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UNRELATIVISED STANDARD COMMUTATOR FORMULA

Abstract. In the present note, which is a marginalia to the previous papers by Roozbeh Hazrat, Alexei Stepanov, Zuhong Zhang, and the author, I observe that for any ideals $A, B \trianglelefteq R$ of a commutative ring R and all $n \geqslant 3$ the birelative standard commutator formula also holds in the unrelativised form, as $[E(n,A), \mathrm{GL}(n,B)] = [E(n,A), E(n,B)]$ and discuss some obvious corollaries thereof.

Nikt nic nie czyta, a jeśli czyta,
to nic nie rozumie, a jeśli nawet rozumie,
to nic nie pamięta.
Stanisław Lem

§1. Introduction

Let R be a commutative ring with 1, $G = \mathrm{GL}(n,R)$ be the general linear group of degree $n \geqslant 3$ over R. For an ideal $I \trianglelefteq R$ denote by $E(n,I)$ the corresponding elementary subgroup, generated by the elementary transvections of level I:

$$E(n,I) = \big\langle t_{ij}(\xi),\ \xi \in I,\ 1 \leqslant i \neq j \leqslant n \big\rangle.$$

The corresponding relative elementary subgroup $E(n,R,I)$ is defined as the normal closure of $E(n,I)$ in the absolute elementary subgroup $E(n,R)$.

Further, consider the reduction homomorphism

$$\rho_I : \mathrm{GL}(n,R) \longrightarrow \mathrm{GL}(n,R/I)$$

modulo I. By definition, the principal congruence subgroup $\mathrm{GL}(n,I) = \mathrm{GL}(n,R,I)$ is the kernel of ρ_I. In other words, $\mathrm{GL}(n,I)$ consists of all matrices g conruent to e modulo I.

One of the keynote results of the structure theory of linear groups over rings is the following birelative standard commutator formula

$$[E(n,R,A), \mathrm{GL}(n,R,B)] = [E(n,R,A), E(n,R,B)],$$

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which unifies a great number of preceeding results by Bass, Mason and Stothers, Vaserstein, Borewicz and myself, and others [1, 17, 16, 27, 3]; see the works by Hong You, Stepanov and myself, Hazrat and Zuhong Zhang [34, 31, 12, 32].

Here we observe that one can remove R everywhere in the above formula, in other words, replace the relative elementary subgroups $E(n, R, A)$ and $E(n, R, B)$ by their naive unrelativised analogues $E(n, A)$ and $E(n, B)$.

Theorem 1. *Let A and B be two ideals of a commutative ring R, $n \geqslant 3$. Then*

$$[E(n, A), \mathrm{GL}(n, B)] = [E(n, A), E(n, B)].$$

The proof is again exactly the same argument by Alexei Stepanov and the author as in [31], based on the Theme of [24], combined with the observation by Roozbeh Hazrat and Zuhong Zhang that the mixed commutator subgroup $[E(n, A), E(n, B)]$ is normal in $E(n, R)$, see Corollary 16 in [11].

Theorem 1 becomes less baffling, if you confront it with the following corollary.

Theorem 2. *Let A and B be two ideals of a commutative ring R, $n \geqslant 3$. Then*

$$[E(n, A), E(n, B)] = [E(n, R, A), E(n, R, B)].$$

Proof. Obviously, the left hand side is contained in the right hand side. To prove the opposite inclusion, observe that [11], Corollary 1A on page 493 (= Lemma 4 below) asserts that

$$[E(n, R, A), E(n, R, B)] = [E(n, A), E(n, R, B)] \leqslant [E(n, A), \mathrm{GL}(n, R, B)],$$

while the right hand side of the last inclusion equals $[E(n, A), E(n, B)]$ by Theorem 1. $\square$

I start with a question that prompted me to look for such a generalisation. After that §§3 and 4 recall the necessary notation and background. The proof of Theorem 1 is then concluded in §5, followed by scattered remarks in §6.

§2. Mennicke's question

In his paper [19] Jens Mennicke returns to his 1965 approach to the congruence subgroup problem [18]. Namely, he defines the following subgroup of the principal congruence subgroup $\mathrm{GL}(n,R,I)$:

$$F(n,R,I)=\big\{g=(g_{ij})\in\mathrm{GL}(n,R)\mid g_{ij}\equiv 0\,(\mathrm{mod}\ I),\\ i\neq j,\ g_{ii}\equiv 1\,(\mathrm{mod}\ I^2)\big\}.$$

Clearly, again this group depends on the ideal I alone, and not on the ambient ring R, and can be denoted simply by $F(n,I)$. The main result [19], Theorem 1, is the proof, in the spirit of [18], combined with some ideas from [26], of the fact that over $\mathbb{Z}$ one has $E(n,I)=F(n,I)$, provided $n\geqslant 3$. Mennicke asks, whether the results of his paper hold for more general rings.

Apparently, Mennicke was not aware of my ancient papers [28, 29], where, also with the use of the ideas from [26], much broader results were established. Namely, there I prove, in particular, that for all ideals $I\neq 0$ in any Dedekind ring $R=R_S$ of arithmetic type one has $F(n,I)/E(n,I)\cong \mathrm{SK}_1(R,I^2)$.

In fact, the main results of [29] were much more general than that. They applied to the elementary subgroup $E(\sigma)$ corresponding to any net $\sigma=(\sigma_{ij})$, $1\leqslant i,j\leqslant n$, of non-zero ideals $\sigma_{ij}\trianglelefteq R$. Denote by $F(\sigma)$ the minimal congruence subgroup containing $E(\sigma)$. Then

$$F(\sigma)/E(\sigma)\cong \mathrm{SK}_1\left(R,\sum_{i\neq j}\sigma_{ij}\sigma_{ji}\right),$$

which specialises to Vaserstein's formula in the case of $n=2$. In Mennicke's case is $E(n,I)\,\mathrm{GL}(n,I^2)=F(n,I)$, hence the occurence of I^2 in the above answer.

As another historical curiousity I could mention that my original proofs in [28] relied not just on the main results of [26], but rather on some of the inside machinery, such as, for instance, multiplicativity of Vaserstein's birelative Mennicke symbols. Later, it was discovered that Vaserstein's lemmas contained irredeemable mistakes, in particular, there are examples where such multiplicativity fails as stated. Luckily, already weeker forms of multiplicativity of Mennicke symbols, established by Armin Leutbecher [14] and [15] to save the main results of [26], sufficed for my purposes. Thus, the proofs in [29] relied on [14] and [15] instead.

However, the question still remains. One of the main technical ingredients in [19] was the proof of Theorem 2, asserting that $E(n,I)\trianglelefteq F(n,I)$. Once you see such a result for $n\geqslant 3$, depending on a single ideal I, it immediately occurs to you that it should hold not just over $\mathbb{Z}$, but at least over all commutative rings R. This is indeed the case. In [20] Bogdan Nica established the following result.

Theorem 3. *Let A be an ideal of a commutative ring R, $n \geqslant 3$. Then $E(n, A)$ is normal in $\mathrm{GL}(n, A)$.*

Observe, that it is an obvious corollary of our Theorem 1. Indeed, setting $A = B$, we see that $[E(n, A), \mathrm{GL}(n, A)]$ is contained in $[E(n, A), E(n, A)] \leqslant E(n, A)$. This is how I started to fancy that Theorem 1 might be true. The rest was an exercise.

As an aside, one could notice that [20] does not cite not just [28, 29] or [24], but even [3], where original Suslin's calculations were presented both in exactly the same form, as in [20], and also in another such form in $\mathrm{GL}(4, R)$.

Actually, [3] contained a much broader generalisation in the same spirit, a proof of normality of the elementary subgroup in the corresponding congruence subgroup $E(\sigma) \trianglelefteq G(\sigma)$ (or even in its normaliser $N_{\mathrm{GL}(n,R)}(G(\sigma))$, for that matter), not for individual ideals, but for nets of ideals $\sigma = (\sigma_{ij})$, $1 \leqslant i, j \leqslant n$, provided that each index $1 \leqslant i \leqslant n$ is contained in an equivalence class of cardiality $\geqslant 3$, such that σ_{ij} depends only on equivalence classes of i and j, not on i and j themselves. Nominally, in the statement of Theorem 3 it was stipulated that $\sigma_{ij} = R$ for $i \sim j$, but examining the proof, it is easy to see that such extra condition is not needed at this place (it is heavily used in other results of [3], of course). Compare also the proof of [30], Theorem 2.

At about the same time as Mennicke submitted his paper, Alexei Stepanov and I were discussing a possible major sequel to our paper [24]. I remember sketching to Alexei a birelative extension of the argument we used to prove [24], Theorem 1 – essentially the same argument as reproduced below in the proof of Theorem 1 of the present paper. This occurred in a room at V5 of *Uni Bielefeld*, just a few meters from Mennicke's office.

Unfortunately, the intended sequel never materialised, some pieces thereof made their way to separate opuscula, such as [31, 32]. However, even while writing [31] and [32] we have not uncovered the veritable purport of that argument (of course, at that point one of the key components, normality of $[E(n, A), E(n, B)]$, was not there yet). This teaches us two lessons, how difficult it is to convey mathematics, and how difficult it is to state a theorem, even when you do have its proof for a long time.

§3. Notation and antecedent results

For two subgroups $F, H \leqslant G$, we denote by $[F, H]$ their mutual commutator subgroup spanned by all commutators $[f, h]$, where $f \in F$, $h \in H$. Observe that our commutators are always left-normed, $[x, y] = xyx^{-1}y^{-1}$. The double commutator $[[x, y], z]$ will be denoted simply by $[x, y, z]$. Further, ${}^{x}y = xyx^{-1}$ denotes the left conjugate of y by x. In the sequel we repeatedly use obvious commutator identities such as $[y, x] = [x, y]^{-1}$, or $[xy, z] = {}^{x}[y, z]\cdot[x, z]$ and $[x, yz] = [x, y]\cdot{}^{y}[x, z]$, mostly without any specific reference.

As usual, e denotes the identity matrix and e_{ij} is a standard matrix unit. For $\xi \in R$ and $1 \leqslant i \neq j \leqslant n$, we denote by $t_{ij}(\xi) = e + \xi e_{ij}$, we denote the corresponding [elementary] transvection. A matrix $g \in \mathrm{GL}(n, R)$ is written as $g = (g_{ij})$, $1 \leqslant i, j \leqslant n$, where g_{ij} is its entry in the position (i, j). Entries of the inverse matrix $g^{-1} = (g'_{ij})$, $1 \leqslant i, j \leqslant n$, are denoted by g'_{ij}.

The present paper is a marginalia to [24, 31, 12, 32, 5, 13, 7, 11], where similar problems were considered for $\mathrm{GL}(n, R)$, and their follow-ups for unitary groups, and Chevalley groups [8, 9, 6, 21, 22, 10, 23], ...

The following lemma is the birelative standard commutator formula. For $\mathrm{GL}(n, R)$ there are three entirely different published proofs:

• By Alexei Stepanov and myself [31], based on decomposition of unipotents, which, as also the proof of Theorem 1 in the present paper, is essentially a clone of the proof of Theorem 1 in [24].

• By Roozbeh Hazrat and Zuhong Zhang [12], relying on double relative localisation, developed expressly with this purpose, to answer [31], Problem 2.

• By Alexei and myself again [32], reducing (via level calculations) birelative case to the relative/absolute one, that was already known for some 30 years from the works of Andrei Suslin, Leonid Vaserstein, Zenon Borewicz and myself [25, 27, 4].

Luckily, when writing [31, 32] we were not aware of the work by Hong You [34], where essentially the same argument as in [32] was already used for Chevalley groups. For otherwise relative localisation with all its upshots and ramifications might have never been discovered.

Lemma 1. *Let A and B be two ideals of a commutative ring R, $n \geqslant 3$. Then*
$$[E(n, R, A), \mathrm{GL}(n, R, B)] = [E(n, R, A), E(n, R, B)].$$

The following results are [11], Lemma 1A, and Corollary 16 on page 493, respectively.

Lemma 2. *Let A and B be two ideals of a commutative ring R, $n \geqslant 3$. Then*

$$E(n, R, AB) \leqslant [E(n, A), E(n, B)].$$

Lemma 3. *Let A and B be two ideals of a commutative ring R, $n \geqslant 3$. Then $[E(n, A), E(n, B)]$ is normal in $E(n, R)$.*

Lemmas 2 and 3 were not immediately obvious (at least to me). It seems that Lemma 2 is a routine calculation in terms of the unipotent radicals of two opposite parabolic subgroups, reproduced *hundreds* of times since its first occurence in [1, 2] (compare Lemma 6 below). However, the real difficulty is that while proving Lemma 2 we still do not know that $[E(n, A), E(n, B)]$ is normal in $E(n, R)$. In fact, this is exactly what we are attempting to accomplish at this stage.

I remember first hearing these statements from Roozbeh Hazrat and Zuhong Zhang sometime in 2012. In fact, Lemma 3 is an immediate corollary of Lemmas 1 and 2 but it is so counter-intuitive, that initially I could not believe it and asked Zuhong for ulterior elucidations.

Since these lemmas are closely related to determining generators of the birelative commutator subgroup $[E(n, R, A), E(n, R, B)]$, I expected to find them in [12, 13]. But trying to track a precise reference, I discovered that instead of Lemma 3 even [7], Corollary 1A on page 750, announced only the following result.

Lemma 4. *Let A and B be two ideals of a commutative ring R, $n \geqslant 3$. Then*

$$[E(n, A), E(n, R, B)] = [E(n, R, A), E(n, R, B)].$$

Thus, apparently the first occurences of Lemma 3 in print are [10], Corollary 5.2, in the context of Chevalley groups[1], and [11], Corollary 16, for $\mathrm{GL}(n, R)$ over quasi-finite (in particular, commutative) rings.

[1]There is a misprint in the first display formula in the proof of that corollary on page 406. The second occurence of $E(\Phi, R, IJ)$ should read as $G(\Phi, R, IJ)$.

§4. Decomposition of transvections

By R^n we denote the free right R-module, consisting of columns of height n with components in R and by ${}^nR$, we denote the free left R-module, consisting of rows of length n with components in R. Standard bases in R^n and ${}^nR$, are denoted by $e_1, \dots, e_n$ and $f_1, \dots, f_n$, respectively.

A *transvection* is a matrix of the form $e + uv$, where $u \in R^n$, $v \in {}^nR$ are a column and a row such that $vu = 0$. Classically, [the line spanned by] u is called the *centre* of the transvection $e + uv$, while [the hyperplane orthogonal to] v is called its *axis*, see, for instance [3].

Clearly, if $u_j = 0$, one has $e + uf_j = \prod t_{ij}(u_i)$, where the product is taken over all $i \neq j$. Similarly, if $v_i = 0$, one has $e + e_i v = \prod t_{ij}(v_j)$, where the product is taken over all $j \neq i$. If we additionally assume that $u \in A^n$ and $v \in {}^nB$ then, clearly, $e + uf_j \in E(n, B)$ and $e + e_i v \in E(n, A)$.

The following properties are classically known (and obvious!), and will be used in the sequel without any specific reference.

• A conjugate of a transvection is again a transvection

$$g(e + uv)g^{-1} = e + (gu)(vg^{-1}).$$

• Transvections with the same centre are additive with respect to their axes. In other words, if $v, w \in {}^nR$ are such that $vu = wu = 0$, then

$$e + u(v + w) = (e + uv)(e + uw).$$

• By the same token, transvections with the same axis are additive with respect to their centres. In other words, if $u, z \in R^n$ are such that $vu = vz = 0$, then

$$e + (u + z)v = (e + uv)(e + zv).$$

• The commutator of two transvections that are not opposite is again a transvection. In other words, if $u, z \in R^n$ are two columns, whereas $v, w \in {}^nR$ are two rows such that $vu = wz = wu = 0$, then

$$[e + uv, ezw] = e + u(vz)w = e + u(vz)w.$$

As in [31] our proof depends on the Theme of [24]. Of course, there the following lemma is stated only in the absolute case, but replacing R by an ideal $A \trianglelefteq R$ and requesting $\xi \in A$ does not make any difference, see [31], Lemma 4.

Lemma 5. *Let R be a commutative ring, $n \geqslant 3$, and $A \trianglelefteq R$. Then, for any matrix $g \in \mathrm{GL}(n, R)$, the elementary group $E(n, A)$ is generated by* *transvections $e + e_i v$, where $1 \leqslant i \leqslant n$, whereas row $v \in {}^nA$ is such that $v_i = 0$ and the row vg^{-1} contains at least one zero entry, say $(vg^{-1})_j = 0$.*

The following result is a routine computation in terms of two opposite unipotent radicals, based on the above properties of transvections. It is known since [1, 2]. For details, see the proof of either [24], Lemma 3, or [33], Lemma 21.

Lemma 6. *Let A be an ideal of R. Further, let, $u \in R^n$ and $v \in {}^nA$ be a column and a row such that $vu = 0$ and $v_j = 0$ for some j. Then*

$$e+uv = (e+u_je_jv)(e+(u-u_je_j)v) = (e+u_je_jv)[e+(u-u_je_j)f_j, e+e_jv].$$

§5. Proof of Theorem 1

Now we are all set to prove our main result. Since the right-hand side is obviously contained in the left-hand side, it remains to verify the inverse inclusion. We proceed exactly as in the proof of Theorem of [31]. For any matrix $g \in \mathrm{GL}(n, B)$, the elementary subgroup $E(n, A)$ is generated by transvections of the form $e + e_iv$ described in Lemma 5.

Observe that, again as in [31], it suffices to verify that all commutators of the form $[e+e_iv, g]$, where $v \in {}^nA$, $v_i = 0$, and $(vg^{-1})_j = 0$ for some i and j belong to $[E(n, A), E(n, B)]$. Indeed, $[E(n, A), E(n, B)]$ being normal in the absolite elementary group $E(n, R)$ by Lemma 3, the formula $[xy, g] = {}^x[y, g]\cdot[x, g]$ then implies that $[x, g] \in [E(n, A), E(n, B)]$ for all $x \in E(n, A)$.

Now, we can repeat the rest of the proof *almost* verbatim, just being slightly more cautious about elementary conjugations. Namely, now we cannot pull conjugations inside the individual factors of $[E(n, A), E(n, B)]$ and have to keep track of them as exponents.

As in [31] we follow the proof of [24], Theorem 1. However, here, as in [31], all products are considered modulo the normal subgroup $E(n, R, AB)$. This is still possible since $E(n, R, AB)$ is contained in the new right hand side by Lemma 2.

- In the case where $j \neq i$ plugging

$$[e + e_iv, g] = (e + e_iv)\big(e - (ge_i)(vg^{-1})\big)$$

into the formula in Lemma 6, we see that

$$[e + e_iv, g] = (e + e_iv)\big(e - g_{ji}e_j(vg^{-1})\big)\big[e - (ge_i - g_{ji}e_j)f_j, e - e_j(vg^{-1})\big].$$

Since $g_{ji} \equiv 0 \pmod B$, whereas $vg^{-1} \equiv 0 \pmod A$ and $(vg^{-1})_j = 0$, the second factor on the right-hand side of the above expression for $[e+e_iv, g]$ is contained in $E(n, R, AB)$.

Now, let us take a closer look at the commutator in the right hand side. Since $ge_i - g_{ji}e_j \equiv e_i \pmod B$ and $(ge_i - g_{ji}e_j)_j = 0$, we can express the column $ge_i - g_{ji}e_j$ as $ge_i - g_{ji}e_j = e_i + u$, where $u \in B^n$, $u_j = 0$. Therefore, in this case the commutator in the right hand side of the above expression equals

$$\begin{aligned}&\big[t_{ij}(1)(e+uf_j), e - e_j(vg^{-1})\big]\\ &\qquad\qquad = t_{ij}(1)\big[e+uf_j, e-e_j(vg^{-1})\big]\,\big[t_{ij}(1), e-e_j(vg^{-1})\big],\end{aligned}$$

where $e + uf_j \in E(n, B)$. Since $vg^{-1} \in {}^nA$, $(vg^{-1})_j = 0$, one has $e'e_j(vg^{-1}) \in E(n, A)$.

This means that the first commutator in the right hand of this last expression, and thus by Lemma 3 also its elementary conjugate, belongs to $[E(n, A), E(n, B)]$.

It only remains to compute $[t_{ij}(1), e - e_j(vg^{-1})]$. Recalling that $vg \equiv v \equiv vg^{-1} \pmod{AB}$ and $v_i = (vg^{-1})_j = 0$, we see that $v_j, (vg^{-1})_i \in AB$. Thus, we can express the row vg^{-1} as $vg^{-1} = f_i(vg^{-1})_i + w$, where the complementary row $w = vg^{-1}{}' f_i(vg^{-1})_i \in {}^nA$ is still congruent to v modulo AB. But now we are much better off, since w has two zeros, $w_i = w_j = 0$.

This means that the second commutator in the last expression can be rewritten as

$$\big[t_{ij}(1), (e-e_jw)t_{ji}((vg^{-1})_i)\big] = [t_{ij}(1), e-e_jw]\cdot{}^{e-e_jw}\big[t_{ij}(1), t_{ji}((vg^{-1})_i)\big].$$

By the very definition the commutator of two elementary transvections in the last formula – and thus also all of its [elementary] conjugates – sit in $E(n, R, AB)$. This means that modulo $E(n, R, AB)$ we have

$$\begin{aligned}\big[t_{ij}(1), e-e_j(vg^{-1})\big] &\equiv [t_{ij}(1), e-e_jw] \equiv e - e_iw\\ &\equiv e - e_iv \pmod{E(n, R, AB)}.\end{aligned}$$

Summarising the above, we see that modulo $E(n, R, AB)$ the unaccounted part of the commutator in the initial expression for $[e + e_iv, g]$ is exactly the inverse of the first factor therein. Thus, they simply result in another elementary conjugation, which (again by Lemma 3) still leaves us inside $[E(n, A), E(n, B)]$.

• In the case where $j = i$ the formula in Lemma 6 boils down to

$$[e + e_i v, g] = (e + e_i v)\big(e - g_{ii} e_i (vg^{-1})\big)\,\big[e - (ge_i - g_{ii}e_i)f_i, e - e_i(vg^{-1})\big].$$

Since $ge_i - g_{ii}e_i \equiv 0 \pmod B$ and $(ge_i{}' g_{ii}e_i)_i = 0$, ane has $(ge_i - g_{ii}e_i)f_i \in E(n, B)$. Similarly, since $vg^{-1} \equiv 0 \pmod A$ and $(vg^{-1})_i = 0$, one has $e - e_i(vg^{-1}) \in E(n, A)$. Hence, the commutator on the right-hand side of the last equality is itself in $[E(n, A), E(n, B)]$.

On the other hand, $g_{ii}v \equiv v \pmod{AB}$ and $vg^{-1} \equiv e \pmod{AB}$. Hence, $e - g_{ii}e_i(vg^{-1}) \equiv e - e_i v \pmod{E(n, R, AB)}$. Thus, the first two factors cancel out modulo $E(n, R, AB)$.

§6. Final remarks

There is no doubt that the main result of the present paper should hold in any situation, where we know the birelative standard commutator formula and enough commutator calculus to guarantee that the commutator of two unrelativised elementary subgroups is normal in the absolute elementary group.

This is the case, for instance in Chevalley groups [9, 10, 23], and for Bak's hyperbolic unitary groups [8, 11]. We plan to return to this problem in forthcoming publications. Most probably, especially in the noncommutative case, instead of decomposition of unipotents one will have to recourse to a version of localisation. But possibly even a more sophisticated version of level calculations might suffice.

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