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ONE DIMENSIONAL INVERSE PROBLEM IN PHOTOACOUSTIC. NUMERICAL TESTING

Abstract. We consider the problem of reconstruction of Cauchy data for the wave equation in $\mathbb{R}^1$ by the measurements of its solution on the boundary of the finite interval. This is a one-dimensional model for the multidimensional problem of photoacoustics, which was studied in [2]. We adapt and simplify the method for one-dimensional situation and provide the results on numerical testing to see the rate of convergence and stability of the procedure. We also give some hints on how the procedure of reconstruction can be simplified in 2d and 3d cases.

Dedicated to the memory of A. P. Kachalov

§1. Introduction

Photoacoustic is a method of visualization for obtaining optical images in a scattering medium. Ultrashort pulses of laser radiation causes thermoelastic stresses in the light absorption region. Such thermal expansion causes the propagation of ultrasonic waves in the medium, which can be registered and measured. After that, using the measurements, mathematical algorithms and computers one can produce an image, in this case the technology called Photoacoustic tomography.

In what follows we assume that the visualized medium is homogeneous with respect to ultrasound, put the speed of sound equal to 1 and write down the standard wave equation for a pressure wave

$$\begin{cases} u_{tt} - \Delta u = 0, \quad x \in \mathbb{R}^n, \ t > 0, \\ u\big|_{t=0} = a, \ u_t\big|_{t=0} = b, \end{cases} \tag{1.1}$$

Key words and phrases: inverse problem, photoacoustic, wave equation.

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where the functions $a = a(x)$, $b = b(x)$ are the initial conditions describing spatial distributions of the absorption of laser radiation energy and of its temporal change. Also we assume that there is a set of transducers running over a surface S measuring the pressure wave and collecting data. A transducer at at point x at time t measures the value $u(x,t)$. Then we can suppose that

$$F := u\big|_{S\times[0,T]}. \tag{1.2}$$

is known for some $T > 0$. The inverse problem (IP) is to find the initial data a, b, using data $F(x,t)$ measured by transducers. In our setting we assume that the surface S is a unit sphere and $\operatorname{supp} a, \operatorname{supp} b$ are located inside S.

In the present paper we continue the the work started in [2]. Our goal here is to develop new algorithms that improve the speed of calculations and the clarity of the resulting image. In [2] we presented such algorithms for three ($n = 3$) and two ($n = 2$) dimensional situation. The numerical realization of these algorithms is quite involved in multidimensional situation, that is why here we restrict ourselves for the case $n = 1$. On the one hand, the one-dimensional situation is rather simple, on the other hand, for numerical simulation in $n = 1$ we preset the simplified version of an algorithm described in [2]. And we also point out the way how this simplifications can be made in cases $n = 2$ and $n = 3$.

In the second section we derive the Fourier representation for the forward problem in the cases $n = 1, 2, 3$. In the third section we describe the algorithm of solving IP in one-dimensional situation. In the forth section the modifications needed for simplifications of the algorithm in multidimensional cases are described. In the last section we demonstrate the results of numerical simulation in the case $n = 1$.

§2. Solving of forward problem

Since $\operatorname{supp} a, b \subset B_1 = \{x \in \mathbb{R}^n \mid |x| < 1\}$ and the speed of sound in (1.1) equal to one, we have that

$$u(x,t) = 0, \quad |x| = T + 1,\ t \in (0,T). \tag{2.1}$$

We use the separation of variables (Fourier) method to solve the initial boundary value problem (1.1), (2.1), we look for the solution in a form

$$u(x,t) = \sum_{k=1}^{\infty} X_k(x) T_k(t), \tag{2.2}$$

where $X_k(x)$ are solution to the following spectral problem

$$\begin{cases} -\Delta X_k = \lambda_k X_k, & |x| < 1+T, \\ X_k\big|_{|x|=1+T} = 0, \end{cases} \tag{2.3}$$

and $T_k(t)$ are defined by

$$\begin{cases} -T_k'' = \lambda_k T_k, & 0 < t < T, \\ \sum\limits_{k=1}^{\infty} X_k(x) T_k(0) = a(x), & |x| < 1+T, \\ \sum\limits_{k=1}^{\infty} X_k(x) T_k'(0) = b(x), & |x| < 1+T. \end{cases} \tag{2.4}$$

From (2.4) we have that

$$T_k(t) = a_k \cos\sqrt{\lambda_k} t + \frac{b_k}{\sqrt{\lambda_k}} \sin\sqrt{\lambda_k} t, \tag{2.5}$$

where

$$a_k = \frac{(a, X_k)_{L_2}}{\|X_k\|_{L_2}}, \quad b_k = \frac{(b, X_k)_{L_2}}{\|X_k\|_{L_2}}. \tag{2.6}$$

Solution to (2.3) has the following form in different dimensions:

$$X_k(x) = \sin\Big(\frac{x + (1+T)}{2(1+T)} k\pi\Big), \quad \lambda_k = \Big(\frac{k\pi}{2(1+T)}\Big)^2, \quad n = 1, \tag{2.7}$$

$$X_{kl}(x) = J_k\Big(\frac{\nu_k^l r}{1+T}\Big) e^{ik\varphi}, \quad \lambda_{kl} = \Big(\frac{\nu_k^l}{1+T}\Big)^2, \quad n = 2, \tag{2.8}$$

$$X_{klm}(x) = \frac{1}{\sqrt{r}} J_{k+1/2}\Big(\frac{\xi_k^l r}{1+T}\Big) e^{im\varphi} P_k^m(\cos\theta), \quad \lambda_{klm} = \Big(\frac{\xi_k^l}{1+T}\Big)^2, \ n=3, \tag{2.9}$$

where J_k, $J_{k+1/2}$ – Bessel functions of the first kind, ν_k^l and ξ_k^l – their roots, P_k^m – associated Legendre polynomials:

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)} \Big(\frac{x}{2}\Big)^2,$$

$$P_k^m(x) = \frac{(-1)^m}{2^k k!} (1-x^2)^{m/2} \frac{d^{l+k}}{dx^{l+k}} (x^2-1)^k.$$

Thus the representation (2.2) gives a solution to (1.1), (2.1) in the region $\{(x,t)\,|\,|x| < T+1,\, t \in (0,T)\}$, with X_k defined by (2.7)–(2.9) and T_k by (2.5).

§3. Solving of IP for $n = 1$

The IP is to find the initial values $a(x)$, $b(x)$ of (1.1) using the observation $F(x,t)$ measured on unit sphere:

$$F(x,t) = u(x,t), \quad |x| = 1,\ 0 < t < T.$$

In 1d case the solution u in $\{(x,t) \,|\, |x| < T+1,\ t \in (0,T)\}$ is given by (2.2), (2.5), (2.6), (2.7). The latter yields the following equality

$$\sum_{k=1}^{\infty} X_k(x)(a_k \cos\sqrt{\lambda_k}t + \frac{b_k}{\sqrt{\lambda_k}} \sin\sqrt{\lambda_k}t) = F(x,t), \ \ |x| = 1,\ t < T. \quad (3.1)$$

We would like to determine coefficients a_k and b_k from (3.1) and restore functions a and b as Fourier series:

$$a(x) = \sum_k a_k X_k(x), \quad b(x) = \sum_k b_k X_k(x).$$

We restrict ourselves to the case $n = 1$ and $b = 0$, the numerical testing in the last section was conducted specifically in this situation. Other cases require some modifications, we postpone them for further consideration.

In one dimensional case (3.1) becomes

$$\begin{cases} \sum\limits_{k=1}^{\infty} \sin\Big(\frac{2+T}{2(1+T)}k\pi\Big)(a_k \cos\frac{k\pi}{2(1+T)}t + \frac{b_k}{\sqrt{\lambda_k}} \sin\frac{k\pi}{2(1+T)}t) = F(1,t), \\ \sum\limits_{k=1}^{\infty} \sin\Big(\frac{T}{2(1+T)}k\pi\Big)(a_k \cos\frac{k\pi}{2(1+T)}t + \frac{b_k}{\sqrt{\lambda_k}} \sin\frac{k\pi}{2(1+T)}t) = F(-1,t). \end{cases} \quad (3.2)$$

Using the equality $\frac{T}{2(1+T)}k\pi = k\pi - \frac{2+T}{2(1+T)}k\pi$ we get what

$$F(1,t) = u_1(t) + u_2(t), \quad F(-1,t) = -u_1(2(T+1)-t) + u_2(2(T+1)-t),$$

where

$$u_1(t) = \sum_{k=1}^{\infty} a_k \sin\Big(\frac{2+T}{2(1+T)}k\pi\Big) \cos\frac{k\pi}{2(1+T)}t,$$

$$u_2(t) = \sum_{k=1}^{\infty} b_k \sin\Big(\frac{2+T}{2(1+T)}k\pi\Big) \sin\frac{k\pi}{2(1+T)}t.$$

On assuming that $b = 0$ and introducing the notation

$$\widehat{F}(t) = \begin{cases} F(1,t), & 0 < t < 1+T, \\ F(-1, 2(1+T)-t), & 1+T < t < 2(1+T), \end{cases}$$

we then arrive at the following relation

$$\widehat{F}(t) = \sum_{k=1}^{\infty} a_k \sin\Big(\frac{2+T}{2(1+T)}k\pi\Big)\cos\frac{k\pi}{2(1+T)}t. \tag{3.3}$$

The left hand side of the last equatity is known, therefore we can find a_k as a Fourier coefficients of the basis $\left\{\cos\frac{k\pi}{2(T+1)}t\right\}_{k=0}^{\infty}$ in $L_2(0,2(T+1))$: .

$$a_k = \frac{\widehat{F}_k}{\sin\Big(\frac{2+T}{2(1+T)}k\pi\Big)}, \quad \text{where } \widehat{F}_k = \frac{(\widehat{F}, \cos\frac{k\pi}{2(1+T)}t)_{L_2(0,2(T+1))}}{T+1}. \tag{3.4}$$

From (3.4) we can easily see that if, we take $T = 2$ then the denominator in the first formula vanishes for $k = 3, 6, 9, \ldots$. Therefore we could not use (3.4) to reconstruct a. It is also the case if $k = (T+1)n$ for some $n \in \mathbb{N}$. To improve this we need the following lemma:

Lemma 1. *Let $T \in \mathbb{N}$ be fixed, the function a with $\operatorname{supp} a \subset (-1,1)$ admits the expansion*

$$a(x) = \sum_{n=1}^{\infty} a_n \sin\Big(\frac{x+1}{2}n\pi\Big).$$

Define $\widetilde{a}(x)$ by the rule:

$$\widetilde{a}(x) = \begin{cases} a(x), & -1 < x < 1, \\ 0, & 1 < |x| < T+1, \end{cases}$$

so $\widetilde{a}(x)$ admits the representation

$$\widetilde{a}(x) = \sum_{k=1}^{\infty} \widetilde{a}_k \sin\Big(\frac{x+1+T}{2(T+1)}k\pi\Big).$$

Then

$$a(x) = \begin{cases} \frac{T+1}{T}A(x), & T \textit{ is even} \\ A(x), & T \textit{ is odd}, \end{cases} \tag{3.5}$$

where

$$A(x) = \sum_{\substack{k\in\mathbb{N},\\ k\neq(T+1)n}} \widetilde{a}_k \sin\Big(\frac{x+1+T}{2(T+1)}k\pi\Big). \tag{3.6}$$

Proof. We express $\widetilde{a}_k$ via a_n:

$$\begin{aligned}\widetilde{a}_k &= \frac{1}{T+1}\int\limits_{-(T+1)}^{T+1} \widetilde{a}(x)\sin\Big(\frac{x+1+T}{2(T+1)}k\pi\Big)\,dx \\ &= \frac{1}{T+1}\sum_{n=1}^{\infty}\int\limits_{-1}^{1} a_n \sin\Big(\frac{x+1}{2}n\pi\Big)\sin\Big(\frac{x+1+T}{2(T+1)}k\pi\Big)\,dx \\ &= \begin{cases}\frac{2}{\pi}\sin\Big(\frac{k\pi T}{2(T+1)}\Big)\sum\limits_{n=1}^{\infty}\frac{na_n}{(T+1)^2n^2-k^2}, & n+k \text{ is even},\\ 0, & n+k \text{ is odd}.\end{cases}\end{aligned}$$

The last formula works if only $(T+1)n \neq k$ for any n, if for some n_0 we have $(T+1)n_0 = k$, then

$$\widetilde{a}_k = \frac{1}{T+1}a_{n_0}\cos\Big(\frac{Tn_0}{2}\pi\Big).$$

From the last equality we derive that

$$\begin{aligned}\widetilde{a}(x) &= A(x) + \frac{1}{T+1}\sum_{n=1}^{\infty} a_n \cos\Big(\frac{Tn}{2}\pi\Big)\sin\Big(\frac{x+1+T}{2}n\pi\Big) \\ &= A(x) + \frac{1}{T+1}\sum_{n=1}^{\infty}\frac{a_n}{2}\Big(\sin\Big(\frac{x+1}{2}n\pi\Big) + \sin\Big(\frac{x+1}{2}n\pi + Tn\pi\Big)\Big) \\ &= A(x) + \frac{1}{(T+1)2}\Big(a(x) + (-1)^T a(x)\Big).\end{aligned}$$

Taking here $|x| < 1$ yields (3.5), which completes the proof. □

Applying this result to our situation, we can make the following

Proposition 1. *Let $T \in \mathbb{N}$ be fixed. The unknown function a can be recovered by formula* (3.5), *where coefficients $\widetilde{a}_k$ in representation* (3.6) *are given by* (3.4).

§4. Remarks on solving of IP for $n = 2, 3$.

We make use of (2.8), (2.5), (2.2) to write down an Fourier expansion of observation (1.2) in 2d case:

$$\sum_{k,\,l} J_k\Big(\frac{\nu_k^l}{1+T}\Big)e^{ik\varphi}\Big(a_{kl}\cos\Big(\frac{\nu_k^l}{1+T}t\Big) + b_{kl}\frac{1+T}{\nu_k^l}\sin\Big(\frac{\nu_k^l}{1+T}t\Big)\Big) = F(t,\varphi),$$

where $F(t,\varphi)$ is a given boundary measurements and a_{kl} and b_{kl} are Fourier coefficients subjected to determination. Multiplying in $L_2(0,2\pi)$ by $\frac{1}{\sqrt{2\pi}}e^{ik\varphi}$ we obtain:

$$\sum_l J_k\Big(\frac{\nu_k^l}{1+T}\Big)\Big(a_{kl}\cos\Big(\frac{\nu_k^l}{1+T}t\Big)+b_{kl}\frac{1+T}{\nu_k^l}\sin\Big(\frac{\nu_k^l}{1+T}t\Big)\Big)=F_k(t), \quad (4.1)$$

where $F_k(t)$ is a Fourier coefficient of $F(t,\varphi)$ w.r.t. family $\left\{\frac{1}{\sqrt{2\pi}}e^{ik\varphi}\right\}_{k\in\mathbb{Z}}$ in $L_2(0,2\pi)$.

Equation (4.1) is a 2d analog of (3.3), but the arguments of cosine and sine functions are different: instead of $\frac{k\pi}{2(T+1)}$ we have $\frac{\nu_k^l}{T+1}$, where ν_k^l is a root of Bessel function J_k.

On observing that the system $\left\{\cos\Big(\frac{k\pi}{2(T+1)}t\Big)\right\}_{k=0}^{\infty}$ is orthogonal in $L_2(0,2(T+1))$, we derived formula for a_k (3.4) from (3.3). But now the system $\left\{\cos\Big(\frac{\nu_k^l}{T+1}t\Big)\right\}_{l=1}^{\infty}$ is not a basis. At the same time this family is not "very bad".

We remind that

$$J_k(x)=\sqrt{\frac{2}{\pi x}}\cos\Big(x-\frac{k\pi}{2}-\frac{\pi}{4}\Big)+\mathcal{O}_{x\to\infty}\Big(\frac{1}{x^{3/2}}\Big),$$

and therefore

$$\nu_k^l\approx\frac{k\pi}{2}+\frac{\pi}{4}+\frac{2l-1}{2}\pi,\quad l=1,2,\dots,$$

which means that

$$\nu_k^l-\nu_k^{l+1}\to\pi \ \text{ if } \ l\to\infty.$$

All aforesaid and [1], (see Chapter 4, Theorem II.4.1) allows us to conclude that the system $\left\{\cos\Big(\frac{\nu_k^l}{T+1}t\Big)\right\}_{l=1}^{\infty}$ is minimal in $L_2(0,T+1)$ and system $\left\{\cos\Big(\frac{\nu_k^l}{T+1}t\Big),\ \sin\Big(\frac{\nu_k^l}{T+1}t\Big)\right\}_{l=1}^{\infty}$ is minimal in $L_2(-T-1,T+1)$. If $b=0$ then

$$a_{kl}=\frac{F_{kl}}{J_k\Big(\frac{\nu_k^l}{1+T}\Big)},\quad F_{kl}=\frac{\Big(F_k(t),u_{kl}(t)\Big)_{L_2(0,T+1)}}{\|u_{kl}(t)\|_{L_2(0,T+1)}}, \quad (4.2)$$

where the system $\{u_{kl}(t)\}_{l=1}^{\infty}$ is bi-orthogonal to $\left\{\cos\Big(\frac{\nu_k^l}{T+1}t\Big)\right\}_{l=1}^{\infty}$ in $L_2(0,T{+}1)$. Formula (4.2) is 2d analog of (3.4), but instead of $\sin\big(\frac{2+T}{2(1+T)}k\pi\big)$

in denominator we have $J_k\left(\frac{\nu_k^l}{1+T}\right)$. It has an advantage over one-dimensional case because the denominator of (4.2) does not vanish for integer values of T. Disadvantage is that denominator tend to zero while $l \to \infty$ which makes calculation sensitive to accuracy.

Similar reasoning can be used in a three-dimensional situation, but instead of (2.8) we use (2.9).

§5. Numerical experiment

In this section we provide the results on numerical testing for the one-dimensional problem. Namely, we consider (1.1) with $n = 1$, $b = 0$, take $T = 2$ in (2.1). First we generate data for IP i.e. observation (1.2), by solving forward problem with the use of (2.2), (2.5), (2.6), (2.7).

After solving forward problem we discretized obtained data (function $F(\pm 1, t)$) to get closer to the case where the data is the result of a real experiment, i.e. we replace $F(\pm 1, t)$ by $F(\pm 1, \frac{k}{N})$, $k = 1, \ldots, 2N$ for some large N, we also add some noise to the data to be more realistic.

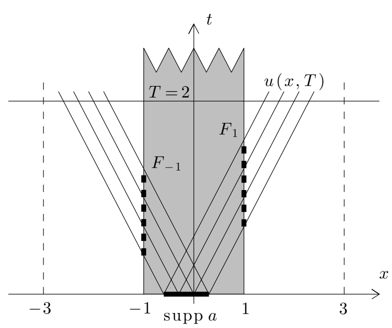

Figure 1. Space-time cylinder.

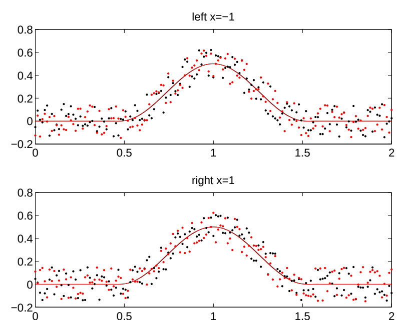

Figure 2. Observation.

For $a(x) = 1/2 + 1/2\cos(2\pi x)$ observation $F_{\pm 1}$ is represented on a picture. Before applying our algorithm, we checked the possibility of solving this problem by brute force, i.e. minimizing the least-square-error for a Fourier polynomial. As it is shown on the picture, the result was negative:

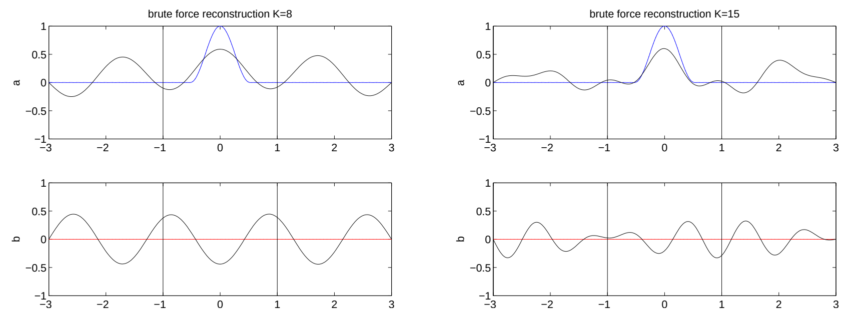

Figure 3. Minimizing least-square-error.

We applied our algorithm, described in section 3 with $T = 2$. As was mentioned, the reconstruction of Fourier coefficients a_k using (3.4) is possible only for $k \neq 3n$ where $n = 1, 2, \ldots$ because for $k = 3n$ the denominator in (3.4) vanishes. Making use of Lemma 1 and Proposition 1 we get the following answer:

$$a(x) = \frac{3}{2}A(x), \quad \text{where} \quad A(x) = \sum_{k \neq 3n} a_k \sin\Big(\frac{x+3}{6}k\pi\Big).$$

Reconstructed data in the 'error' case with a 50 Fourier coefficients are represented on figures

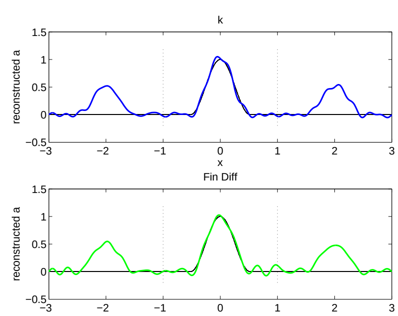

Figure 4. Smooth wave.

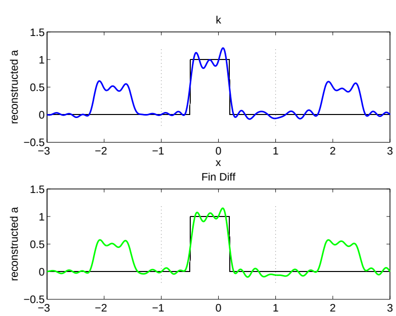

Figure 5. Step function.

The last two figures demonstrate comparison of our algorithm (first row) with the reconstruction by solving wave equation backward in time by finite-difference scheme (second row). On the right figure we reconstruct initial condition $a(x)$ having nonsmooth profile.

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