surface free energy near commensurate-incommensurate transitions*, J. Phys. A **24**, 257 (1991).
4. L. D. Faddeev, *Quantum Inverse Scattering Method*, Sov. Sci. Rev. Math. **C1**, 107 (1980).
5. V. E. Korepin, N. M. Bogoliubov, A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions*, Cambridge University Press, Cambridge, 1993.
6. N. M. Bogoliubov, *Four-vertex model and random tilings*, Theor. Math. Phys. **155**, 523 (2008).
7. N. M. Bogoliubov, *Four vertex model*, J. Math. Sc., New York **151**, 2816 (2008).
8. N. M. Bogoliubov, C. Malyshev, *Integrable models and combinatorics*, Russian Math. Surveys **70**, 789 (2015).
9. R. Stanley, *Enumerative combinatorics*, Vols. 1, 2, Cambridge University Press, Cambridge, 1996, 1999.
10. D. M. Bressoud, *Proofs and Confirmations. The Story of the Alternating Sign Matrix Conjecture*, Cambridge University Press, Cambridge, 1999.

11. R. P. Stanley, *The Conjugate Trace and Trace of a Plane Partition*, J. Comb. Theor. A **14**, 53 (1973).
12. N. M. Bogoliubov, C. Malyshev, *The phase model and the norm-trace generating function of plane partitions*, J. Stat. Mech.: Theory and Experiment, 083101 (2018).

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