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ON THE BATEMAN–HÖRMANDER SOLUTION OF THE WAVE EQUATION, HAVING A SINGULARITY AT A RUNNING POINT

Abstract. Hörmander have presented a remarkable example of a solution of the homogeneous wave equation, which has a singularity at a running point. We are concerned with analytic investigation of this solution for the case of three spatial variables. We describe its support, study its behavior near the singular point and establish its local integrability. We observe that the Hörmander solution is a specialization of a solution found by Bateman five decades in advance.

§1. Introduction

We are concerned with solutions of the wave equation with three spatial variables

$$\Box u \equiv u_{xx} + u_{yy} + u_{zz} - \frac{1}{c^2} u_{tt} = 0, \qquad c = \mathrm{Const} > 0, \tag{1}$$

of which we speak as of functions of spatial variables $\mathbf{r} = (x, y, z) \in \mathbb{R}^3$, dependent on the parameter t. Hörmander's theory of wave front propagation [1] admits the existence of solutions of (1) which are smooth with respect to $\mathbf{r}$ at each instant of time $t \in \mathbb{R}$ everywhere except for a single spatial point which runs with the velocity c along a spatial straight line. An example of such a solution was presented in [1] for the case of $m > 1$ spatial variables. The approach was based on abstract PDE theory and analytic properties of the solution were not discussed. Here, we consider this solution in the case of $m = 3$ and establish that it is a specification of the classical relatively undistorted Bateman solution, see [2,3]. We investigate in detail its behavior near the singular point. Our approach is purely analytical and quite elementary.

We start by recalling the result due to Bateman.

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§2. The classical Bateman solution

Denote a quadratic form known in physics as the space–time interval (see, e.g., [4]) by

$$\ell^2 = \ell^2(x, y, z, t) = c^2t^2 - x^2 - y^2 - z^2. \tag{2}$$

On the basis of group theory, Bateman established that the transformation

$$u(x, y, z, t) \mapsto U = \frac{1}{\ell^2} u\left(\frac{x}{\ell^2}, \frac{y}{\ell^2}, \frac{z}{\ell^2}, \frac{t}{\ell^2}\right) \tag{3}$$

bearing now his name[1] maps a solution of the wave equation (1) into a solution of the same equation. Applying this transformation to the plane wave

$$u = \mathcal{F}(z - ct),$$

where $\mathcal{F}(\cdot)$ is an arbitrary function of one real variable, we get a solution $u = \frac{1}{\ell^2}\mathcal{F}(\frac{z-ct}{\ell^2})$. Further, putting $\mathfrak{f}(s) = s\mathcal{F}(1/s)$, we come up with a solution of the form

$$u = \frac{1}{z - ct}\mathfrak{f}\left(\frac{\ell^2}{z - ct}\right) \tag{4}$$

with arbitrary function $\mathfrak{f}(\cdot)$, see [2, 3]. The expression (4) is known as the Bateman relatively undistorted wave solution, the function $\mathfrak{f}(\cdot)$ is called the waveform.

Bateman was not concerned with imposing any condition on the function $\mathfrak{f}(\cdot)$ and was not interested whether the function (4) (which is singular at $z = ct$) satisfies the equation (1) in the whole space–time $\mathbb{R}^3 \times \mathbb{R}$.

Other approaches to derivation of the Bateman solution (4) and its generalizations are described, e.g., in [5–8]. Until recently, this solution found no direct application whereas its complexified versions play an important role in description of highly localized solutions of the wave equation (see, e.g., [6–9]).

§3. Hörmander's construction

Henceforth, we understand the equation (1) in the sense of generalized functions. Recall the Hörmander derivation. [1] Let $\mathscr{E}_+$ and $\mathscr{E}_-$ be the

[1] Some authors, however, call it the Kelvin transformation.

advanced and retarded fundamental solutions of the wave equation (1), satisfying

$$\Box\mathscr{E}_\pm = \delta(y)\delta(z)\delta(x)\delta(t), \qquad (5)$$
$$\mathscr{E}_+|_{t<0} = 0, \quad \mathscr{E}_-|_{t>0} = 0,$$

where $\delta(\cdot)$ is the one-dimensional delta function. Explicit expressions for $\mathscr{E}_+$ and $\mathscr{E}_-$ are well known (see, e.g., [10]):

$$\mathscr{E}_\pm = \frac{h(\pm t)}{2\pi c}\delta\left(\ell^2\right). \qquad (6)$$

Here, ℓ is given by (2) and $h(\cdot)$ is the Heaviside function, $h(t) = 0$ for $t \leqslant 0$, $h(t) = 1$ for $t > 0$. Obviously, the function

$$\mathscr{E} = \mathscr{E}_+ - \mathscr{E}_- = \frac{\operatorname{sgn}(t)}{2\pi c}\delta\left(\ell^2\right) \qquad (7)$$

satisfies the homogeneous wave equation in whole space–time $\mathbb{R}^3 \times \mathbb{R}$. Therefore, the convolution of $\mathscr{E}$ and an arbitrary generalized function F,

$$U = \mathscr{E} * F, \qquad (8)$$

also satisfies homogeneous equation

$$\Box U = 0. \qquad (9)$$

Taking in (8)

$$F = F(x, y, z, \tau) = 2\pi c\delta(x)\delta(y)\delta(z - ct)f(z), \qquad (10)$$

where $f(\,\cdot\,)$ is an arbitrary compactly supported infinitely differentiable function of one variable, yields

$$U(x, y, z, t) = \int_{\mathbb{R}^4} \mathscr{E}(x - \xi, y - \eta, z - \zeta, t - \tau)F(\xi, \eta, \zeta, \tau)d\xi d\eta d\zeta d\tau$$
$$= \int_{\mathbb{R}^2} \delta\left(c^2(\tau - t)^2 - x^2 - y^2 - (\zeta - z)^2\right) \operatorname{sgn}(t - \tau)\delta(\zeta - c\tau)f(\zeta)d\zeta d\tau. \qquad (11)$$

Hörmander [1] was satisfied with the observation that the function (11) belongs to the class $C^\infty(\mathbb{R}^3 \times \mathbb{R}\backslash\{x = y = 0, z = ct\})$ and is not infinitely differentiable for $x = y = 0, z = ct$. We aim at finding and exploring a simple expression for it.

§4. Evaluation of the integral in (11)

Evaluating the integral with respect to ζ, we get

$$U = \int_{-\infty}^{\infty} \delta\left(\ell^2 + 2c\tau(z-ct)\right) \operatorname{sgn}(t-\tau) f(c\tau) d\tau. \tag{12}$$

Observing that the argument of the delta function in (12) vanishes when $\tau = -\frac{\ell^2}{2c(z-ct)}$ and that $\delta(C\tau) = \frac{1}{|C|}\delta(\tau)$, where $C = \text{Const} \in \mathbb{R}$, we obtain

$$U = \frac{1}{2c|z-ct|} f\left(\frac{\ell^2}{2(z-ct)}\right) \operatorname{sgn}\left(t + \frac{\ell^2}{2c(z-ct)}\right). \tag{13}$$

Note that

$$\operatorname{sgn}\left(t + \frac{\ell^2}{2c(z-ct)}\right) = \operatorname{sgn}\left(\frac{-x^2 - y^2 - (z-ct)^2}{2c(z-ct)}\right) = -\operatorname{sgn}(z-ct),$$

whence

$$U = -\frac{1}{2c(z-ct)} f\left(\frac{\ell^2}{2(z-ct)}\right). \tag{14}$$

The expression (14) can be recognized as the Bateman solution (4) with a compactly supported smooth function

$$\mathfrak{f}(\sigma) = -\frac{1}{2c} f\left(\frac{\sigma}{2}\right). \tag{15}$$

For this reason we find it fair to call the function (14) the Bateman–Hörmander solution of the wave equation.

The above argument proves that in spite of the considerable singularity at $z = ct$, the Bateman solution (4) satisfies homogeneous wave equation (1) in the whole space-time $\mathbb{R}^3 \times \mathbb{R}$.[2]

The function (14) is obviously smooth for $z \neq ct$. On the plane $z = ct$, since the waveform $f(\cdot)$ is compactly supported, $U = 0$ unless $\ell^2 = 0$, which holds at the running point

$$\mathbf{s}(t) = \{\mathbf{r} : \ x = y = 0, \ z = ct\}. \tag{16}$$

The behavior of the function (14) near the point (16) will be discussed in Section 7. First we describe its support.

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[2] This fact was recently established in [11] by a different method.

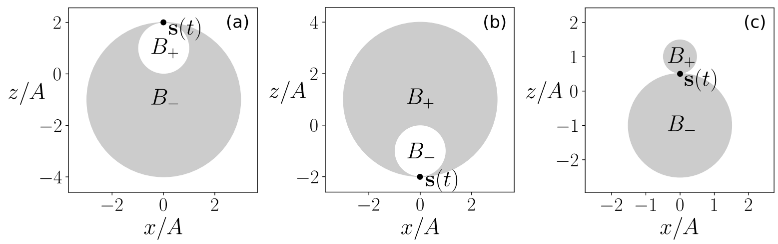

Figure 1. Support of U at different instants of time: (a) $t = -2A$, (b) $t = A/2$, (c) $t = 2A$.

§5. The support of U

Let the function of one variable $f(\cdot)$ have the support supp $f = [-A, A] \subset \mathbb{R}$, $A > 0$. Then supp $U \subset \mathbb{R}^3$ is described by inequalities

$$-A \leqslant \frac{\ell^2}{2(z - ct)} \leqslant A. \tag{17}$$

Let t be an arbitrarily fixed instant of time. Introduce two balls $B_\pm(t)$ by

$$B_\pm(t_0) = \{\mathbf{r} : \ x^2 + y^2 + (z \mp A)^2 \leqslant (ct_0 \mp A)^2\}.$$

and denote by $b_\pm(t) = \partial B_\pm(t)$ their boundaries, i.e., the spheres

$$b_\pm(t) = \{\mathbf{r} : x^2 + y^2 + (z \mp A)^2 = (ct \mp A)^2\}. \tag{18}$$

Consider the intersections of supp U with the half-spaces $z \geqslant ct$ and $z \leqslant ct$,

$$S_> = \text{supp } U \cap \{\mathbf{r} : \ z \geqslant ct\}, \quad S_< = \text{supp } U \cap \{\mathbf{r} : \ z \leqslant ct\}.$$

For $z \geqslant ct$, the inequality (17) is equivalent to

$$\begin{cases} x^2 + y^2 + (z - A)^2 \leqslant (ct - A)^2, \\ x^2 + y^2 + (z + A)^2 \geqslant (ct + A)^2, \end{cases} \tag{19}$$

whence $S_> = \overline{B_+(t) \setminus B_-(t)}$, where $\overline{}$ stands for closure. Analogously, for $z \leqslant ct$, the inequality (17) implies that $S_< = \overline{B_-(t) \setminus B_+(t)}$.

At any instant, t, supp U is a compact domain, restricted by spheres $b_+(t)$ and $b_-(t)$, which are tangent to each other and to the plane $z = ct$ at the point $\mathbf{s}(t)$ described by (16). Depending on whether $ct < -A$, $ct > A$ or $-A < ct < A$, these spheres may lie one in the other (see Figs. 1(a), 1(c)) or have a single common point $\mathbf{s}(t)$ (see Fig. 1(b)).

§6. Local integrability

It is easy matter to establish that the function U (14) is integrable on $\mathbb{R}^3$. One needs to show that

$$I := \int_{\mathbb{R}^3} |U(x,y,z,t)|\, dxdydz < \mathrm{Const} = \mathrm{Const}(t). \tag{20}$$

Consider the case of $ct < -A$, where the domain of integration in (20) reduces to the ball $B_+(t)$. After the shift $z \mapsto z + A$, the left-hand side of the equation (20) becomes

$$I = \int_{x^2+y^2+z^2 \leqslant R^2} |U(x,y,z+A,t)|\, dxdydz,$$

where $R = |ct - A|$. Let

$$M(t) = \max_{\mathbf{r} \in B_+(t)} f\left(\frac{\ell^2}{2(z-ct)}\right).$$

Then

$$\begin{aligned} I &\leqslant \frac{M(t)}{2c} \int_{x^2+y^2+z^2 \leqslant R^2} \frac{1}{|ct-A|-z} dxdydz \\ &= \frac{M(t)}{2c} \int_{x^2+y^2+z^2 \leqslant R^2} \frac{1}{R-z} dxdydz \\ &= \frac{M(t)}{2c} \int_{-R}^{R} \frac{1}{R-z} \left(\int_{x^2+y^2 \leqslant R^2 - z^2} dxdy \right) dz \\ &= \frac{\pi M(t)}{2c} \int_{-R}^{R} \frac{R^2-z^2}{R-z} dz = \frac{\pi R^2}{c} M(t), \end{aligned}$$

which completes the proof for the case under consideration.

The proof of the integrability for $ct < -A$ and $|ct| < A$ is similar to above, and we omit it.

§7. Analytic behavior of the solution U in a neighborhood of the singular point

The function (14) is obviously infinitely differentiable with respect to x, y, and z at each value of t, except for the running spatial point (16). Consider the behavior of the function (14) in its vicinity. We will demonstrate that the solution (14) has no limit as we approach this point.

Fix a time instant t. Consider the plane

$$y = 0. \tag{21}$$

First, discuss the case of

$$|ct| \leqslant A, \tag{22}$$

see Fig. 1(b). Let us approach $\mathbf{s}(t)$ along a straight line

$$z = ct + kx, \quad k \neq 0. \tag{23}$$

The argument of $f(\,\cdot\,)$ takes the form

$$\theta = \frac{\ell^2}{2(z-ct)} = \frac{-kx(kx+2ct) - x^2}{2kx} = -ct + O(x), \quad x \to 0. \tag{24}$$

Because of (22), in the case of a general position

$$f(-ct) \neq 0. \tag{25}$$

Under this assumption

$$U = -\frac{f(-ct)}{2c(z-ct)} = \frac{f(-ct)}{2ckx} + O(1), \quad x \to 0. \tag{26}$$

We see that when approaching the singular point along the line (23), U tends to $\pm\infty$, depending on the sign of kx. Therefore, for those t where (25) holds $\lim_{\mathbf{r}\to\mathbf{s}(t)} U$ along lines (23) does not exist. For the case of $|ct| > A$, as immediately seen from Figs. 1(a), 1(c), the limit along lines (23) equals zero.

For any value of t, let us approach the point $\mathbf{s}(t)$ along a parabola

$$\mathcal{P}(p) = \{\mathbf{r}:\ z = px^2 + ct\}, \quad p \neq 0. \tag{27}$$

The argument of $f(\,\cdot\,)$ becomes

$$\theta = \frac{\ell^2}{2(z-ct)} = \frac{1}{2}\left(z + ct + \frac{x^2}{z-ct}\right) = ct + \frac{1}{2p} + O(x^2), \quad x \to 0. \tag{28}$$

Having in mind (17), we note that for p satisfying

$$-(A + ct) < \frac{1}{2p} < A - ct, \tag{29}$$

in the vicinity of $\mathbf{s}(t)$, the curve (27) belongs to $\operatorname{supp} U$ and

$$U = -\frac{1}{2cp\,x^2}\, f\left(ct + \frac{1}{2p} + O(x^2)\right) \to \infty, \quad x \to 0. \tag{30}$$

Due to the fact that $f(\,\cdot\,)$ is compactly supported, we conclude that for p not satisfying (29), the limit along the curve (27) exists and equals zero.

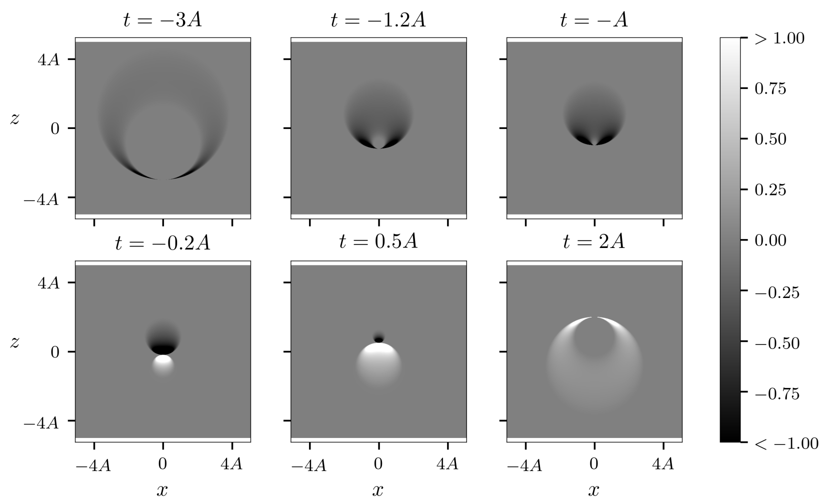

Figure 2. Snapshots of the function (14) at the plane $y = 0$.

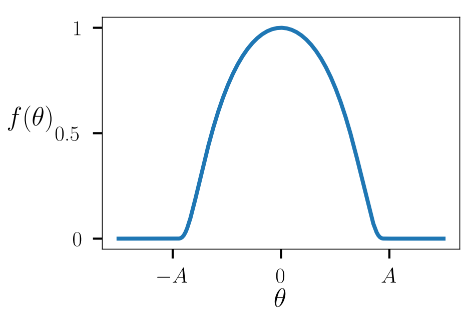

Figure 3. The function (31).

We thus established that the function (14) is not continuous at the point $\mathbf{s}(t)$ described by (16).

Fig. 2 illustrates the behavior of the solution (14) for a nonnegative waveform

$$f(\theta) = \begin{cases} \exp\left(1 + \frac{A^2}{\theta^2 - A^2}\right), & |\theta| < A, \\ 0, & \text{otherwise}, \end{cases} \tag{31}$$

pictured in Fig. 3.

Behavior of the solution (4) near the point $\mathbf{s}(t)$ has much in common with that of another simple solution with singularity at a running point, described by a specialized complexified Bateman solution presented in [12].

§8. Conclusions

The above results can be extended to the case of arbitrary number of spatial variables, where the Bateman solution is described in [9]. The condition that the function $f(\cdot)$ is compactly supported can be replaced by a requirement that $f(\theta) \to 0$ quickly enough as $\theta \to \pm\infty$.

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