Записки научных
семинаров ПОМИ
Том 473, 2018 г.

M. V. Babich

ON PARAMETRIZATION OF SYMPLECTIC QUOTIENT OF CARTESIAN PRODUCT OF COADJOINT ORBITS OF COMPLEX GENERAL LINEAR GROUP WITH RESPECT TO ITS DIAGONAL ACTION

Abstract. A problem of the coordinatization of the manifold constructed via the Marsden–Weinstein quotient is considered. Rational canonical coordinates on the symplectic reduction with respect to the diagonal action of the general linear group on the Cartesian product of coadjoint orbits in the case of the complex general linear group are constructed. The coordinates on the algebraically-open subset of the quotient space are presented. The method is based on the iteration process used for the construction of the projection-flag coordinates, and works if the matrices forming the orbits have a rich enough set of the invariant subspaces.

Dedicated to M. A. Semenov-Tian-Shansky on the occasion of his 70th birthday

§1. Momentum map

Let Lie group $\mathfrak{G}$ acts on the Poisson manifold $\mathcal{M}$ in such a way that the vector-fields generated by the elements $G \in \mathfrak{g}$ of the Lie algebra of $\mathfrak{G}$ are Hamiltonian vector-fields generated by Hamiltonians H_G, and the map $G \to H_G$ is the homomorphism to the Lie algebra of the Hamilton functions. Such action of $\mathfrak{G}$ is called *a Poisson action*.

This action define *a momentum map* $\mu: \ \mathcal{M} \to \mathfrak{g}^*$ of the manifold to the dual algebra. By definition a value $\mu(X) \in \mathfrak{g}^*$ at point $X \in \mathcal{M}$ is a linear map $\mu(X) : G \to H_G(X)$.

Consider a set of Poisson manifolds $\mathcal{M}^{(1)}, \mathcal{M}^{(1)}, \dots \mathcal{M}^{(N)}$ and their Cartesian product

$$\mathcal{M} := \mathcal{M}^{(1)} \times \mathcal{M}^{(1)} \times \cdots \times \mathcal{M}^{(N)}.$$

Key words and phrases: symplectic reduction, momentum map, Lie–Poisson–Kirillov–Kostant form, Deligne–Simpson problem, projection-flag coordinates.

This work is supported by the Russian Science Foundation (project 14-11-00598).

Manifold $\mathcal{M}$ inherits the Poisson structure. Consider a function $f^{(k)}$: $\mathcal{M}^{(k)} \to \mathbb{C}$. It can be lifted to a function on $\mathcal{M}$ as the pullback of $f^{(k)}$ by the projection of the Cartesian product on factor $\mathcal{M}^{(k)}$. We set a Poisson bracket $\{,\}$ on $\mathcal{M}$ between such functions lifted from the different orbits equal to zero, and equal to the Poisson bracket between pre-images if the functions were lifted from one orbit. This definition we extend on the algebra of functions on $\mathcal{M}$ by the linearity and Leibnitz rule.

Let group $\mathfrak{G}$ act on each $\mathcal{M}^{(k)}$, and the actions are Poisson actions. It is easy to see that the diagonal action is Poisson too. The tangent space in every point $X = (X^{(1)}, \dots, X^{(N)}) \in \mathcal{M}$ is a direct sum of the tangent spaces $\mathrm{T}_{X^{(k)}}\mathcal{M}^{(k)}$. The Hamiltonian $H_G^{(k)}$ lifted from each $\mathcal{M}^{(k)}$ defines the Hamiltonian field. Vectors of this field belong to the corresponding summand $\mathrm{T}_{X^{(k)}}\mathcal{M}^{(k)}$, they have no projections on other $\mathrm{T}_{X^{(i)}}\mathcal{M}^{(i)}$, $i \neq k$.

The diagonal action generates such fields in all summands $\mathrm{T}_{X^{(k)}}\mathcal{M}^{(k)}$, their Hamiltonians are $H_G^{(k)}$. The sum of these fields is produced by the Hamiltonian $H_G := \sum\limits_{k=1}^{N} H_G^{(k)}$ that is the sum of the lifted on $\mathcal{M}$ functions. Each summand generates the flow on its own cartesian summand $\mathcal{M}^{(k)}$.

Proposition. *The momentum map $\mu : \mathcal{M} \to \mathfrak{g}^*$ for the diagonal Poisson action of the Lie group on the Cartesian sum of manifolds $\mathcal{M}^{(1)} \times \cdots \times \mathcal{M}^{(N)} =: \mathcal{M}$ is the sum of the momentum maps of the Cartesian summands $\mathcal{M}^{(k)}$:*

$$\mu = \sum_{k=1}^{N} \mu^{(k)}$$

Let $\mathfrak{G}$ be a (semi)simple matrix group. Consider the coadjoint action of $\mathfrak{G}$ on its orbit $\mathcal{O}$. An element $G \in \mathfrak{g}$ generates the vector-field $[*, G]$, on the orbit. It is the Poisson action, its Hamiltonian is $G \in \mathfrak{g}$, treated as the (linear) function on $\mathfrak{g}^*$. The momentum map in this case $\mu_{ad}(A) = A$, where $A \in \mathfrak{g}^* : G \to \operatorname{tr} AG$. It implies:

Proposition. *The momentum map $\mu : \mathcal{M} \to \mathfrak{g}^*$ for the diagonal (co)adjoint action of the simple matrix Lie group on the Cartesian sum coadjoint orbits $\mathcal{O}^{(1)} \times \cdots \times \mathcal{O}^{(N)} \ni \{A^{(1)}, \dots, A^{(N)}\}$ is the sum of the matrices:*

$$\mu(\{A^{(1)}, \dots, A^{(N)}\}) = \sum_{k=1}^{N} A^{(k)}.$$

§2. Projection-flag coordinates

In the works [1,2] projection-flag coordinates on the coadjoint orbits of the general linear group were introduced. The method of their construction is based on the observation that the representation of matrix A from the orbit:

$$A = \begin{pmatrix} \mathrm{I} & 0 \\ Q & I \end{pmatrix} \begin{pmatrix} \lambda & P \\ 0 & \widetilde{A} \end{pmatrix} \begin{pmatrix} \mathrm{I} & 0 \\ Q & I \end{pmatrix}^{-1}$$

produce the skew-orthogonal with respect to Lie-Poisson-Kirillov-Kostant structure $\{\,,\,\}_{LP}$ families of functions on the orbit:

$$\{P_{ij}, P_{kl}\}_{LP} = \{Q_{ij}, Q_{kl}\}_{LP} = \{P_{ij}, \widetilde{A}_{kl}\}_{LP} = \{Q_{ij}, \widetilde{A}_{kl}\}_{LP} = 0.$$

Matrix $\widetilde{A}$ belongs to the orbit of the smaller dimension that makes possible to organize the iteration process.

The geometrical interpretation of the flight of the iteration is the projection of the action of $A \in \mathrm{End}(\mathbb{C}^n)$ along the eigenspace corresponding to λ on a coordinate subspace. The projection induce $\widetilde{A} \in \mathrm{End}(\mathbb{C}^{\widetilde{n}})$, and the pair P, Q.

Let us construct Q. Consider the projection of a subset $\mathbf{e}'$ of the set $(\mathbf{e}', \mathbf{e}'')$ of the basic vectors on the eigenspace corresponding to the eigenvalue λ parallel to the coordinate subspace $\mathscr{L}(\mathbf{e}'')$ and then project a result to $\mathscr{L}(\mathbf{e}'')$ parallel to $\mathscr{L}(\mathbf{e}')$. It gives matrix Q:

$$\mathbf{e}' \to \mathbf{e}' + \mathbf{e}''Q \to \mathbf{e}''Q.$$

The projection on the eigenspace and the subsequent projection on $\mathscr{L}(\mathbf{e}'')$ can be treated as a linear map $\mathcal{Q} \in \mathrm{Hom}(\mathscr{L}(\mathbf{e}'), \mathscr{L}(\mathbf{e}''))$. The family of the conjugated functions $(P)_{ij}$ can be treated as coordinates coming from the opposite direction map $\mathcal{P} \in \mathrm{Hom}(\mathscr{L}(\mathbf{e}''), \mathscr{L}(\mathbf{e}'))$:

$$A = \begin{pmatrix} * & P \\ * & * \end{pmatrix}.$$

The pairing $\mathrm{tr}\,\mathcal{P}\mathcal{Q}$ coincides with the pairing of functions on the orbit generated by the Lie-Poisson-Kirillov-Kostant structure.

The transposed equality

$$B = \begin{pmatrix} \mathrm{I} & -Q_b \\ 0 & I \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ P_b & \widetilde{B} \end{pmatrix} \begin{pmatrix} \mathrm{I} & -Q_b \\ 0 & I \end{pmatrix}^{-1}$$

can be treated as the contraction of B on the co-eigenspace $\mathrm{im}(B-\lambda\mathrm{I})$ and the corresponding transformations of the coordinate subspaces that give P_b, Q_b.

These interpretations are fundamental for the symplectic reduction

$$\mathcal{O}^{(1)}\times\dots\times\mathcal{O}^{(N)}\to\mathcal{O}^{(1)}\times\dots\times\mathcal{O}^{(N)}/\!\!/\mathrm{GL}(n,\mathbb{C}),$$

$$\sum_{k=1}^{N}A^{(k)}=A^{\Sigma}. \tag{1}$$

Consider the iteration process of the construction of the projection-flag coordinates on the orbits $\mathcal{O}^{(k)}$. The lengths of the flights of the iterations may be arbitrary. These lengths are equal to the dimensions of the eigenspaces used. Nevertheless it is possible to treat such long flights as a series of the flights of the unique length, but with the dependent components of the vectors.

Let a flight use Δ-dimensional subspace, for example. The Grassmanian coordinates $q_{\dots}$ form the matrix elements of $(n-\Delta)\times\Delta$ matrix Q. Let us denote its columns by q_i':

$$\begin{pmatrix}\mathrm{I} & 0\\ Q & I\end{pmatrix}=\begin{pmatrix}1 & 0 & \dots & 0 & 0\\ 0 & 1 & \dots & 0 & 0\\ 0 & 0 & \ddots & 0 & 0\\ 0 & 0 & \dots & 1 & 0\\ q_1' & q_2' & \dots & q_\Delta' & \mathrm{I}\end{pmatrix},$$

and represent this matrix as a product of Δ (comuting) matrices

$$\begin{pmatrix}1 & 0 & \dots & 0 & 0\\ 0 & 1 & \dots & 0 & 0\\ 0 & 0 & \ddots & 0 & 0\\ 0 & 0 & \dots & 1 & 0\\ q_1' & 0 & \dots & 0 & \mathrm{I}\end{pmatrix}\begin{pmatrix}1 & 0 & \dots & 0 & 0\\ 0 & 1 & \dots & 0 & 0\\ 0 & 0 & \ddots & 0 & 0\\ 0 & 0 & \dots & 1 & 0\\ 0 & q_2' & \dots & 0 & \mathrm{I}\end{pmatrix}\cdots\begin{pmatrix}1 & 0 & \dots & 0 & 0\\ 0 & 1 & \dots & 0 & 0\\ 0 & 0 & \ddots & 0 & 0\\ 0 & 0 & \dots & 1 & 0\\ 0 & 0 & \dots & q_\Delta' & \mathrm{I}\end{pmatrix}.$$

We introduce vectors q_i's in such a way that the factors in this product become $\begin{pmatrix}\mathrm{I} & 0 & 0\\ 0 & \mathrm{I} & 0\\ 0 & q_i & \mathrm{I}\end{pmatrix}$. It is the same form as a unit-length flight has: $q_i\in\mathbb{C}^{n-i}$. The only difference is the first $\Delta-i$ vanishing components of q_i.

The similar situation take place for p-coordinates too. The first $\Delta-i$ components of the corresponding p_i will be a linear combinations of the last Δ components. The specific form of the dependence is not essential for us. We just keep in mind that the first $\Delta - i$ elements of the vectors q_i, p_i constructed using multidimensional subspace are dependent. These components do not belong the coordinate set of functions.

Nevertheless all the components of *the last pair* p_Δ, q_Δ constructed from the Δ-dimensional subspace are independent. It is very important for the presenting theory. Such last vectors p_Δ, q_Δ do not differ from the vectors constructed using one-dimensional eigenspaces and will be used for the solving of the constant-momentum-equation (1).

Let all N orbits are parameterized, and all the flights of the iterations are rewritten as series of unit-length flights. Let the set of matrices $\{A^{(k)}\}_{k=1}^N$ is "enough rich". Namely, *for each of* $n-1$ *steps* $m = 1, 2, \ldots, n-1$ *of the iteration process*

$$A_{m-1}^{(k)} = \begin{pmatrix} \mathrm{I} & 0 \\ q_m^{(k)} & I \end{pmatrix} \begin{pmatrix} \lambda_m^{(k)} & p_m^{(k)} \\ 0 & A_m^{(k)} \end{pmatrix} \begin{pmatrix} \mathrm{I} & 0 \\ q_m^{(k)} & I \end{pmatrix}^{-1}$$

or

$$A_{m-1}^{(k)} = \begin{pmatrix} \mathrm{I} & -q_m^{(k)} \\ 0 & I \end{pmatrix} \begin{pmatrix} \lambda_m^{(k)} & 0 \\ p_m^{(k)} & A_m^{(k)} \end{pmatrix} \begin{pmatrix} \mathrm{I} & -q_m^{(k)} \\ 0 & I \end{pmatrix}^{-1},$$

there are at least two matrices, say $A_{m-1}^{(\kappa_m)}$ *and* $A_{m-1}^{(\widehat{\kappa}_m)}$, *such that the eigenspaces corresponding* $\lambda_m^{(\kappa_m)}$ *and* $\lambda_m^{(\widehat{\kappa}_m)}$ *are one-dimensional.*

It means that they are either actually unit-length flights of the process or they are the last flights of the representation of the long-length flight as a series of unit-length flights, like the step number Δ in the example.

Moreover, in the initial set $A^{(k)}, k = 1, \ldots, N$ *there must be* not two, but ***three*** *matrices with one-dimensional eigenspaces.* We denote them $A^{(\kappa_1)}$, $A^{(\widehat{\kappa}_1)}$ and one more matrix $A^{(\kappa_0)}$ with the one-dimensional eigenspace corresponding to some $\lambda_1^{(\kappa_0)}$. This matrix will play a special role in the process.

Finally we distinguish a set of matrices with the one-dimensional kernels. The set consists of the triple of $n \times n$ matrices $A^{(\kappa_0)} - \lambda_1^{(\kappa_0)}\mathrm{I}$, $A^{(\kappa_1)} - \lambda_1^{(\kappa_1)}\mathrm{I}$, $A^{(\widehat{\kappa}_1)} - \lambda_1^{(\widehat{\kappa}_1)}\mathrm{I}$ from the initial set, a couple of $(n-1) \times (n-1)$ matrices $A_1^{(\kappa_2)} - \lambda_2^{(\kappa_2)}\mathrm{I}$, $A_1^{(\widehat{\kappa}_2)} - \lambda_2^{(\widehat{\kappa}_2)}\mathrm{I}$ constructed after the first flight of the iteration, a couple of $(n-2) \times (n-2)$ matrices $A_2^{(\kappa_3)} - \lambda_3^{(\kappa_3)}\mathrm{I}$, $A_2^{(\widehat{\kappa}_3)} - \lambda_3^{(\widehat{\kappa}_3)}\mathrm{I}$ constructed on the second flight, and so on. The last couple is two degenerated but non-vanishing 2×2 matrices $A_{n-2}^{(\kappa_{n-1})} - \lambda_{n-1}^{(\kappa_{n-1})}\mathrm{I}$, $A_{n-2}^{(\widehat{\kappa}_{n-2})} - \lambda_{n-1}^{(\widehat{\kappa}_{n-1})}\mathrm{I}$.

We fix the following representation of the matrices from the set:

$$A^{(\kappa_m)}_{m-1} = \begin{pmatrix} \mathrm{I} & 0 \\ q_m^{(\kappa_m)} & \mathrm{I} \end{pmatrix} \begin{pmatrix} \lambda_m^{(\kappa_m)} & p_m^{(\kappa_m)} \\ 0 & A_m^{(\kappa_m)} \end{pmatrix} \begin{pmatrix} \mathrm{I} & 0 \\ q_m^{(\kappa_m)} & \mathrm{I} \end{pmatrix}^{-1}$$

$$A^{(\widehat{\kappa}_m)}_{m-1} = \begin{pmatrix} \mathrm{I} & -q_m^{(\widehat{\kappa}_m)} \\ 0 & \mathrm{I} \end{pmatrix} \begin{pmatrix} \lambda_m^{(\widehat{\kappa}_m)} & 0 \\ p_m^{(\widehat{\kappa}_m)} & A_m^{(\widehat{\kappa}_m)} \end{pmatrix} \begin{pmatrix} \mathrm{I} & -q_m^{(\widehat{\kappa}_m)} \\ 0 & \mathrm{I} \end{pmatrix}^{-1}$$

for $m = 1, 2, \ldots, n-1$, and

$$A^{(\kappa_0)} = \begin{pmatrix} \mathrm{I} & 0 \\ q_1^{(\kappa_0)} & \mathrm{I} \end{pmatrix} \begin{pmatrix} \lambda_1^{(\kappa_0)} & p_1^{(\kappa_0)} \\ 0 & A_1^{(\kappa_0)} \end{pmatrix} \begin{pmatrix} \mathrm{I} & 0 \\ q_0^{(\kappa_0)} & \mathrm{I} \end{pmatrix}^{-1}.$$

Theorem 1. *Equations*

$$q_m^{(\kappa_m)} = 0, \quad q_m^{(\widehat{\kappa}_m)} = 0, \quad m = 1, \ldots, n-1, \quad q_1^{(\kappa_0)} = (1, \ldots, 1)^T$$

define a section of the fiber-bundle

$$\mathcal{O}^{(1)} \times \mathcal{O}^{(2)} \times \cdots \times \mathcal{O}^{(N)} \longrightarrow \mathcal{O}^{(1)} \times \mathcal{O}^{(2)} \times \cdots \times \mathcal{O}^{(N)}/\mathrm{GL}(N, \mathbb{C})$$

on the algebraically-open set of configurations $\{A^{(1)}, A^{(2)}, \ldots, A^{(N)}\} \in \mathcal{O}^{(1)} \times \mathcal{O}^{(2)} \times \cdots \times \mathcal{O}^{(N)}$.

Proof. These equations have a simple geometrical sense. Namely this set of the conditions uniquely fix a projective frame of $\mathbb{C}^n : A^{(n)} \in \mathrm{End}(\mathbb{C}^n)$.

Matrices $A^{(k)}_m$ act on the coordinate subspace that envelopes basic vectors $\mathbf{e}_{m+1}, \mathbf{e}_{m+1}, \ldots, \mathbf{e}_n$. They are the projections of $A^{(k)}$ along the sum of m eigenspaces corresponding to $\lambda^{(k)}_1, \lambda^{(k)}_2, \ldots, \lambda^{(k)}_m$, or they are the projections along the sums of the kernels of the corresponding powers $(A^{(k)} - \lambda^{(k)}_i \mathrm{I})^j$ in the Jordanian case.

The equalities $q_m^{(\kappa_m)} = 0, \quad q_m^{(\widehat{\kappa}_m)} = 0$ uniquely fix the direction of $\mathbf{e}_m$ and the hyperplane containing $\mathbf{e}_{m+1}, \ldots \mathbf{e}_n$ on the coordinate subspace enveloping $\mathbf{e}_m, \ldots \mathbf{e}_n$.

For example the equality $q_1^{(\kappa_1)} = 0$ means that the direction of the first basic vector is chosen to be parallel to the kernel $A^{(\kappa_1)} - \lambda_1^{(\kappa_1)}\mathrm{I}$. The equality $q_1^{(\widehat{\kappa}_1)} = 0$ means that all the basic vectors except $\mathbf{e}_1$ belong to the image of $A^{(\widehat{\kappa}_1)} - \lambda_1^{(\widehat{\kappa}_1)}\mathrm{I}$.

After the finishing the process, the directions of all vectors will be fixed. The scales on the axes are fixed by the last condition, that is the direction of the kernel of $A^{(\kappa_0)} - \lambda_1^{(\kappa_0)}\mathrm{I}$ is a direction of the vector $(1, \ldots, 1)^T$.

The projective frame is fixed in $\mathbb{C}^n$. □

Let us write down the constant-momentum-equation (1):

$$\sum_{k=1}^{N} A^{(k)} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^{(\kappa_0)} & p_1^{(\kappa_0)} \\ 0 & \\ \vdots & A_1^{(\kappa_0)} \\ 0 & \end{pmatrix} \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}^{-1}$$

$$+ \begin{pmatrix} \lambda_1^{(\kappa_1)} & p_1^{(\kappa_1)} \\ 0 & A_1^{(\kappa_1)} \end{pmatrix} + \begin{pmatrix} \lambda_1^{(\widehat{\kappa}_1)} & 0 \\ p_1^{(\widehat{\kappa}_1)} & A_1^{(\widehat{\kappa}_1)} \end{pmatrix} + \sum_{\substack{1 \leqslant k \leqslant N \\ k \notin \{\kappa_0, \kappa_1, \widehat{\kappa}_1\}}} A^{(k)} = A^{\Sigma} = \mathfrak{const} \in \mathrm{gl}(n, \mathbb{C}).$$

It is n^2 scalar equations, but we must satisfy not n^2, but $n^2 - 1$ equations. One equation, namely $\sum_k \operatorname{tr} A^{(k)} = \operatorname{tr} A^{\Sigma}$, should be satisfied in advance, we call it *a traces identity*. Let it be.

Let us denote the lower-right $(m-1) \times (m-1)$ block of any $m \times m$ matrix B by $[B]$. It is evident that the constant-momentum-equation is equivalent to one scalar equation

$$\left(A^{\Sigma}\right)_{11} = \sum_{k=1}^{N} \left(A^{(k)}\right)_{11},$$

two vector equations:

$$p_1^{(\kappa_1)} = \left(A^{\Sigma}\right)_{1,*} - \sum_{k=1}^{N} \left(A^{(k)}\right)_{1,*} \quad p_1^{(\widehat{\kappa}_1)} = \left(A^{\Sigma}\right)_{*,1} - \sum_{k=1}^{N} \left(A^{(k)}\right)_{*,1},$$

and one $(n-1) \times (n-1)$ matrix equation:

$$[A^{\Sigma}] = \sum_{k=1}^{N} A_1^{(k)} + \sum_{\substack{1 \leqslant s \leqslant N \\ s \notin \{\kappa_0, \kappa_1, \widehat{\kappa}_1\}}} q_1^{(s)} p_1^{(s)} + \vec{\mathbf{1}} p_1^{(\kappa_0)}. \tag{2}$$

In these formulas indexes "$1, *$" and "$*, 1$" mean the first row and the first column of the off-diagonal part of the corresponding matrix, $q_1^{(s)} p_1^{(s)}$ and $\vec{\mathbf{1}} p_1^{(\kappa_0)}$ are matrices of the unit rank: $(q_1^{(s)} p_1^{(s)})_{ij} = (q_1^{(s)})_i (p_1^{(s)})_j$, $(\vec{\mathbf{1}} p_1^{(\kappa_0)})_{ij} = (p_1^{(\kappa_0)})_j$.

First of all let us discuss the scalar equation. If the sums $(\sum_{k=1}^{n} A^{(k)})_{ii}$ are equal to the corresponding $(A^{\Sigma})_{ii}$ for all i except one, then the diagonal elements of $\sum_{k=1}^{n} A^{(k)}$ and A^{Σ} coincide for this missed element too because of the traces identity. We leave the scalar equation $\left(A^{\Sigma}\right)_{11}=\sum\limits_{k=1}^{N}\left(A^{(k)}\right)_{11}$ for this "automatic" solution.

The couple of the vector equations we set as definitions of vectors $p_1^{(\kappa_1)}, p_1^{(\widehat{\kappa}_1)}$.

Consider the matrix equation (2). It does not contain the components of vectors $p_1^{(\kappa_1)}, p_1^{(\widehat{\kappa}_1)}$ because the corresponding $q_1^{(\kappa_1)}, q_1^{(\widehat{\kappa}_1)}$ vanish, consequently $q_1^{(\kappa_1)}p_1^{(\kappa_1)}=p_1^{(\widehat{\kappa}_1)}q_1^{(\widehat{\kappa}_1)}=0$ and the summation index "s" does not equal to $\kappa_1, \widehat{\kappa}_1$. We solve equation (2) first and then substitute the result, that is some expressions for $(A_1^{(k)})_{i,j}, (q_1^{(s)})_i, (p_1^{(s)})_i,\ \kappa_1\neq s\neq\widehat{\kappa}_1$ into the definition of $p_1^{(\kappa_1)}, p_1^{(\widehat{\kappa}_1)}$. No additional equations arise.

Let us rewrite equation (2) in the following form:

$$\sum_{k=1}^{N} A_1^{(k)} = A_1^{\Sigma} - \vec{\mathbf{1}}p_1^{(\kappa_0)}, \tag{3}$$

where $A_1^{\Sigma} := [A^{\Sigma}] - \sum\limits_{\substack{1\leqslant s\leqslant N\\ s\notin\{\kappa_0,\kappa_1,\widehat{\kappa}_1\}}} q_1^{(s)}p_1^{(s)}$. It has the same form as (1) except the summand $\vec{\mathbf{1}}p_1^{(\kappa_0)}$. The dimension of this matrix equation is $(n-1)\times(n-1)$. We will make the same trick once more and write (3) as

$$\sum_{\substack{1\leqslant k\leqslant N\\ k\notin\{\kappa_2,\widehat{\kappa}_2\}}} A_1^{(k)} + \begin{pmatrix} \lambda_2^{(\kappa_2)} & p_2^{(\kappa_2)} \\ 0 & A_2^{(\kappa_2)} \end{pmatrix} + \begin{pmatrix} \lambda_2^{(\widehat{\kappa}_2)} & 0 \\ p_2^{(\widehat{\kappa}_2)} & A_2^{(\widehat{\kappa}_2)} \end{pmatrix} = A_1^{\Sigma} - \vec{\mathbf{1}}p_1^{(\kappa_0)}. \tag{4}$$

The difference between this case $(n-1)\times(n-1)$ and the previous one is the presence of $\vec{\mathbf{1}}p_1^{(\kappa_0)}$ in the right-hand side, and the absence of the distinguished value of index like κ_0 for $A_1^{(k)}$. One more difference is the fact that the trace identity is already used, consequently we need to solve the scalar equation

$$\sum_{\substack{1\leqslant k\leqslant N\\ k\notin\{\kappa_2,\widehat{\kappa}_2\}}} \left(A_1^{(k)}\right)_{11} + \lambda_2^{(\kappa_2)} + \lambda_2^{(\widehat{\kappa}_2)} = \left(A_1^{\Sigma}\right)_{11} - (p_1^{(\kappa_0)})_1.$$

It fix the value of $(p_1^{(\kappa_0)})_1$ as a function of the matrix elements of $A_1^{(k)}$ and $\left(A_1^{\Sigma}\right)_{11}$. The correctness of the theory is based on the fact that the values of the components of $(p_1^{(\kappa_0)})_1$ do not take part in another steps of the process. Some matrix values of $A_1^{(k)}$ will be expressed as functions of another values but that expressions do not use $(p_1^{(\kappa_0)})_1$. The value $s = \kappa_0$ of the index in the definition of A_1^Σ containing the sum $\sum_s q_1^{(s)} p_1^{(s)}$ do not used too.

Let us denote a set of matrices A_m^Σ:

$$A_m^\Sigma = [A_{m-1}^\Sigma] - \sum_{\substack{1\leqslant s\leqslant N\\ s\notin\{\kappa_m,\widehat{\kappa}_m\}}} q_m^{(s)} p_m^{(s)}, \quad A_1^\Sigma := [A^\Sigma] - \sum_{\substack{1\leqslant s\leqslant N\\ s\notin\{\kappa_0,\kappa_1,\widehat{\kappa}_1\}}} q_1^{(s)} p_1^{(s)}$$

with the decreasing sizes $(n-m)\times(n-m)$, $m = 2, 3, \dots, n-2$.

The constant-momentum-equation generates a sequence of equations

$$\sum_{\substack{1\leqslant k\leqslant N\\ k\notin\{\kappa_m,\widehat{\kappa}_m\}}} A_{m-1}^{(k)} + \begin{pmatrix} \lambda_m^{(\kappa_m)} & p_m^{(\kappa_m)} \\ 0 & A_m^{(\kappa_m)} \end{pmatrix} + \begin{pmatrix} \lambda_m^{(\widehat{\kappa}_m)} & 0 \\ p_m^{(\widehat{\kappa}_m)} & A_m^{(\widehat{\kappa}_m)} \end{pmatrix}$$
$$= A_{m-1}^\Sigma - \vec{\mathbf{1}}[p_1^{(\kappa_0)}]_{m-1},$$

where $[p_1^{(\kappa_0)}]_r := \Big((p_1^{(\kappa_0)})_r, (p_1^{(\kappa_0)})_{r+1}, \dots, (p_1^{(\kappa_0)})_n\Big)$ denote a vector that is a last part of vector $p_1^{(\kappa_0)}$, $[p_1^{(\kappa_0)}]_1 := p_1^{(\kappa_0)}$. Any equation from the sequence is equivalent to one scalar equation, two vector equations and the equation that is the next in the sequence. It means that the following theorem take place.

Theorem 2. *The constant-momentum equation* $\sum_k A^{(k)} = A^\Sigma$ *is satisfied if we set*

$$\Big(p_1^{(\kappa_0)}\Big)_m = \Big(A_m^{(\Sigma)}\Big)_{11} + \sum_{\substack{1\leqslant s\leqslant N\\ s\notin\{\kappa_{m+1},\widehat{\kappa}_{m+1}\}}} p_{m+1}^{(s)} q_{m+1}^{(s)} - \sum_{k=1}^N \lambda_{m+1}^{(k)}, \quad m = 1, \dots, n-2,$$

$$\Big(p_1^{(\kappa_0)}\Big)_{n-1} = \Big(A_{n-2}^{(\Sigma)}\Big)_{22} - \sum_{k=1}^N \lambda_n^{(k)} - \sum_{\substack{1\leqslant s\leqslant N\\ s\notin\{\kappa_{n-1},\widehat{\kappa}_{n-1}\}}} p_{n-1}^{(s)} q_{n-1}^{(s)},$$

and

$$p_m^{(\kappa_m)} := \big(A_{m-1}^\Sigma\big)_{1,*} - \sum_{\substack{1\leqslant s\leqslant N\\ s\notin\{\kappa_m,\widehat{\kappa}_m\}}} \Big(A_{m-1}^{(s)}\Big)_{1,*} - [p_1^{(\kappa_0)}]_m$$

$$p_m^{(\widehat{\kappa}_m)} := \left(A_{m-1}^{\Sigma}\right)_{*,1} - \sum_{\substack{1\leqslant s\leqslant N\\ s\notin\{\kappa_m,\widehat{\kappa}_m\}}} \left(A_{m-1}^{(s)}\right)_{*,1} - (p_1^{(\kappa_0)})_{m-1}\vec{\mathbf{1}}.$$

Proof. The constant-momentum-equation follows from these equalities by the construction, so what we need to prove is the correctness of the procedure. It is the absence of additional equations.

We solve the scalar equations first. It gives the dependence of $p_1^{(\kappa_0)}$ from the different variables, but there are no $p_m^{(\kappa_m)}, p_m^{(\widehat{\kappa}_m)}$ among them, because the components of the vectors $p_m^{(\kappa_m)}, p_m^{(\widehat{\kappa}_m)}$ do not take place in the first two lines (the scalar equations). That is why the substitution of $p_1^{(\kappa_0)}$ into the vector equations do not give new equations. □

References

1. M. V. Babich and S. E. Derkachov, *On rational symplectic parametrization of coajoint orbit of $GL(N)$. Diagonalizable case.* — Algebra Analiz, **22**, No. 3 (2010), 16–30 (in Russian).
2. M. V. Babich, *Birational Darboux Coordinates on (Co)Adjoint Orbits of $GL(N,\mathbb{C})$.* — Funct. Analys. Applic., **50**, No. 1 (2016), 17–30.
3. M. V. Babich, *On birational Darboux coordinates on coadjoint orbits of classical complex Lie groups.* — Zap. Nauchn. Semin. POMI, **432** (2015), 36–57. ftp://ftp.pdmi.ras.ru/pub/publicat/znsl/v432/p068.pdf.
4. M. V. Babich, *Young tableaux and stratification of space of complex square matrices.* — Zap. Nauchn. Semin. POMI, **433** (2015), 41–64. ftp://ftp.pdmi.ras.ru/pub/publicat/znsl/v433/p041.pdf.
5. S. E. Derkachov, A. N. Manashov, *R-matrix and Baxter Q-operators for the noncompact SL(N;C) invariant spin chain SIGMA* 2, (2006). 084 arXiv:nlin.SI/0612003.
6. M. V. Babich, *About Coordinates on the Phase Spaces of the Schlesinger System and the Garnier-Painlevé* 6 *System.* — Dokl. Math., **75**, No. 1 (2007), 71.
7. A. G. Reiman, M. A. Semenov-Tyan'-Shanskii, *Integrable systems. Group-theoretic approach.* — Sovremennaya Matematika, Inst. for Computer Research, Moscow–Izhevsk (2003). (in Russian).
8. I. M. Gelfand, M. I. Najmark, *Unitary representation of classical groups.* — Trudy Mat. Instituta V. A. Steklova, **36** (1950).

St.Petersburg Department
of V. A. Steklov Mathematical Institute
of Russian Academy of Science.
Saint Petersburg State University,
St.Petersburg, Russia
E-mail: mbabich@pdmi.ras.ru,
m.babich@spbu.ru

Поступило 22 сентября 2018 г.