Записки научных
семинаров ПОМИ
Том 470, 2018 г.

G. A. Noskov

PLOTKIN'S GEOMETRIC EQUIVALENCE, MAL'CEV'S CLOSURE AND INCOMPRESSIBLE NILPOTENT GROUPS

Abstract. In 1997 B. I. Plotkin introduced the notion of geometric equivalence of algebraic structures and posed the question: Is it true that every nilpotent torsion-free group is geometrically equivalent to its Mal'cev's closure? A negative answer was given by V. V. Bludov and B. V. Gusev in 2007 in the form of three counterexamples. In this paper we present an infinite series of counterexamples of unbounded Hirsch rank and nilpotency degree.

§1. Introduction

A great deal of "universal algebraic geometry" have been developed by B. I. Plotkin and V. N. Remeslennikov and their collaborators in the last two decades, see [21–24, 26, 27] and [4, 20]. In particular, in [27] Plotkin introduced an important concept of geometrically equivalent algebraic structures. It turns out that two groups are geometrically equivalent if and only if every finitely generated subgroup of each of these groups can be embedded into a suitable Cartesian power of the other group [23].

In this paper we study the geometric equivalence of nilpotent groups. Following P. Hall, we call a finitely generated torsion-free nilpotent group an f-group. An f-group of nilpotence degree c is called an f_c-group. The following is known due to [29]:

1. (1) Two nilpotent groups are geometrically equivalent if and only if they satisfy the same quasi-identities.
2. (2) If Γ is an f_2-group then it is geometrically equivalent to its Mal'cev's closure $\sqrt{\Gamma}$.

Key words and phrases: geometric equivalence, Mal'cev's closure, incompressible nilpotent groups.

The work was supported by the program of fundamental scientific research of the SB RAS I.1.1., project 0314-2016-0004. The author is thankful to referee for constructive comments and recommendations.

(3) Let Γ_1, Γ_2 be two f_2-groups with cyclic center. Then Γ_1, Γ_2 are geometrically equivalent if and only if their Mal'cev's closures are isomorphic.
(4) Let Γ_1, Γ_2 be two f_2-groups whose centers have rank 2. Then $\Gamma_1 \sim \Gamma_2$ if and only if either $\sqrt{\Gamma_1} \simeq \sqrt{\Gamma_2}$ or else there is an f_2-group N with the cyclic center and such that $\Gamma_1 \sim N \sim \Gamma_2$.
(5) A nilpotent torsion free relatively free group is geometrically equivalent to its Mal'cev's closure.

For the case of two Abelian groups it is shown that they are geometrically equivalent if and only if for every prime number p the exponents of their p-Sylow subgroups coincide and if one of these groups is not periodic then the other group is not periodic either [2]. Every two non-abelian 2-step nilpotent graph $\mathbb{Q}$-groups are geometrically equivalent [19]. A torsion-free nilpotent group Γ is geometrically equivalent to its Mal'cev's closure $\sqrt{\Gamma}$ if and only if $\Gamma \sim n\Gamma$ for every natural n, where $n\Gamma$ denotes the subgroup of Γ generated by all n-th powers of all of elements of Γ [3].
In [27] Plotkin and his coauthors formulate the following problem.

Problem 1.1. *Is it true that every torsion-free nilpotent group is geometrically equivalent to its Mal'cev's closure? If not, consider the conditions when geometrical equivalence takes place.*

Examples of finitely generated nilpotent groups of degree three, four and five, which are not geometrically equivalent to their Mal'cev's closures, were constructed by V. V. Bludov and B. V. Gusev in [3].

We can now formulate our main result.

Theorem 1.2. *In each odd dimension $n \geqslant 7$ there is an f-group Γ of Hirsch rank n and of nilpotence degree $n-1$, which is not geometrically equivalent to its Mal'cev's closure $\sqrt{\Gamma}$.*

§2. Radicals, closure and geometric equivalence

For an arbitrary collection of groups S_α, $\alpha \in I$, we denote by $S = \Pi_{\alpha\in I} S_\alpha$ the set of functions $f : I \to \cup_{\alpha\in I} S_\alpha$ satisfying the condition that $f(\alpha) \in S_\alpha$ for all $\alpha \in I$. It is readily checked that the set S with multiplication defined by the rule $(fg)(\alpha) = f(\alpha)g(\alpha)$, is a group; it is also called the Cartesian product of the groups S_α. The value of a function f at the element α is called the projection of f on the factor S_α, or the component of f in S_α. In case all S_α coincide, say $S_\alpha = S$, we call $\Pi_{\alpha\in I} S_\alpha$

the Cartesian power and denote it by S^I. Thus S^I is the set of all functions $f : I \to S$.

A group G is approximable by a group S if there is an embedding $\iota : G \rightarrowtail S^I$ of G into some Cartesian power of S. Equivalently, there is a family of homomorphisms $s_i : G \to S$ $(i \in I)$ such that $\cap_I \ker(s_i) = \{1\}$. In fact it is the "universal" Cartesian power

$$S^{\mathrm{Hom}(G,S)} \tag{2.1}$$

that is responsible for the approximability.

Lemma 2.1. *Let $\iota : G \to S^{\mathrm{Hom}(G,S)}$ be the canonical homomorphism, given by*

$$(\iota g)(\sigma) = \sigma(g) \tag{2.2}$$

for all $\sigma \in \mathrm{Hom}(G,S)$. Then G is approximable by S if and only if ι is injective.

Proof. If $\iota : G \to S^{\mathrm{Hom}(G,S)}$ is injective then G is clearly approximable by S. Conversely, suppose that there is an embedding $G \rightarrowtail S^I$. Then $\cap_I \ker(s_i) = \{1\}$, and thus $\cap \{\ker(\sigma) : \sigma \in \mathrm{Hom}(G,S)\} = \{1\}$, which means that ι is injective. □

If the canonical homomorphism $\iota : G \to S^{\mathrm{Hom}(G,S)}$ is not injective then the non-approximability of G by S can be measured by the kernel $\ker \iota$. The following definition is due to Plotkin (see [10]).

Definition 2.2. *The kernel of the canonical homomorphism $\iota : G \to S^{\mathrm{Hom}(G,S)}$ is called the S-radical of the group G and is denoted by $R_S G$,*

$$R_S G = \ker \iota = \cap \{\ker \sigma : \sigma \in \mathrm{Hom}(G,S)\}. \tag{2.3}$$

Lemma 2.3 (minimality). *The radical $R_S G = \ker \iota$ is the "minimal" possible kernel among the kernels of the homomorphisms of the form $\kappa : G \to S^I$. Precisely, for every homomorphism $\kappa : G \to S^I$ the kernel $\ker \kappa$ contains the kernel $\ker \iota$.*

Proof. We may assume that the set I is a singleton, so κ is a homomorphism $\kappa : G \to S$. Then clearly $\ker \kappa \supseteq \cap \{\ker \sigma : \sigma \in \mathrm{Hom}(G,S)\} = \ker \iota$. □

Another important definition by Plotkin is that of "S-closure operator" Cl_S, which acts on the set of normal subgroups of a fixed group G. The operator Cl_S assigns to each normal subgroup N of an arbitrary group G a certain normal subgroup $Cl_S(N,G) \leqslant G$, called the S-closure of N

in G. Precisely, $R_S(G/N)$ is a (normal) subgroup of G/N, hence it is of the form $R_S(G/N) = G_0/N$ for uniquely defined subgroup G_0 and we set

$$Cl_S(N,G) \stackrel{\text{def}}{=} G_0.$$

Thus, we have the formula

$$R_S(G/N) = Cl_S(N,G)/N, \tag{2.4}$$

which expresses the closure via the radical. Conversely, putting $N = \{1\}$ in this formula we obtain

$$R_S(G) = Cl_S(1,G). \tag{2.5}$$

The direct formula for the closure is:

$$Cl_S(N,G) = \cap\{\ker\sigma : \sigma \in \operatorname{Hom}(G,S), \ker\sigma \supseteq N\}. \tag{2.6}$$

A subgroup $N \leqslant G$ is S-closed in G if $Cl_S(N) = N$. For instance, $\ker\sigma$ is S-closed for every $\sigma \in \operatorname{Hom}(G,S)$.

Remark 2.4. In general the S-closed sets do not form a topology on G. For instance, take as S the additive group of the field Z/pZ and $G = (\mathbb{Z}/p\mathbb{Z})^n, n \geqslant 2$. Then the S-closed sets are precisely the linear subspaces in G and the union of two linear subspaces is not linear in general.

The closure operator immediately leads to Plotkin's definition of geometrically equivalent groups, see [27]. The groups S and T are geometrically equivalent, written $S \sim T$, if for every free group F of finite rank the closure operators Cl_S, Cl_T coincide on F.

Lemma 2.5. *The groups S and T are geometrically equivalent if and only if $R_S K = R_T K$ for every finitely generated group K.*

Proof. Suppose S and T are geometrically equivalent and let $K = F/N$ be a finitely generated group with F a free group of finite rank. Then by definition $R_S(F/N) = Cl_S(N,F)/N$ and $R_T(F/N) = Cl_T(N,F)/N$, so $R_S K = R_T K$. Conversely, suppose $R_S K = R_T K$ for every finitely generated group K. Let F be a free group of finite rank and N be a normal subgroup in F. Then by definition

$$R_S(F/N) = Cl_S(N,F)/N, \quad R_T(F/N) = Cl_T(N,F)/N, \tag{2.7}$$

so

$$Cl_S(N,F)/N = R_S(F/N) = R_T(F/N) = Cl_T(N,F)/N, \tag{2.8}$$

that is the closure operators Cl_S, Cl_T coincide on F. □

Lemma 2.6 ([23]). *The groups S, T are geometrically equivalent if and only if every finitely generated subgroup of S can be approximated by T and vice versa.*

Proof. Suppose $S \sim T$ and let S_0 be a finitely generated subgroup of S. Fix a free group F of finite rank and an epimorphism $\phi : F \twoheadrightarrow S_0$. By definition $\ker\phi$ is S-closed, hence $\ker\phi$ is also T-closed. Consequently,

$$\ker\phi = \cap\{\ker\phi_i |\ \phi_i : F \to T,\ i \in I\} \tag{2.9}$$

for some index set I. Hence the family of homomorphisms (ϕ_i) forms a homomorphism $F \overset{(\phi_i)}{\to} T^I$ whose kernel equals $\ker\phi$ in view of (2.9). Therefore, (ϕ_i) induces an embedding $S_0 = F/\ker\phi \rightarrowtail T^I$, thus S_0 is approximable by T. Similarly, one shows that every finitely generated subgroup of T can be approximated by S.

Conversely, suppose that every finitely generated subgroup of S can be approximated by T and vice versa. By symmetry of S, T it is enough to show that $R_S G \geqslant R_T G$ for every finitely generated group G. By definition

$$R_S G = \cap\{\ker\phi : \phi \in \mathrm{Hom}\,(G, S)\} \tag{2.10}$$

hence the inclusion $R_S G \geqslant R_T G$ would follow from the inclusion $\ker\phi \geqslant R_T G$ for every $\phi \in \mathrm{Hom}\,(G, S)$. For every such ϕ by assumption there is an embedding $\phi\,(G) \rightarrowtail T^I$. The composition $\iota' : G \to \phi\,(G) \rightarrowtail T^I$ has a kernel $\ker\iota' = \ker\phi$ so by minimality Lemma 2.3 $\ker\phi \geqslant R_T G$ as desired. □

§3. Mal'cev's closure and geometric equivalence

Let Γ be a torsion-free nilpotent group. Recall that a group Γ is divisible if for every element g and every positive integer m the equation $x^m = g$ can be solved in Γ. Extraction of n-th roots in Γ is a "partial operation" on the group; i.e. it is single-valued but not everywhere defined. We call a divisible, torsion-free nilpotent group a (nilpotent) Mal'cev's (=divisible) closure of Γ if it contains Γ but has no proper divisible subgroups containing Γ. Divisible closure of Γ exists, has the same nilpotency class, and further any two nilpotent divisible closures of Γ (i.e. "minimal" divisible overgroups of Γ) are isomorphic and consist entirely of roots of elements of Γ. Moreover, given any automorphism ϕ of Γ there is an isomorphism between the two divisible closures which extends ϕ (for all above statements see [13, Theorem 16.2.8]).

Define

$$\zeta_0\Gamma = 1, \quad \zeta_{i+1}\Gamma/\zeta_i\Gamma = Z\left(\Gamma/\zeta_i\Gamma\right), \quad i = 0, 1, 2, \ldots; \tag{3.11}$$

The groups $\zeta_i\Gamma$ are called the higher centers of Γ. The ascending series

$$1 \leqslant \zeta_1\Gamma \leqslant \zeta_2\Gamma \leqslant \cdots \tag{3.12}$$

is the upper (or ascending) central series of Γ.

Theorem 3.1 ([13], Theorem 17.3.1)**.** *Let Δ be a subgroup of a divisible, torsion-free nilpotent group Γ. The set $\sqrt{\Delta}$ of all elements of Γ some powers of which lie in Δ (the "radical closure" of Δ in Γ), is a subgroup of Γ, and therefore a divisible closure of Δ. The higher centers of the subgroups Δ and $\sqrt{\Delta}$ are related in the following way:*

$$\zeta_i\sqrt{\Delta} = \sqrt{\zeta_i\Delta}, \quad \zeta_i\Delta = \Delta \cap \sqrt{\zeta_i\Delta}. \tag{3.13}$$

The following theorem by Bludov and Gusev is the key to proving the geometric non-equivalence [3].

Theorem 3.2. *A torsion-free nilpotent group Γ is geometrically equivalent to its Mal'cev's closure $\sqrt{\Gamma}$ if and only if $\Gamma \sim n\Gamma$ for every natural n.*

A family $\mathfrak{a} = \{\mathfrak{a}_{ij} : 1 \leqslant i < j \leqslant n, \mathfrak{a}_{ij} \subseteq \mathbb{Q}\}$ of additive subgroups of $\mathbb{Q}$ is called a carpet if $\mathfrak{a}_{ij}\mathfrak{a}_{jk} \subseteq \mathfrak{a}_{ik}$ for all i, j, k. It is easily verified that the set

$$\Delta_n\left(\mathfrak{a}\right) = \{x |\, x \in \mathrm{Mat}_n\left(\mathbb{Q}\right), x_{ij} \in \mathfrak{a}_{ij} \text{ for } i < j \text{ and } x_{ij} = 0 \text{ for } i \geqslant j\} \tag{3.14}$$

is a nilpotent matrix ring. The set

$$\Gamma_n\left(\mathfrak{a}\right) = e + \Delta_n\left(\mathfrak{a}\right) \tag{3.15}$$

is a nilpotent matrix group; we shall call it the (unitriangular) congruence subgroup modulo the carpet $\mathfrak{a}$. If every $\mathfrak{a}_{ij}$ is either zero or $\mathbb{Q}$ then $\Gamma_n\left(\mathfrak{a}\right)$ is divisible; indeed if $g = e + x$, then $g^{\frac{1}{m}} = \sum\limits_{i=0}^{n-1} \binom{1/m}{i} x^i$ is an mth root of g. The closure of a group $\mathfrak{a}_{ij}$ is defined as $\sqrt{\mathfrak{a}_{ij}} = \mathbb{Q}$ if $\mathfrak{a}_{ij} \neq \{0\}$ and $\sqrt{\mathfrak{a}_{ij}} = \{0\}$ otherwise. Note that if $\mathfrak{a}_{ij} \neq \{0\}$ then the quotient group $\mathbb{Q}/\mathfrak{a}_{ij}$ is periodic. Define the closure carpet of $\mathfrak{a}$ as $\sqrt{\mathfrak{a}} = \left\{\sqrt{\mathfrak{a}_{ij}} : 1 \leqslant i < j \leqslant n,\ \mathfrak{a}_{ij} \subseteq \mathbb{Q}\right\}$.

Theorem 3.3. *For every carpet $\mathfrak{a}$ the group $\Gamma_n\left(\sqrt{\mathfrak{a}}\right)$ is the Mal'cev's closure of $\Gamma_n\left(\mathfrak{a}\right)$.*

Proof. Indeed $\Delta_n\left(\sqrt{\mathfrak{a}}\right)$ is a $\mathbb{Q}$-algebra thus $\Gamma_n\left(\sqrt{\mathfrak{a}}\right)$ is closed under root extraction, so is divisible. It remains to show that for each $x \in \Delta_n\left(\sqrt{\mathfrak{a}}\right)$ some power of $e+x$ lies in $\Gamma_n(\mathfrak{a})$. There is a natural m such that $x=\frac{1}{m}y$, where $y \in \Delta_n(\mathfrak{a})$. We have

$$\left(e+\frac{1}{m}y\right)^N = e+\frac{N}{m}y+\frac{N(N-1)}{2m^2}y^2+\ldots+\frac{N(N-1)\cdots(N-n+1)}{m^{n-1}(n-1)!}y^{n-1}. \tag{3.16}$$

Now, take $N=m^{n-1}(n-1)!$, then $\left(e+\frac{1}{m}y\right)^N \in e+\Delta_n(\mathfrak{a})=\Gamma_n(\mathfrak{a})$. □

Theorem 3.4. *For every carpet* $\mathfrak{a}$ *the group* $\Gamma_n(\mathfrak{a})$ *is geometrically equivalent to* $\Gamma_n\left(\sqrt{\mathfrak{a}}\right)$.

Proof. It is enough to show that every finitely generated subgroup Γ of $\Gamma_n\left(\sqrt{\mathfrak{a}}\right)$ embeds into $\Gamma_n(\mathfrak{a})$. For a matrix $d_r=\operatorname{diag}\left\{1,r,r^2,\ldots,r^{n-1}\right\}$ we have

$$d_r\Gamma_n(\mathfrak{a})\,d_r^{-1}=\left\{x \mid\ x\in UT_n(\mathbb{Q}),\, x_{ij}\in\frac{r^i}{r^j}\mathfrak{a}_{ij},\, i<j\right\}. \tag{3.17}$$

It follows that for sufficiently large r the group Γ is contained entirely in $d_r\Gamma_n(\mathfrak{a})\,d_r^{-1}$. Hence the homomorphism $g\mapsto d_r^{-1}gd_r$ embeds Γ into $\Gamma_n(\mathfrak{a})$. □

Conclusion. Geometric equivalence follows from the fact that Γ is "compressible" from $\Gamma_n\left(\sqrt{\mathfrak{a}}\right)$. This means that negative examples must be searched among "incompressible" groups.

§4. Mal'cev's correspondence and incompressibility

An f-group is called incompressible (=co-Hopfian) if it contains no proper subgroup isomorphic to itself or, equivalently, if every its injective endomorphism is in fact an automorphism. Yet another formulation is that every endomorphism either is automorphic or else has a nontrivial kernel.

A. I. Mal'cev established the correspondence between f-groups and finite-dimensional nilpotent Lie algebras over $\mathbb{Q}$. It allows to translate certain group-theoretic questions into the Lie algebra language, where they are often easier to handle. As an example I. O. Belegradek managed to give a criterion for incompressibility in terms of Lie algebras [1].

Recall that the Campbell–Hausdorff formula asserts that, e.g. if one considers the algebra of formal power series in two non-commuting indeterminants x, y, then

$$e^x e^y = e^{F(x,y)} \tag{4.18}$$

where

$$F(X,Y) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \frac{1}{12}[Y,[Y,X]] + \dots \tag{4.19}$$

is a (universal) formal Lie series in X, Y. Write $F(X,Y) = X+Y+\tau(X,Y)$. By the identity (4.19) one sees that τ has no terms in X alone or in Y alone.

Given a nilpotent Lie algebra $\mathfrak{g}$ over $\mathbb{R}$, we define the Lie group structure on $\mathfrak{g}$ by the Campbell–Hausdorff formula. For $x, y \in \mathfrak{g}$, define $\tau(x,y)$ by substitution in the formal Lie series $\tau(X,Y)$. Since $\mathfrak{g}$ is nilpotent, this defines a polynomial map $\tau : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ with $\tau(0,y) = \tau(x,0) = 0$. Set

$$x \cdot y = x + y + \tau(x,y) \tag{4.20}$$

By the formal identity (4.18) it follows that this composition gives a group structure to $\mathfrak{g}$. The straight line $t \mapsto tx, t \in \mathbb{R}$, is the one-parameter group generated by $x \in \mathfrak{g}$. Denote by $G_{\mathfrak{g}} = (\mathfrak{g}, \cdot)$ the space $\mathfrak{g}$ with this Lie group structure. Note that $G_{\mathfrak{g}}$ is connected and simply connected nilpotent Lie group. The exponential mapping $\exp : \mathfrak{g} \to G_{\mathfrak{g}}$ is identical and so the inverse log is also identical.

Suppose $\mathfrak{g}$ has rational structure constants, i.e. $\mathfrak{g} \simeq \mathbb{R} \otimes \mathfrak{g}_{\mathbb{Q}}$ for some Lie algebra $\mathfrak{g}_{\mathbb{Q}}$ over $\mathbb{Q}$. Then the group $G_{\mathfrak{g}_{\mathbb{Q}}} = (\mathfrak{g}_{\mathbb{Q}}, \cdot)$ is a divisible subgroup of $G_{\mathfrak{g}}$. According to the classical papers by Mal'cev [17, 18] the group $G_{\mathfrak{g}}$ has cocompact discrete f-subgroup $\Gamma_{\mathfrak{g}_{\mathbb{Q}}}$containing in $G_{\mathfrak{g}_{\mathbb{Q}}}$. More precisely, $\Gamma = \Gamma_{\mathfrak{g}_{\mathbb{Q}}}$ is of the form

$$\Gamma_{\mathfrak{g}_{\mathbb{Q}}} = \sum \frac{1}{n_i} \mathbb{Z} e_i \ , \tag{4.21}$$

for a suitable basis (e_i) of $\mathfrak{g}_{\mathbb{Q}}$ and suitable tuple (n_i) of natural numbers. It follows immediately from this description that the $\mathbb{Q}$-span of Γ is $\mathfrak{g}_{\mathbb{Q}}$ and that $G_{\mathfrak{g}_{\mathbb{Q}}}$ is Mal'cev's closure of Γ, i.e. $\sqrt{\Gamma} = G_{\mathfrak{g}_{\mathbb{Q}}}$. The group Γ is uniquely determined by $\mathfrak{g}_{\mathbb{Q}}$ up to commensurability.

This correspondence between f-groups and nilpotent Lie algebras over $\mathbb{Q}$ allowed I. O. Belegradek to obtain the following criterion of incompressibility of an f-group [1] (definitions are found below).

Theorem 4.1 ([1] (see [11] for an alternative proof)). *If a Lie algebra $\mathfrak{g}_{\mathbb{Q}}$ over $\mathbb{Q}$ is derivationally nilpotent, then the group $\Gamma_{\mathfrak{g}_{\mathbb{Q}}}$ is incompressible.*

A Lie algebra $\mathfrak{g}$ is called derivationally nilpotent (= characteristically-nilpotent) if the algebra of its derivations Der $\mathfrak{g}$ consists of nilpotent operators [8]. If this is so, then the inner derivations are nilpotent as well and by Engel's theorem we conclude that $\mathfrak{g}$ is nilpotent. If $\mathfrak{g}$ is a finite-dimensional Lie algebra over the field $K = \mathbb{R}$ or $\mathbb{C}$, then it is derivationally nilpotent if and only if the Lie group $(\mathrm{Aut}\ \mathfrak{g})^{\circ}$ is unipotent [8].

As observed in [16], the property of being characteristically nilpotent does not depend on the ground field. More precisely: if the Lie algebra $\mathfrak{g}$ is derivationally nilpotent as F-algebra (here it is not necessary to suppose that it has characteristic zero) and K/F is a field extension, then $\mathfrak{g}$ is also derivationally nilpotent as K-algebra.

§5. Strong incompressibility and filiform Lie algebras

It is plausible that incompressibility does not suffice for an f-group Γ to provide the non-equivalence $\Gamma \not\sim \sqrt{\Gamma}$. An f-group $\Gamma \neq 1$ is called strongly incompressible if every its endomorphism ϕ is either automorphic or else $\ker\phi$ contains the center $Z\Gamma$. A strongly incompressible group is incompressible since then center of nontrivial nilpotent group is nontrivial. This notion has already occurred implicitly in [3].

Theorem 5.1. *If Γ is a strongly incompressible f-group then Γ is not geometrically equivalent to every of its proper subgroup. Furthermore, Γ is not geometrically equivalent to Mal'cev's closure $\sqrt{\Gamma}$.*

Proof. If there were a geometric equivalence $\Gamma \sim \Delta$ for a proper subgroup $\Delta < \Gamma$ then we could get the equality

$$R_\Gamma \Gamma = R_\Delta \Gamma. \tag{5.22}$$

By definition

$$R_\Gamma \Gamma = \cap \{\ker\phi : \phi \in \mathrm{Hom}\,(\Gamma, \Gamma)\} = 1 \tag{5.23}$$

since $\ker id_\Gamma = 1$. On the other hand

$$R_\Delta \Gamma = \cap \{\ker\phi : \phi \in \mathrm{Hom}\,(\Gamma, \Delta)\}. \tag{5.24}$$

Every homomorphism $\phi \in \mathrm{Hom}\,(\Gamma, \Delta)$ is not injective by incompressibility of Γ, so it has a non-trivial kernel $\ker\phi$ which, by assumption, contains $Z\Gamma$. But this implies that $R_\Delta \Gamma \geqslant Z\Gamma \neq 1$ that would contradict (5.22).

As for the second assertion is concerned by Bludov and Gusev Theorem 3.2 the equivalence $\Gamma \sim \sqrt{\Gamma}$ would imply $\Gamma \sim n\Gamma$ for all n. But $n\Gamma$ is a proper subgroup for all $n > 1$, contradicting the first assertion. $\square$

Lemma 5.2. *If* Γ *is an incompressible* f*-group and* $Z\Gamma \simeq \mathbb{Z}$ *then* Γ *is strongly incompressible.*

Proof. Let $\phi : \Gamma \to \Gamma$ be a homomorphism with nontrivial kernel $\ker\phi$. Since $\ker\phi$ is a normal subgroup of Γ, the intersection $\ker\phi \cap Z\Gamma$ is nontrivial [13]. Hence $\ker\phi \cap Z\Gamma = n\,(Z\Gamma)$ for some natural number n. The embedding $\bar{\phi} : \Gamma/\ker\phi \rightarrowtail \Gamma$ takes the quotient

$$Z\Gamma \ker\phi/\ker\phi \simeq Z\Gamma/\ker\phi \cap Z\Gamma \simeq \mathbb{Z}/n\mathbb{Z} \tag{5.25}$$

into finite subgroup of Γ. Since Γ is torsion-free, $n = 1$ and $\ker\phi \geqslant Z\Gamma$ as required. □

The last Lemma shows that it is important to work in the class of nilpotent algebras with 1-dimensional center. This class is not well behaved under natural operations. For this reason we find out that the subclass of filiform Lie algebras is more appropriate. Let $\mathfrak{g}$ be a Lie algebra over a field and $Z\,(\mathfrak{g})$ its center. Define

$$\zeta_0\mathfrak{g} = 0, \quad \zeta_{i+1}\mathfrak{g}/\zeta_i\mathfrak{g} = Z\,(\mathfrak{g}/\zeta_i\mathfrak{g})\,, \quad i = 0, 1, 2, \ldots; \tag{5.26}$$

The subalgebras $\zeta_i\mathfrak{g}$ are called the higher centers of the algebra $\mathfrak{g}$. The ascending series

$$0 \leqslant \zeta_1\mathfrak{g} \leqslant \zeta_2\mathfrak{g} \leqslant \cdots \tag{5.27}$$

is the upper (or ascending) central series of $\mathfrak{g}$. A Lie algebra $\mathfrak{g}$ is nilpotent if and only if $\zeta_k\mathfrak{g} = \mathfrak{g}$ for some $k \geqslant 0$. The smallest number k such that $\zeta_k\mathfrak{g} = \mathfrak{g}$ is said to be the nilpotency class [1] or the nilpotence degree of the Lie algebra of $\mathfrak{g}$. Similar notions and definitions may be introduced for (Lie) groups.

It is easy to see that the nilpotence degree of $\mathfrak{g}$ is smaller than or equal to $\dim\mathfrak{g} - 1$. A nilpotent Lie algebra $\mathfrak{g}$ of dimension n and nilpotence degree $n-1$ is called filiform or of maximal degree. Equivalently, a nilpotent Lie Algebra $\mathfrak{g}$ is said to be filiform if $\dim\zeta_i\mathfrak{g} = i$ for $0 \leqslant i \leqslant n-2$ ($n = \dim\mathfrak{g}$) and $\zeta_{n-1}\mathfrak{g} = \mathfrak{g}$.

The sequence of integers $p_i = \dim\,(\zeta_i\mathfrak{g}/\zeta_{i-1}\mathfrak{g})$ is called the type of a nilpotent Lie algebra $\mathfrak{g}$. The filiform Lie algebras are just the algebras of type {1, 1, . . . , 1, 2}. (This explains the name ”filiform” which means

[1]"...The use of the word "class" for this number is unfortunate but firmly established; some adroitness is needed in order to avoid stylistic masterpieces like "class of nilpotent groups of a given class."" [13].

threadlike.) In particular, the center of the filiform Lie algebra of dimension $\geqslant 3$ is 1-dimensional.

5.1. Constructing $\Gamma_{\mathfrak{g}_\mathbb{Q}}$ with infinite cyclic center.

Proposition 5.3 ([8]). *Let G be a Lie group and let $\mathfrak{g}$ be its tangent Lie algebra. The tangent algebras to the groups $\zeta_k G$ are $\zeta_k \mathfrak{g}$, the terms of the uppercentral series.*

Proposition 5.4 ([28, Chap. II, Prop. 2.17]). *Let Γ be a cocompact discrete subgroup in a simply connected nilpotent Lie group G. Then $\zeta_i \Gamma$ is a lattice in $\zeta_i G$ for all i. In particular, the intersection of $Z\Gamma$ with the centre of G is a cocompact discrete subgroup in the centre.*

Theorem 5.5. *If $\mathfrak{g}_\mathbb{Q}$ is a filiform Lie $\mathbb{Q}$-algebra of dimension $\geqslant 3$ then the center of $\Gamma_{\mathfrak{g}_\mathbb{Q}}$ is infinite cyclic.*

Proof. The algebra $\mathfrak{g} = \mathbb{R} \otimes \mathfrak{g}_\mathbb{Q}$ is a filiform Lie $\mathbb{R}$-algebra since the nilpotence degree can not increase under scalar extensions. Thus $\dim Z\mathfrak{g} = 1$. Hence $ZG_\mathfrak{g} \simeq \mathbb{R}$ by Proposition 5.3. Thus $Z\Gamma_{\mathfrak{g}_\mathbb{Q}} \simeq \mathbb{Z}$ by Proposition 5.4. □

§6. Proof of the main result

Theorem 6.1. *If $\mathfrak{g}_\mathbb{Q}$ is a rational filiform derivationally nilpotent Lie algebra of dimension $\geqslant 3$ then the group $\Gamma_{\mathfrak{g}_\mathbb{Q}}$ is not geometrically equivalent to its Mal'cev's closure.*

Proof. By Theorem 4.1, $\Gamma_{\mathfrak{g}_\mathbb{Q}}$ is incompressible. By Theorem 5.5, the center of $\Gamma_{\mathfrak{g}_\mathbb{Q}}$ is infinite cyclic. Hence, by Lemma 5.2, $\Gamma_{\mathfrak{g}_\mathbb{Q}}$ is strongly incompressible. By Theorem 5.1, $\Gamma_{\mathfrak{g}_\mathbb{Q}}$ is not geometrically equivalent to Mal'cev's closure $\sqrt{\Gamma_{\mathfrak{g}_\mathbb{Q}}}$. □

The above theorem implies that for the proof of the main Theorem 1.2 we need to construct for every odd $n \geqslant 7$ a $\mathbb{Q}$-rational Lie algebra $\mathfrak{g}_n$ of dimension n which satisfies

(1) $\mathfrak{g}_n$ is derivationally nilpotent,
(2) $\mathfrak{g}_n$ is filiform i.e. $\mathfrak{g}_n$ has nilpotence degree $n-1$.

By definition the algebra $\mathfrak{g}_n$ has a basis $e_1, \ldots, e_n$, $n \geqslant 7$ and relations

$$\begin{array}{ll} [e_1, e_k] = e_{k+1} & k = 2, \ldots, n \\ [e_2, e_k] = e_{k+3} & k = 3, \ldots, n-3 \\ [e_k, e_{n-k}] = (-1)^{k+1} e_n & k = 1, \ldots, m \\ [e_i, e_j] = 0 & \text{for all other } i < j. \end{array} \tag{6.28}$$

In fact, the real nilpotent Lie algebra $\mathfrak{r}_n = \mathbb{R} \otimes \mathfrak{g}_n$ was defined by S. Yamaguchi [30].

Lemma 6.2. *For every odd $n \geqslant 7$ the Lie algebra $\mathfrak{r}_n$ is derivationally nilpotent in the sense that every its derivation is nilpotent.*

Proof of Lemma 6.2. It is shown in [30] that $A = \mathrm{Aut}\,(\mathfrak{r}_n)$ is a connected unipotent Lie group. Thus $A - E$ consists of n-nilpotent elements, i.e. $(a - E)^n = 0$ for each $a \in A$. The Lie algebra $\mathrm{Lie}\,(A)$ is isomorphic to the algebra of derivations $\mathrm{Der}\,(\mathfrak{r}_n)$. Let δ be a derivation of $\mathfrak{r}_n$. An automorphism $\exp\,(t\delta)$ of $\mathfrak{r}_n$ is unipotent for all real t hence $(\exp\,(t\delta) - 1)^n = 0$. We have

$$\lim_{t \to 0} \left(\frac{\exp\,(t\delta) - 1}{t} \right)^n = \delta^n = 0, \tag{6.29}$$

thus δ is nilpotent. Hence $\mathrm{Der}(\mathfrak{r}_n)$ consists of nilpotent derivations so $\mathfrak{r}_n$ is derivationally nilpotent. □

Lemma 6.3. *For every odd $n \geqslant 7$ the Lie $\mathbb{Q}$-algebra $\mathfrak{g}_n$ is derivationally nilpotent.*

Proof of Lemma 6.3. Let δ be a derivation of $\mathfrak{g}_n$. Extend δ to a derivation $\bar{\delta}$ of $\mathfrak{r}_n$ by continuity. Thus $\bar{\delta}$ is nilpotent hence δ is nilpotent also. □

Lemma 6.4. *For every odd $n \geqslant 7$ the Lie $\mathbb{Q}$-algebra $\mathfrak{g}_n$ is filiform.*

Proof of Lemma 6.4. The relation $[e_1, [e_1, ..., [e_1, e_2]]] = e_n$ of length $n - 2$ shows that the nilpotence degree of $\mathfrak{g}_n$ equals $n - 1$. □

Theorem 1.2 is proved.

Remark 6.5. We note that Yamaguchi also considered the even-dimensional algebras $\mathfrak{g}_n$ for $n \geqslant 8$, introduced by analogous relations. However, they are not Lie algebras (this fact was not noticed by the author). Precisely, in $\mathfrak{g}_8$ the Jacobi identity is not satisfied, for instance,

$$[e_1, [e_3, e_4]] + [e_3, [e_4, e_1]] + [e_4, [e_1, e_3]] = [e_3, -e_5] = -e_8 \neq 0, \tag{6.30}$$

see [14]. Unfortunately, it is stated in [8, page 60] that odd dimensional $\mathfrak{g}_n$ also do not satisfy the Jacobi identity, precisely, that in $\mathfrak{g}_7$ the identity is not satisfied for e_1, e_2, e_4. However,

$$[e_1,[e_2,e_4]] + [e_2,[e_4,e_1]] + [e_4,[e_1,e_2]] \tag{6.31}$$

$$= [e_1,e_7] + [e_2,-e_5] + [e_4,e_3] = 0 - (-e_7) - (e_7) = 0. \tag{6.32}$$

§7. Questions and problems

(1) We will denote by $\mathcal{L}_m$ the set of Lie algebra laws on $\mathbb{C}^m, m \geqslant 1$. The set $\mathcal{L}_m$ is naturally an affine algebraic $\mathbb{Q}$-defined variety. Let $\mathcal{N}_m \subseteq \mathcal{L}_m$ be a ($\mathbb{Q}$-defined) subvariety of all m-dimensional nilpotent Lie algebras. Let $\mathcal{F}_m \subseteq \mathcal{N}_m$ be a (Zariski open) subset of filiform Lie algebras. As $\mathcal{F}_m = \mathcal{N}_m - \mathcal{N}_m^{(m-2)}$, this subset is a Zariski open subset of $\mathcal{N}_m$. For any $m \geqslant 7$ every irreducible component C of $\mathcal{F}_m$ contains a non-empty Zariski open set $\mathcal{DNF}_m$ consisting of derivationally nilpotent Lie algebras [7, Chapter 7, Theorem 2]. Even more is obtained, namely that in every nonempty open set a derivationally nilpotent Lie algebra can be found.

(2) Do all the properties from paragraph 1 hold also over reals ? For instance, is $\mathcal{DNF}_m(\mathbb{R})$ non-empty and open in the real Zariski topology on $\mathcal{N}_m(\mathbb{R})$? Is the set $\mathcal{DNF}_m(\mathbb{Q})$ is dense in $\mathcal{DNF}_m(\mathbb{R})$ in the Euclidean topology? As J. M. Ancochea explained to the author of [1], in each dimension $m \geqslant 7$ there are infinitely many pairwise non-isomorphic characteristically nilpotent Lie algebras over $\mathbb{Q}$. Does there exist an algebra $L \in \mathcal{DNF}_m(\mathbb{R})$ that is isolated in $\mathcal{N}_m(\mathbb{R})$ in the Euclidean topology?

(3) Show the example of incompressible f-group Γ such that $\Gamma \sim \sqrt{\Gamma}$.

(4) Let Γ be an f-group and G_Γ is its Mal'cev's completion. For simply-connected nilpotent Lie groups the exponential map exp is a diffeomorphism with globally defined inverse log. The $\mathbb{Q}$-span of $\log(\Gamma)$ is a nilpotent Lie subalgebra over $\mathbb{Q}$, which we denote by $L(\Gamma)$. Can one decide whether an f-group Γ is incompressible by looking at $L(\Gamma)$ (I. Belegradek)? A related question is whether the incompressible property for Γ can be read off $\mathbb{R} \otimes L(\Gamma)$, which is the Lie algebra of G_Γ.

(5) The input of the Group Isomorphism problem GroupIso consists of two finite groups G_1 and G_2 of order n given by multiplication tables ($n \times n$ matrices of integers between 1 and n) and it is asked

whether the groups are isomorphic. A partial outstanding question is whether there exists a polynomial (in n) algorithm solving GroupIso. Let GeomEquiv be the problem whose input consists of two finite groups G_1 and G_2 and it is asked whether the groups are geometrically equivalent. An easier problem EmbedPower is to find a number m such that G_1 is embedable into G_2^m.

References

1. I. Belegradek, *On co-Hopfian nilpotent groups.* — Bull. London Math. Soc. **35** (2003), 805–811.
2. A. Berzins, *Geometrical equivalence of algebras.* — Int. J. Algebra Comput. **11**, No. 4 (2001), 447–456.
3. V. V. Bludov, B. V. Gusev, *Geometric equivalence of groups.* — Tr. Inst. Mat. Mekh. UrO RAN **13**, No. 1 (2007), 57–78.
4. G. Baumslag, A. Myasnikov, V. Remeslennikov, *Algebraic geometry over groups.* I. *Algebraic sets and ideal theory.* — J. Algebra **219** (1999), 16–79.
5. R. Camm, *Simple free products.* — J. London Math. Soc. **28** (1953), 66–76.
6. R. C. Lyndon, P. E. Schupp, *Combinatorial Group Theory*, Springer Ergebinsberichte **89**, Berlin–Heidelberg–New York, 1977.
7. M. Goze, You. B. Khakimdjanov, *Nilpotent Lie algebras*, Kluwer Academics Publishers, 1996.
8. É. B. Vinberg, V. V. Gorbatsevich, A. L. Onishchik, *Structure of Lie groups and Lie algebras*, Encyclopaedia of Math. Sciences **41**, Springer-Verlag, 1994.
9. R.W Goodman, *Nilpotent Lie Groups: Structure and Applications to Analysis.* — Lect. Notes Math. **562**, Springer-Verlag, Berlin/New York (1976).
10. R. Göbel, S. Shelah, *Radicals and Plotkin's problem concerning geometrically equivalent groups.*— Proc. Amer. Math. Soc. **130**, No. 3, 673–674.
11. H. Hamrouni, *Euler characteristics on a class of finitely generated nilpotent groups.* — Osaka J. Math. **50** (2013), 339–346.
12. G. Hochschild, *The structure of Lie groups*, Holden-Day Inc., San Francisco, 1965.
13. M. I. Kargapolov and Ju. I. Merzljakov *Fundamentals of the theory of groups.* Grad. Texts Math. **62**, Springer-Verlag, New York, 1979.
14. Yu. Khakimdjanov, *Characteristically nilpotent Lie algebras.* — Math. Sb. **181**, No. 5 (1990), 642–655.
15. E. I. Khukhro, V. D. Mazurov, *Unsolved problems in group theory. The Kourovka Notebook*, Russian Acad. Sci., Novosibirsk, 16th ed., 2006.
16. G. Leger, S. Tôgô, *Characteristically nilpotent Lie algebras.* — Duke Math. J. **26** (1959), 623–628.
17. A. I. Mal'cev, *Nilpotent torsion-free groups.*— Izv. Akad. Nauk. SSSR. Ser. Mat. **13** (1949), 201–212.
18. A. I. Mal'cev, *On a class of homogeneous spaces.*— Izv. Akad. Nauk. SSSR. Ser. Mat. **13** (1949), 9–32.

19. A. A. Mishchenko, *Model-theoretic and algebra-geometric problems for nilpotent partial commutative groups*, Phd thesis, Omsk, 2009.
20. A. Myasnikov, V. Remeslennikov, *Algebraic geometry over groups* II, *Logical foundations.* — J. Algebra **234**, No. 1 (2000) 225–276.
21. D. Nikolova, B. Plotkin, *Some notes on universal algebraic geometry.* In: "Algebra. Proc. Int. Conf. on Algebra on the Occasion of the 90th Birthday of A. G. Kurosh, Moscow, Russia, 1998." Walter De Gruyter Publ., Berlin, 1999, 237 – 261.
22. B. Plotkin, *Algebraic logic, varieties of algebras and algebraic varieties.* In: Proc. Int. Alg. Conf., St. Petersburg, 1995, St.Petersburg, 1999, p. 189–271.
23. B. Plotkin, *Varieties of algebras and algebraic varieties.* — Israel J. Math. **96**, No. 2 (1996), 511–522.
24. B. Plotkin, *Some notions of algebraic geometry in universal algebra.* — Algebra Analysis **9**, No. 4 (1997), 224–248.
25. B. Plotkin, *Varieties of algebras and algebraic varieties: Categories of algebraic varieties.* — Siberian Adv. Math. **7**, No. 2 (1997), 64–97.
26. B. Plotkin, *Radicals in groups, operations on classes of groups, and radical classes.* — Transl., II Ser. Amer. Math. Soc. **119** (1983) 89–118.
27. B. Plotkin, E. Plotkin, A. Tsurkov, *Geometrical equivalence of groups.* — Commun. Algebra **27**, No. 8 (1999), 4015–4025.
28. M. S. Raghunathan, *Discrete Subgroups of Lie Groups*, Springer, 1972.
29. A. Tsurkov, *Geometrical equivalence of nilpotent groups.* — Zap. Nauchn. Semin. POMI **330** (2006), 259–270.
30. S. Yamaguchi, *On some classes of nilpotent Lie algebras and their automorphism groups.* — Mem. Fac. Sci. Kyushu Univ. 1981. V. 35. P. 341–351.

Sobolev Institute of Mathematics,
644043 Omsk, Russia
E-mail: g.noskov@googlemail.com

Поступило 22 декабря 2016 г.