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V. Abgaryan, A. Khvedelidze, A. Torosyan

ON MODULI SPACE OF THE WIGNER QUASIPROBABILITY DISTRIBUTIONS FOR N-DIMENSIONAL QUANTUM SYSTEMS

Abstract. A mapping between operators on the Hilbert space of N-dimensional quantum system and the Wigner quasiprobability distributions defined on the symplectic flag manifold is discussed. The Wigner quasiprobability distribution is constructed as a dual pairing between the density matrix and the Stratonovich–Weyl kernel. It is shown that the moduli space of the Stratonovich–Weyl kernel is given by an intersection of the coadjoint orbit space of the $SU(N)$ group and a unit $(N-2)$-dimensional sphere. The general consideration is exemplified by a detailed description of the moduli space of 2, 3 and 4-dimensional systems.

§1. Introduction

According to the postulates of the quantum theory, the fundamental description of a physical system is provided by the density operator [1]

$$\varrho = \sum_k p_k |\psi_k\rangle\langle\psi_k|, \tag{1}$$

which represents the quantum statistical ensemble $\{p_k, |\psi_k\rangle\}$, i.e., a set consisting of vectors $|\psi_k\rangle \in \mathcal{H}$ of the Hilbert space $\mathcal{H}$ and their probabilities p_k with a sum equal to one, $\sum\limits_k p_k = 1$. The density operator ϱ determines the expectation value $\mathbb{E}(\widehat{A})$ of a Hermitian operator $\widehat{A}$ acting on $\mathcal{H}$,

$$\mathbb{E}(\widehat{A}) = \operatorname{Tr}\left[\widehat{A}\varrho\right], \quad \text{with} \quad \operatorname{Tr}\left[\varrho\right] = 1. \tag{2}$$

The latter is assigned to a physical observable associated with the operator $\widehat{A}$. On the other hand, an ensemble of a classical mechanical system is characterized by a probability distribution function $\rho(q,p)$, i.e., the density of the probability to find the system in a state localized in the vicinity of a phase space point with coordinates q and p. Correspondingly, the statistical

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average, i.e., the expectation value $\mathbb{E}(A)$ of a physical quantity described by the function $A(q,p)$ on a phase space is given by the following convolution:

$$\mathbb{E}(A) = \int \mathrm{d}\Omega\, A(q,p)\,\rho(q,p), \quad \text{with} \quad \int \mathrm{d}\Omega\, \rho(q,p) = 1, \tag{3}$$

where $\mathrm{d}\Omega$ denotes the normalized volume form of a classical phase space.

Aiming to collate two representations of observables, the classical (3) and the quantum (2), the so-called Weyl–Wigner invertible mapping between Hilbert space operators and functions on a phase space has been introduced in the early stages of the development of quantum mechanics [2–6]. The primary elements of this map are two notions: the *symbol of operator*, i.e., a function $A_W(q,p)$ corresponding to the operator A, and the *quasi-distribution function* $W(q,p)$ defined over a phase space. As a result, the quantum analogue of the statistical average (3) reads

$$\mathbb{E}(\widehat{A}) = \int \mathrm{d}\Omega\, A_W(q,p)\, W(q,p), \quad \text{with} \quad \int \mathrm{d}\Omega\, W(q,p) = 1. \tag{4}$$

However, even a quick-look at this attempt to build a bridge between classical and quantum statistical pictures shows a lack of their equivalence. Indeed, one can point out the following observations:

- Because of Heisenberg's uncertainty principle, the function $W(q,p)$ has negative values for certain quantum states. Hence it is not a true probability density and is referred to as quasiprobability distribution.
- Dirac's quantization rule based on the canonical commutator relations makes the interplay between operators and their symbols highly sophisticated. Replacement of canonical variables by their quantum counterparts in expressions of functions over the phase-space faces an ambiguity of ordering of the corresponding canonical operators. [1]

[1] According to Weyl's rule of quantization [2], any classical observable $A(\boldsymbol{p},\boldsymbol{q})$, i.e., a function on the phase space $\mathbb{R}^{2n}$ with a standard canonical symplectic structure, is associated with an operator $\widehat{A}_\omega$ on the Hilbert space $L^2(\mathbb{R}^n)$ constructed as the "*Weyl quantum Fourier transform*:"

$$A \mapsto \widehat{A}_\omega = \int\limits_{\mathbb{R}^{2n}} \mathrm{d}\Omega(\omega)\, \widetilde{A}(\boldsymbol{u},\boldsymbol{v}) \exp \frac{\imath}{\hbar} \Big(\boldsymbol{u}\widehat{\boldsymbol{P}} + \boldsymbol{v}\widehat{\boldsymbol{Q}} \Big), \quad \mathrm{d}\Omega = \omega(\boldsymbol{u},\boldsymbol{v})\, \mathrm{d}\boldsymbol{u}\mathrm{d}\boldsymbol{v}, \tag{5}$$

where $\widehat{P}$ and $\widehat{Q}$ are operators on $L^2(\mathbb{R}^n)$ obeying canonical commutator relations, $\widetilde{A}(\boldsymbol{u},\boldsymbol{v})$ is Fourier transform of $A(\boldsymbol{u},\boldsymbol{v})$, and the integration measure $\mathrm{d}\Omega$ is defined by a weight function $\omega(\boldsymbol{u},\boldsymbol{v})$. Different choice of $\omega(\boldsymbol{u},\boldsymbol{v})$ is a source of various orderings of

In spite of both flaws, Wigner functions or other formulated quasiprobability distributions, such as Husimi [4] and Glauber–Sudarshan [8, 9] representations, remain today an important tool for understanding of interrelations between quantum and classical statistical descriptions [10]. Moreover, nowadays one can see a growing interest to phase space formulation of quantum mechanics based on the method of quasiprobability distributions for finite dimensional systems (see e.g. [11–18] and references therein). The latter is coming from needs of diverse applications in quantum optics [19] and also in quantum information and communications [20]. Such an intense usage of quasi-distributions again raises an issue of understanding of the above mentioned shortcomings.[2]

In the present note, a problem of construction of quasiprobability distribution functions for generic N-level systems is studied within a purely algebraic approach. The basic mathematical objects in this approach are: a special unitary group $G = SU(N)$, its Lie algebra $\mathfrak{g} = \mathfrak{su}(N)$:

$$\mathfrak{su}(N) = \{X \in M(N, \mathbb{C}) \mid X = -X^\dagger, \quad \mathrm{tr}X = 0\}, \tag{7}$$

non-commutative operators $\widehat{P}$ and $\widehat{Q}$. For example, the factor $\omega(\boldsymbol{u}, \boldsymbol{v}) = \exp\left(-\frac{\imath}{2}\boldsymbol{u}\boldsymbol{v}\right)$ corresponds to a standard ordering of polynomials in mathematical literature when writing first the position coordinate Q, then the momentum P. The so-called normal ordering is related to the weight $\omega(\boldsymbol{u}, \boldsymbol{v}) = \exp\left(-\frac{1}{4}\left(\boldsymbol{u}^2 + \boldsymbol{v}^2\right)\right)$, while the original Weyl, or symmetric, order complies with $\omega(\boldsymbol{u}, \boldsymbol{v}) = 1$. The inverse formula that maps the operator to its symbol belongs to Wigner [3]. For a unit weight factor case, $\omega = 1$, the inverse formula reads:

$$A(\boldsymbol{u}, \boldsymbol{v}) = \frac{1}{(2\pi\hbar)^n}\mathrm{tr}\left[\widehat{A}_1 \exp -\frac{\imath}{\hbar}\left(\boldsymbol{u}\widehat{\boldsymbol{P}} + \boldsymbol{v}\widehat{\boldsymbol{Q}}\right)\right]. \tag{6}$$

A further elaboration of Weyl's quantization scheme leads to the non-commutative formulation of mechanics [6] and finally to the development of the so-called deformation quantization, cf. [7].

[2]History going back to Dirac's idea on negative energies teaches us to pay more attention to a "nonsense" of negative probabilities. In this context it is the best to afford the following words by R.Feynman: "*It is that a situation for which a negative probability is calculated is impossible, not in the sense that the chance for it happening is zero, but rather in the sense that the assumed conditions of preparation or verification are experimentally unattainable*" [21].

and its dual space $\mathfrak{g}^* = \mathfrak{su}(N)^*$.[3] It is well known that the universal covering algebra $\mathfrak{U}(\mathfrak{su}(N))$ of the Lie algebra $\mathfrak{su}(N)$ is an arena of the basic objects of N-level quantum system. Particularly, a state space $\varrho \in \mathfrak{P}_N$ is defined as the space of *positive semidefinite* $N \times N$ Hermitian matrices H_N with a unit trace:

$$\mathfrak{P}_N = \{X \in H_N \mid X \geqslant 0, \quad \mathrm{tr}\,(X) = 1\}. \tag{9}$$

Every state described by the density matrix $\varrho \in \mathfrak{P}_N$ is in correspondence with some element of the Lie algebra $\mathfrak{su}(N)$:

$$\varrho = \frac{1}{N}\mathbb{I}_N + \frac{1}{N}\,\imath\,\mathfrak{su}(N). \tag{10}$$

In order to build up the Wigner function, apart from the quantum state space $\mathfrak{P}_N$, the notion of its dual $\mathfrak{P}_N^*$ is required. Every point of the dual space determines the Stratonovich–Weyl (SW) kernel [22, 23]. As it was shown recently in [24], the space $\mathfrak{P}_N^*$ can be defined as follows:

$$\mathfrak{P}_N^* = \{X \in H_N \mid \mathrm{tr}\,(X) = 1, \quad \mathrm{tr}\left(X^2\right) = N\}. \tag{11}$$

It turns out that the dual pairing (8) of a density matrix $\varrho \in \mathfrak{P}_N$ and SW kernel $\Delta(\Omega_N) \in \mathfrak{P}_N^*$:

$$W_\varrho(\Omega_N) = \mathrm{tr}\,[\varrho\,\Delta(\Omega_N)] \tag{12}$$

enables us with the proper Wigner function which satisfies all the Stratonovich–Weyl postulates [22,23]. Taking into account a unit trace condition, SW kernel $\Delta(\Omega_N)$ can be related to the dual of $\mathfrak{su}(N)$:

$$\Delta(\Omega_N) = \frac{1}{N}\mathbb{I}_N + \kappa\,\frac{1}{N}\,\imath\,\mathfrak{su}(N)^*, \tag{13}$$

where $\kappa = \sqrt{N(N^2-1)/2}$ is a normalization constant. From representations (10) and (13) it follows that all nontrivial information comes from pairing between traceless parts of a density matrix and SW kernel. In the subsequent sections, after a short overview of the Stratonovich–Weyl

[3] Since $\mathfrak{g}$ is a linear space over the real field $\mathbb{R}$, one can define a bilinear map $\langle\,.\,,\,.\,\rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$, and identify algebra with its dual. The conventional inner product on $\mathfrak{g}$,

$$\langle A,\, B\rangle := \mathrm{tr}\left(A^\dagger B\right), \quad A, B \in \mathfrak{su}(N), \tag{8}$$

enables to set up a duality pairing and to realize an isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$.

postulates, algebraic and geometric aspects of the dual space $\mathfrak{P}_N^*$ are discussed. In particular, we establish interrelation between the Wigner functions and the coadjoint orbits [25] $\mathcal{O}_{\boldsymbol{r}}$ of $SU(N)$:

$$\mathcal{O}_{\boldsymbol{r}} = \{UDU^\dagger \, : U \in SU(N)\}, \tag{14}$$

where $\boldsymbol{r}$ denotes N-tuple of real numbers $\boldsymbol{r} = r_1, r_2, \ldots, r_N$ which are elements of the diagonal matrix $D = \mathrm{diag}||r_1, r_2, \ldots, r_N||$ ordered as $r_1 \geqslant r_2 \geqslant \ldots \geqslant r_N$ and summed up to zero, $\sum\limits_{i=1}^{N} r_i = 0$. It is then proved that

$$W_\varrho(\Omega_N) - \frac{I}{N} \, : \, \mathfrak{P}_N \times \mathcal{O}_{\boldsymbol{r}}\big|_{\sum r_i^2 = N/(N-1)} \quad \to \quad \mathbb{R}. \tag{15}$$

Furthermore, in order to describe in unitary invariant way an ambiguity of the Wigner function, we introduce the *moduli space* $\mathcal{P}_N$ of SW kernel as the following coset:

$$\mathcal{P}_N := \frac{\mathcal{O}_{\boldsymbol{r}}}{SU(N)}\Bigg|_{\sum r_i^2 = N/(N-1)} . \tag{16}$$

The moduli space geometrically represents intersections of the orbit space of the $SU(N)$ group coadjoint action with an $(N-2)$-dimensional sphere. Finalizing our note, we give few examples of the moduli space of the Wigner functions for low-level quantum systems, for a qubit (N=2), qutrit (N=3) and quatrit (N=4).

§2. Constructing the Wigner function

Below we give a brief summary of the Wigner quasiprobability distribution construction starting from the basic Stratonovich–Weyl postulates and reformulating them into a set of algebraic constraints on a spectrum of SW kernels $\Delta(\Omega_N)$.

• **The Stratonovich–Weyl principles** • Following to Brif and Mann [23], the postulates known as the Stratonovich–Weyl correspondence can be written as the following constraints on the kernel $\Delta(\Omega_N)$:

(1) **Reconstruction**: a state ϱ is reconstructed from the WF (12) via the integral over a phase space:

$$\varrho = \int\limits_{\Omega_N} \mathrm{d}\Omega_N \, \Delta(\Omega_N) W_\varrho(\Omega_N) \, ; \tag{17}$$

(2) **Hermicity**:

$$\Delta(\Omega_N) = \Delta(\Omega_N)^\dagger\,; \tag{18}$$

(3) **Finite Norm**: a state norm is given by the integral of the Wigner distribution:

$$\mathrm{tr}[\varrho] = \int\limits_{\Omega_N} \mathrm{d}\Omega_N W_\varrho(\Omega_N), \qquad \int\limits_{\Omega_N} \mathrm{d}\Omega_N\, \Delta(\Omega_N) = 1\,; \tag{19}$$

(4) **Covariance**: the unitary transformations $\varrho' = U(\alpha)\varrho U^\dagger(\alpha)$ induce the kernel change:

$$\Delta(\Omega_N') = U(\alpha)^\dagger \Delta(\Omega_N) U(\alpha). \tag{20}$$

Algebraic master equation for SW kernel. The above given axioms allow derivation of algebraic equations for SW kernel of N- level quantum systems. With this goal, following the paper [24], we accomplish next steps:

I. Identification of phase-space Ω_N with complex flag manifold.

Hereinafter, a phase-space Ω_N will be identified with a complex flag manifold, $\Omega_N = \mathbb{F}^N_{d_1,d_2,\dots,d_s}$. The latter emerges as follows: supposing that a spectrum of SW kernel $\Delta(\Omega_N)$ consists of real eigenvalues with the algebraic multiplicity k_i, i.e., the isotropy group H of the kernel is

$$H = U(k_1) \times U(k_2) \times U(k_{s+1}),$$

one can see that the phase space Ω_N can be realized as a coset space $U(N)/H$, the complex flag manifold $\mathbb{F}^N_{d_1,d_2,\dots,d_s}$, where $(d_1, d_2, ..., d_s)$ is a sequence of positive integers with sum N, such that $k_1 = d_1$ and $k_{i+1} = d_{i+1} - d_i$ with $d_{s+1} = N$. Furthermore, since the flag manifold represents a coadjoin orbit of $SU(N)$, its symplectic structure is given by the corresponding Kirillov–Kostant–Souriau symplectic 2-form [25].

II. Enlarging of phase-space Ω_N to $SU(N)$ group manifold.

Owing to the unitary symmetry of N-dimensional quantum system, we can relate a measure $\mathrm{d}\Omega_N$ on the symplectic space Ω_N with the normalized Haar measure $\mathrm{d}\mu_{SU(N)}$ on the $SU(N)$ group manifold:

$$\mathrm{d}\Omega_N = C_N^{-1} \mathrm{d}\mu_{SU(N)}/\mathrm{d}\mu_H.$$

Here C_N is a real normalization constant, $\mathrm{d}\mu_H$ is the Haar measure on the isotropy group H induced by the embedding, $H \subset SU(N)$.

Noting that the integrand in (17) is a function of the coset variables only, the reconstruction integral can be extended to the whole group $SU(N)$,

$$\varrho = Z_N^{-1} \int\limits_{SU(N)} \mathrm{d}\mu_{SU(N)}\, \Delta(\Omega_N) W_\varrho(\Omega_N), \tag{21}$$

where the normalization constant $Z_N^{-1} = C_N^{-1}/\mathrm{vol}(H)$ includes the factor $\mathrm{vol}(H)$ which is the volume of the isotropy group H.

III. Derivation of algebraic equations for SW kernel.
Relations (12) and (21) imply the integral identity

$$\varrho = Z_N^{-1} \int\limits_{SU(N)} \mathrm{d}\mu_{SU(N)}\, \Delta(\Omega_N)\, \mathrm{tr}\left[\varrho\Delta(\Omega_N)\right]. \tag{22}$$

Substituting the singular value decomposition for SW kernel into (22) and evaluating the integral using the Weingarten formula [26–28]:

$$\begin{aligned}\int d\mu U_{i_1j_1}U_{i_2j_2}\bar{U}_{k_1l_1}\bar{U}_{k_2l_2} &= \frac{1}{N^2-1}(\delta_{i_1k_1}\delta_{i_2k_2}\delta_{j_1l_1}\delta_{j_2l_2} + \delta_{i_1k_2}\delta_{i_2k_1}\delta_{j_1l_2}\delta_{j_2l_1}) \\ &- \frac{1}{N(N^2-1)}(\delta_{i_1k_1}\delta_{i_2k_2}\delta_{j_1l_2}\delta_{j_2l_1} + \delta_{i_1k_2}\delta_{i_2k_1}\delta_{j_1l_1}\delta_{j_2l_2}),\end{aligned}$$

we derive the equations:

$$(\mathrm{tr}[\Delta(\Omega_N)])^2 = Z_N N, \quad \mathrm{tr}[\Delta(\Omega_N)^2] = Z_N N^2. \tag{23}$$

IV. Normalization of SW kernel.
The constant Z_N in the equation (21) can be determined with the aid of the so-called standardization condition,

$$Z_N^{-1}\int \mathrm{d}\mu_{SU(N)} W_A(\Omega_N) = \mathrm{tr}[A]. \tag{24}$$

Fixing the normalization constant Z_N, we finally arrive at the **"master equations"** for SW kernel:

$$\mathrm{tr}\left[\Delta(\Omega_N)\right] = 1, \qquad \mathrm{tr}[\Delta(\Omega_N)^2] = N. \tag{25}$$

§3. Moduli space: reckoning up solutions to the "master equations"

Classifying solutions to the master equations (25), we arrive at the notion of a "*moduli space*" as the space $\mathcal{P}_N$, points of which are associated with the unitary equivalent admissible SW kernel of N-dimensional quantum system. Analysis of eq. (25) solutions space displays the following properties of the moduli space $\mathcal{P}_N$:

(1) $\dim(\mathcal{P}_N(\boldsymbol{\nu})) = N-2$, i.e., a maximal number of continuous parameters $\boldsymbol{\nu}$ characterizing the solution $\Delta(\Omega_N\,|\,\boldsymbol{\nu})$ is $N-2$;

(2) geometrically, $\mathcal{P}_N$ is represented as an intersection of an $(N-2)$-dimensional sphere $\mathbb{S}_{N-2}$ with the orbit space $\mathfrak{su}(N)^*/SU(N)$ of $SU(N)$ action on a dual space $\mathfrak{su}(N)^*$:

$$\mathcal{P}_N \cong \mathbb{S}_{N-2} \bigcap \frac{\mathfrak{su}(N)^*}{SU(N)}. \tag{26}$$

In order to become convinced in above statements, consider the singular value decomposition of SW kernel and assume that the kernel is generic with all eigenvalues distinct.[4] Using the orthonormal basis $\{\lambda_1, \lambda_2, \dots, \lambda_{N^2-1}\}$ of $\mathfrak{su}(N)$, the SVD decomposition reads:

$$\Delta(\Omega_N|\boldsymbol{\nu}) = \frac{1}{N} U(\Omega_N) \left[I + \kappa \sum_{\lambda\in H} \mu_s(\boldsymbol{\nu})\lambda_s \right] U(\Omega_N)^\dagger, \tag{27}$$

where $\kappa = \sqrt{N(N^2-1)/2}$, and H is the Cartan subalgebra $H \in \mathfrak{su}(N)$.

From the master equation (25) it follows that the coefficients $\mu_s(\boldsymbol{\nu})$ live on an $(N-2)$-dimensional sphere $\mathbb{S}_{N-2}(1)$ of radius one:

$$\sum_{s=2}^{N} \mu^2_{s^2-1}(\boldsymbol{\nu}) = 1. \tag{28}$$

A generic SW kernel can be parameterized by $N-2$ spherical angles. The parameter $(\boldsymbol{\nu})$ introduced in order to label members of the family of the Wigner functions can be associated with a point on $\mathbb{S}_{N-2}(1)$. More precisely, a one-to-one correspondence between points on this sphere and unitary non-equivalent SW kernels occurs only for a certain subspace of $\mathbb{S}_{N-2}(1)$. This subspace $\mathcal{P}_N(\boldsymbol{\nu}) \subset \mathbb{S}_{N-2}(1)$ represents the moduli space of SW kernel. Its geometry is determined by the $\Delta(\Omega_N\,|\,\boldsymbol{\nu})$ eigenvalues

[4] In this case, the isotropy group of SW kernel is isomorphic to $(N-1)$-dimensional torus $\mathbb{T}^{N-1} = \{g \in SU(N) : g - \text{diagonal}\}$.

ordering. The chosen descending order of the eigenvalues restricts the range of spherical angles parameterizing (28) and cuts out the moduli space of $\Delta(\Omega_N\,|\,\boldsymbol{\nu})$ in the form of a spherical polyhedron. Details of SW kernels parametrization in terms of spherical angles are given in the Appendix A.

§4. The Wigner function as dual pairing between ϱ and Δ

As soon as the space of all possible SW kernels is known, the construction of the Wigner function reduces to a computation of pairing (12). Using the $\mathfrak{su}(N)$ expansions (10) for a density matrix ϱ_ξ of N-level system characterized by (N^2-1)-dimensional Bloch vector $\boldsymbol{\xi}$,

$$\varrho_\xi = \frac{1}{N}\left(I + \sqrt{\frac{N\,(N-1)}{2}}\,(\boldsymbol{\xi},\boldsymbol{\lambda})\right),$$

and SW kernel decomposition (27), we arrive at the general representation for the WF:

$$W^{(\boldsymbol{\nu})}_{\boldsymbol{\xi}}(\theta_1,\theta_2,\dots,\theta_d) = \frac{1}{N}\left[1 + \frac{N^2-1}{\sqrt{N+1}}\,(\boldsymbol{n},\boldsymbol{\xi})\right], \tag{29}$$

where (N^2-1)-dimensional vector $\boldsymbol{n}$ is given by a linear combination of $N-1$ orthonormal vectors $\boldsymbol{n}^{(s^2-1)}$ with coefficients $\mu_{s^2-1}(\boldsymbol{\nu})$, $s = 2,3,\dots,N$,

$$\boldsymbol{n} = \mu_3\boldsymbol{n}^{(3)} + \mu_8\boldsymbol{n}^{(8)} + \cdots + \mu_{N^2-1}\boldsymbol{n}^{(N^2-1)}.$$

The vectors $\boldsymbol{n}^{(s^2-1)}$ are determined by the Cartan subalgebra $\lambda_{s^2-1} \in H$:

$$\boldsymbol{n}^{(s^2-1)}_\mu = \frac{1}{2}\,\mathrm{tr}\left(U\lambda_{s^2-1}U^\dagger\lambda_\mu\right), \quad s = 2,3,\dots,N.$$

As it was mentioned in the Introduction, the number of the symplectic coordinates $\vartheta_1,\vartheta_2,\dots,\vartheta_d$ of the Wigner function depends on the isotropy group of SW kernel (cf. details in [24]).

§5. Examples

Below we present an explicit parametrization for a moduli space of a few low-dimensional quantum systems, including a single qubit, qutrit and quatrit.

5.1. The moduli space of a single qubit SW kernel. For a 2-level quantum system, a qubit, the master equations (25) determine the spectrum (up to permutation) of 2-dimensional SW kernel uniquely:

$$\Delta^{(2)}(\Omega_2) = \frac{1}{2} U(\Omega_2) \begin{pmatrix} 1+\sqrt{3} & 0 \\ 0 & 1-\sqrt{3} \end{pmatrix} U(\Omega_2)^\dagger, \tag{30}$$

with $U(\Omega_2) \in SU(2)/U(1)$. Its connection to the structure of the coadjoint orbits of $SU(2)$ is straightforward. There are two types of the coadjoint orbits of $SU(2)$:

(1) 2-dimensional regular orbits $\mathcal{O}_{\boldsymbol{r}}$,

$$\mathcal{O}_{\{r,-r\}} = \left\{ U \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix} U^\dagger,\, U \in SU(2) \right\},$$

defined for an ordered 2-tuple, $\boldsymbol{r} = \{r, -r\}$, $r > 0$. They are isomorphic to a 2-dimensional sphere $\mathbb{S}_2(r)$ with the radius given by the value of the $SU(2)$ invariant:

$$r^2 = -\det\left(\mathcal{O}_{\boldsymbol{r}}\right); \tag{31}$$

(2) zero-dimensional orbit, point $r = 0$.

Identifying these orbits with the traceless part of SW kernel $\Delta^{(2)} - \frac{1}{2}\mathbb{I}$ and taking into account the expression (30), we get convinced that

$$r^2 = \frac{4}{3}\operatorname{tr}\left[\left(\Delta^{(2)} - \frac{1}{2}\mathbb{I}\right)^2\right] = 2.$$

Thus, the moduli space of SW kernel of a qubit represents the single point, $r^2 = 2$, from the set of equivalence classes of the regular $SU(2)$ orbits, $[\mathcal{O}_{\boldsymbol{r}}] \cong \mathfrak{su}(2)/U(1)$.

5.2. The moduli space of a single qutrit SW kernel. The master equations (25) determine two lowest-degree polynomial $SU(N)$ invariants of SW kernel $\Delta(\Omega_3)$, the linear and the quadratic ones. For the case of a 3-dimensional quantum system, a qutrit, the third algebraically independent polynomial $SU(3)$ invariant remains unfixed, thus allowing a one-parametric family of a qutrit SW kernels to exist.

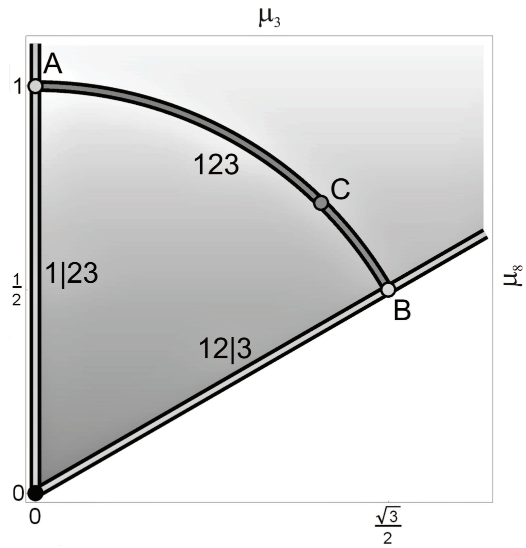

Figure 1. The cone representing the orbit space of $SU(3)$. The interior of the cone represents $\dim \mathcal{O} = 4$ orbits. The apex corresponds to a zero-dimensional orbit, while other points on the ordinate ($\mu_8 = 0$) and on the positive ray $\mu_8 = \mu_3/\sqrt{3}$ also determine $\dim \mathcal{O} = 4$ orbits. The intersection of the cone with a unit circle gives an arc which is the moduli space of a qutrit SW kernel. The point C with $\cos(\zeta_C) = (-1+3\sqrt{5})/8$ describes the singular SW kernel.

Following the normalization convention (27), let us write down the SVD decomposition of a qutrit SW kernel in the following form:

$$\Delta(\Omega_3) = U(\Omega_3)PU(\Omega_3)^\dagger = U(\Omega_3) \left[\frac{1}{3}\mathbb{I} + \frac{2}{\sqrt{3}} \begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix} \right] U(\Omega_3)^\dagger, \tag{32}$$

where $U(\Omega_3) \in SU(3)$, and a 3-tuple $\boldsymbol{r} = \{r_1, r_2, r_3\}$ parameterizes a traceless diagonal part of the SVD decomposition of SW kernel, $r_1+r_2+r_3 = 0$. Expanding P over the Gell-Mann basis elements $\lambda_3 = \mathrm{diag}\|1, -1, 0\|$ and

$\lambda_8 = \frac{1}{\sqrt{3}}\mathrm{diag}\|1, 1, -2\|$ of the $SU(3)$ Cartan subalgebra,

$$P = \frac{1}{3}\mathbb{I} + \frac{2}{\sqrt{3}}\left(\mu_3\lambda_3 + \mu_8\lambda_8\right), \tag{33}$$

we find:

$$r_1 = \mu_3 + \frac{1}{\sqrt{3}}\mu_8, \quad r_2 = -\mu_3 + \frac{1}{\sqrt{3}}\mu_8, \quad r_3 = -\frac{2}{\sqrt{3}}\mu_8. \tag{34}$$

From these relations it follows that the chosen decreasing order of parameters $r_1 \geqslant r_2 \geqslant r_3$ determines on the (μ_3, μ_8)-plane the 2-dimensional polyhedral cone $C_2(\pi/3)$ with the apex angle $\pi/3$:

$$C_2(\pi/3) = \left\{ \boldsymbol{x} \in \mathbb{R}^2 \;\middle|\; \begin{pmatrix} 1 & 0 \\ \frac{-1}{\sqrt{3}} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geqslant 0 \right\}. \tag{35}$$

• **The SU(3) orbits** • The cone $C_2(\pi/3)$ represents the orbit space of $SU(3)$ group action on $\mathfrak{su}(3)$ algebra. Next we identify this algebra times $\imath$ with the traceless part of $\Delta(\Omega_3)$ and classify SW kernel in accordance to the corresponding coadjoint orbits. In order to realize this program, let us consider the tangent space to the $SU(3)$ orbits. It is spanned by the linearly independent vectors built of the commutators: $t_k = [\lambda_k, \Delta],\ \lambda_k \in \mathfrak{su}(3)$. The number of independent vectors t_k determines the dimensionality of the orbits via the rank of the 8×8 Gram matrix:

$$\mathcal{G}_{kl}(\Delta^{(3)}) = \frac{1}{2}\,\mathrm{tr}\,(t_k t_l), \qquad k, l = 1, 2, \ldots, 8. \tag{36}$$

Since the rank of the Gram matrix (36) is $SU(3)$ invariant, one can calculate it for the diagonal representative of SW kernel (33). The straightforward computations give

$$\mathcal{G}(\Delta^{(3)}) = \frac{4}{3}\mathrm{diag}\,||\, g_1, g_1, 0, g_2, g_2, g_3, g_3, 0\,||, \tag{37}$$

where $g_1 = 4\mu_3^2$, $g_2 = \frac{1}{\sqrt{3}}(\mu_3 + \sqrt{3}\mu_8)^2$, $g_3 = \frac{1}{\sqrt{3}}(\mu_3 - \sqrt{3}\mu_8)^2$. From these expressions it follows that there are three types of $SU(3)$ orbits which can be classified according to their symmetry and dimensions:

(1) $\underline{\dim(\mathcal{O}_{\boldsymbol{r}}) = 6}$. These *regular orbits* abbreviated as $\mathcal{O}(123)$ (or simply 123) are labeled by a 3-tuple $\boldsymbol{r}$ with $r_1 > r_2 > r_3$ and have the isotropy group $H_{(123)}$ isomorphic to a 2-dimensional torus, $H_{(123)} \cong \mathbb{T}^2$. They are in one-to-one correspondence with the interior points of the cone $C_2(\pi/3)$ in Fig.1.

(2) $\underline{\dim(\mathcal{O}_{\boldsymbol{r}}) = 4.}$ These *degenerate orbits* represent two subfamilies with degenerate 3-tuples $\boldsymbol{r}$: either $r_1 = r_2 > r_3$ or $r_1 > r_2 = r_3$. Following V. I. Arnold [30], we denote them as $1|23$ and $12|3$ respectively. Geometrically, the equivalence class $[\mathcal{O}]$ of degenerate orbits represents the boundary lines in the $SU(3)$ orbit space:

$$\begin{aligned}\mathcal{O}(1|23) &\mapsto 1|23 : \quad \{\boldsymbol{x} \in C_2(\pi/3)|\ x_2 = 0\ \},\\ \mathcal{O}(12|3) &\mapsto 12|3 : \quad \{\boldsymbol{x} \in C_2(\pi/3)|\ x_2 = x_1/\sqrt{3}\ \}.\end{aligned}$$

Both classes, up to conjugacy in $SU(3)$ have the same isotropy group:

$$H_{(12|3)} \cong H_{(1|23)} = \left\{ h \in \left[\begin{array}{c|c} e^{i\alpha}g & 0 \\ \hline 0 & e^{-i\alpha} \end{array}\right] \;\middle|\; g \in SU(2) \right\}. \tag{38}$$

(3) $\underline{\dim(\mathcal{O}_0) = 0.}$ One orbit $\mathcal{O}_0$, a single point $(0,0)$ which is stationary under the $SU(3)$ group action.

• **The parametrization of a qutrit SW kernels** • We are now in a position to describe the moduli space of a qutrit as a certain subspace of the $SU(3)$ orbit space. Indeed, taking into account that the second order master equation (25) describes a circle of radius one centered at the origin of (μ_3, μ_8)-plane, we convinced that the moduli space of a qutrit SW kernel represents the arc depicted in Fig. 1. More precisely, based on the above classification of the $SU(3)$ orbits, we treat a qutrit moduli space as the union of two strata:

- The regular stratum corresponding to the regular $SU(3)$-orbits. Geometrically it is the arc $\widehat{AB}/\{A, B\}$ with its endpoints A and B excluded. The corresponding Wigner functions have a 6-dimensional support and 1-dimensional family of SW kernels, the spectrum of which can be written as:

$$\operatorname{spec}\left(\Delta^{(3)}(\nu)\right) = \left\{\frac{1-\nu+\delta}{2}, \frac{1-\nu-\delta}{2}, \nu\right\}, \tag{39}$$

where $\delta = \sqrt{(1+\nu)(5-3\nu)}$ and $\nu \in (-1/3, -1)$. The parameter ν is related to the apex angle ζ of the cone $C_2(\pi/3)$: [5]

$$\nu = \frac{1}{3} - \frac{4}{3}\cos(\zeta), \quad \zeta \in [0,\ \pi/3]. \tag{40}$$

- The end points A and B of the arc $\widehat{AB}$ correspond to two degenerate SW kernels, with $\nu = -1$ and $\nu = -\frac{1}{3}$ respectively,

$$\mathrm{spec}\left(\Delta^{(3)}(-1)\right) = \{1, 1, -1\}, \quad \mathrm{spec}\left(\Delta^{(3)}\left(-\frac{1}{3}\right)\right) = \frac{1}{3}\{5, -1, -1\}.$$

It is necessary to point out that the kernel $\Delta^{(3)}(-1)$ was found by Luis [29].
• **The singular SW kernels of qutrit** • Apart from the above categorization of SW kernels, we distinguish the *singular kernels* which have at least one zero eigenvalue. From the expression (39) it follows that for a qutrit case among three zeros of the determinant $\det(\Delta^{(3)}) = \nu(\nu^2 - \nu - 1)$ only one, $\nu = (1-\sqrt{5})/2$, is admissible: [6]

$$\mathrm{spec}\left(\Delta_{(103)}\right) = \left\{\frac{1+\sqrt{5}}{2},\ 0,\ \frac{1-\sqrt{5}}{2}\right\}.$$

§6. The moduli space of a single quatrit SW kernel

The master equations (25) for a four-level system, a quatrit, determine a 2-parametric family of SW kernels. We start, similarly as in the case of qutrit, with the SVD decomposition of a quatrit SW kernel:

$$\begin{aligned}\Delta(\Omega_4) &= U(\Omega_4) P U(\Omega_4)^\dagger \\ &= U(\Omega_4)\left[\frac{1}{4}\mathbb{I} + \frac{\sqrt{30}}{4}\begin{pmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & r_4 \end{pmatrix}\right] U(\Omega_4)^\dagger,\end{aligned}$$

---

[5]The apex angle ζ determines the value of a 3-rd order polynomial SU(3)-invariant:

$$\cos(3\zeta) = -\frac{27}{16}\det\left(\Delta^{(3)} - \frac{1}{3}\mathbb{I}\right) = -\frac{27}{16}\det\left(\Delta^{(3)}\right) - \frac{11}{16}.$$

[6]Traces of powers of this "*golden ratio*" kernel are given by the so-called *Lucas numbers*:

$$\mathrm{tr}\left(\Delta_{(103)}\right)^2 = 3, \quad \mathrm{tr}\left(\Delta_{(103)}\right)^3 = 4, \quad \dots, \quad \mathrm{tr}\left(\Delta_{(103)}\right)^n = L_n.$$

with $U(\Omega_4) \in SU(4)$ and a 4-tuple $\boldsymbol{r} = \{r_1, r_2, r_3, r_4\}$, such that $r_1 + r_2 + r_3 + r_4 = 0$. These parameters expressions in terms of expansion coefficients of P over the Gell-Mann basis elements $\lambda_3 = \mathrm{diag}||1, -1, 0, 0||$, $\lambda_8 = \frac{1}{\sqrt{3}}\mathrm{diag}||1, 1, -2, 0||$ and $\lambda_{15} = \frac{1}{\sqrt{3}}\mathrm{diag}||1, 1, 1, -3||$ of the $SU(3)$ Cartan subalgebra,

$$P = \frac{1}{4}\mathbb{I} + \frac{\sqrt{30}}{4}\left(\mu_3\lambda_3 + \mu_8\lambda_8 + \mu_{15}\lambda_{15}\right), \tag{41}$$

read:

$$r_1 = \mu_3 + \frac{1}{\sqrt{3}}\mu_8 + \frac{1}{\sqrt{6}}\mu_{15}, \qquad r_2 = -\mu_3 + \frac{1}{\sqrt{3}}\mu_8 + \frac{1}{\sqrt{6}}\mu_{15}, \tag{42}$$

$$r_3 = -\frac{2}{\sqrt{3}}\mu_8 + \frac{1}{\sqrt{6}}\mu_{15}, \qquad r_4 = -\frac{3}{\sqrt{6}}\mu_{15}. \tag{43}$$

Due to the order $r_1 \geqslant r_2 \geqslant r_3 \geqslant r_4$, expansion coefficients μ_3, μ_8 and μ_{15} belong to a 3-dimensional polyhedral cone $C_3\,(\pi/6)$ with the apex angle $\pi/6$:

$$C_3\,(\pi/6) = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} 1 & 0 & 0 \\ \frac{-1}{\sqrt{3}} & 1 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \geqslant 0 \right\}. \tag{44}$$

• **The SU(4) orbits** • The cone $C_3\,(\pi/6)$ represents the $SU(4)$ orbit space. The calculated for a diagonal representative 15×15 Gram matrix

$$\mathcal{G}(\Delta^{(4)}) = \frac{5}{2}\,\mathrm{diag}||g_1, g_1, 0, g_2, g_2, g_3, g_3, 0, g_4, g_4, g_5, g_5, g_6, g_6, 0||, \tag{45}$$

where

$$g_1 = 3\mu_3^2, \quad g_2 = \frac{3}{4}\left(\mu_3 + \sqrt{3}\mu_8\right)^2, \quad g_3 = \frac{3}{4}\left(\mu_3 - \sqrt{3}\mu_8\right)^2,$$
$$g_4 = \frac{1}{8}\left(\sqrt{6}\mu_3 + \sqrt{2}\mu_8 + 4\mu_{15}\right)^2, \quad g_5 = \frac{1}{8}\left(-\sqrt{6}\mu_3 + \sqrt{2}\mu_8 + 4\mu_{15}\right)^2,$$
$$g_6 = \left(\mu_8 - \sqrt{2}\mu_{15}\right)^2.$$

Analysis of zeros of the Gram matrix (45) shows the following pattern of the regular and degenerate $SU(4)$ orbits.

- $\underline{\dim(\mathcal{O}_{\boldsymbol{r}}) = 12.}$ The regular orbits have a maximal dimension owing to the smallest isotropy group: $H_{(1234)} = \mathbb{T}^3 \in SU(4)$. The equivalent class of the regular orbits represents an interior of the cone $C_3\,(\pi/6)$;

- The degenerate orbits are divided into subclasses:
  (1) $\underline{\dim(\mathcal{O}_{\boldsymbol{r}}) = 10}$. The equivalence class of these orbits is one of the following faces of the cone $C_3\,(\pi/6)$:

$$\mathcal{O}(1|234) \mapsto 1|234: \quad \{\boldsymbol{x} \in C_3\,(\pi/6) \ \mid x_1 = 0\ \}, \tag{46}$$

$$\mathcal{O}(12|34) \mapsto 12|34: \quad \{\boldsymbol{x} \in C_3\,(\pi/6) \ \mid x_1 = -\sqrt{3}x_2\}, \tag{47}$$

$$\mathcal{O}(123|4) \mapsto 123|4: \quad \{\boldsymbol{x} \in C_3\,(\pi/6) \ \mid x_2 = +\sqrt{2}x_3\}. \tag{48}$$

All the above orbits have the same isotropy group (up to SU(4)-conjugation):

$$H_{(1|234)} = \left\{ h \in \left[\begin{array}{c|c|c} e^{i\alpha}g & 0 & 0 \\ \hline 0 & e^{i\beta} & 0 \\ \hline 0 & 0 & e^{i\gamma} \end{array}\right] \ \Big|\ g \in SU(2),\ \alpha+\beta+\gamma=0 \right\}.$$

The dimension of this stratum is in agreement with the dimension of the corresponding isotropy group,

$$\dim(\mathcal{O}_{\boldsymbol{r}}) = \dim(SU(4)) - \dim(H_{\boldsymbol{r}}) = 15 - (3+2) = 10.$$

  (2) $\underline{\dim(\mathcal{O}_{\boldsymbol{r}}) = 8}$. The equivalence class of these orbits is the following edge of the cone $C_3\,(\pi/6)$:

$$\mathcal{O}(1|23|4) \mapsto 1|23|4: \quad \{\boldsymbol{x} \in C_3\,(\pi/6) \ \mid x_1 = 0,\ x_2 = \sqrt{2}x_3\ \}. \tag{49}$$

The 7-dimensional isotropy group is:

$$H_{(1|23|4)} = \left\{ h \in \left[\begin{array}{c|c} e^{i\alpha}g & 0 \\ \hline 0 & e^{-i\alpha}g' \end{array}\right] \ \Big|\ g, g' \in SU(2) \right\}. \tag{50}$$

  (3) $\underline{\dim(\mathcal{O}_{\boldsymbol{r}}) = 6}$. The equivalence class of these orbits is one of the following edges of the cone $C_3\,(\pi/6)$:

$$\mathcal{O}(1|2|34) \mapsto 1|2|34: \{\boldsymbol{x} \in C_3\,(\pi/6) \ \mid x_1 = 0,\ x_2 = 0\}, \tag{51}$$

$$\mathcal{O}(12|3|4) \mapsto 12|3|4: \{\boldsymbol{x} \in C_3\,(\pi/6) \ \mid x_1 = \sqrt{3}x_2,\ x_2 = \sqrt{2}x_3\}. \tag{52}$$

Both classes have the same up to conjugacy 9-dimensional isotropy group:

$$H_{(1|2|34)} = \left\{ h \in \left[\begin{array}{c|c} e^{i\alpha}g & 0 \\ \hline 0 & e^{-i\alpha} \end{array}\right] \ \Big|\ g \in SU(3) \right\}. \tag{53}$$

  (4) $\underline{\dim(\mathcal{O}_{\boldsymbol{r}}) = 0}$.The apex of cone $C_3\,(\pi/6)$ with the stability group $SU(4)$.

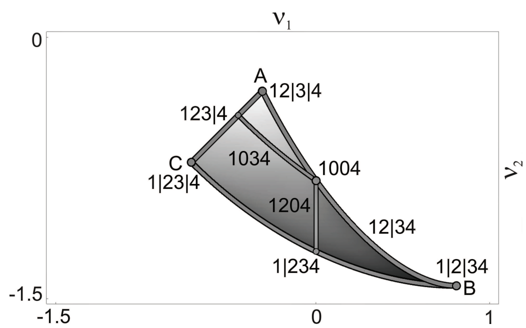

Figure 2. Support of SW kernel of a quatrit on (ν_1, ν_2)-plane. The interior of a curvilinear triangle ABC corresponds to the regular SW kernels. The boundary lines describe the double degeneracy cases. The vertexes A and B describe a quatrit kernels with a triple degeneracy, while the vertex C corresponds to a quatrit kernel with two double degeneracy.

• **The parametrization of a quatrit SW kernels** • Now we are ready to enumerate all SW kernels for a quatrit according to the above given classification of the $SU(4)$ orbits:

(1) The regular 2-dimensional family of SW kernels:

$$\text{spec}\left(\Delta^{(4)}(\nu_1,\nu_2)\right) = \left\{\frac{1-\nu_1-\nu_2+\delta}{2}, \frac{1-\nu_1-\nu_2-\delta}{2}, \nu_1, \nu_2\right\}, \quad (54)$$

where $\delta = \sqrt{7+2\nu_1-3\nu_1^2+2\nu_2-2\nu_1\nu_2-3\nu_2^2}$.

(2) The degenerate 1-dimensional family of SW kernels:

(a) A family of SW kernels of 1|234 type:

$$\text{spec}\left(\Delta_{(1|234)}\right) = \left\{\frac{1-\nu}{3}+\frac{1}{6}\delta_1, \frac{1-\nu}{3}+\frac{1}{6}\delta_1, \nu, \frac{1-\nu-\delta_1}{3}\right\}, \quad (55)$$

where $\delta_1 = \sqrt{22+4\nu-8\nu^2}$ and $\nu \in \left(\frac{1}{4}(1-\sqrt{15}),\ \frac{1}{4}(1+\sqrt{5})\right)$;

(b) A family of SW kernels of 12|34 type:

$$\text{spec}\left(\Delta_{(12|34)}\right) = \left\{\frac{1-2\nu+\delta_2}{2}, \nu, \nu, \frac{1-2\nu-\delta_2}{2}\right\}, \quad (56)$$

where $\delta_2=\sqrt{7+4\nu-8\nu^2}$ and $\nu\in\left(\frac{1}{4}(1-\sqrt{5}),\frac{1}{4}(1+\sqrt{5})\right)$;

(c) A family of SW kernels of 123|4 type:

$$\operatorname{spec}\left(\Delta_{(123|4)}\right)=\left\{\frac{1-2\nu+\delta_2}{2},\frac{1-2\nu-\delta_2}{2},\nu,\nu\right\}, \tag{57}$$

where $\nu\in\left(\frac{1}{4}\left(1-\sqrt{15}\right),\frac{1}{4}\left(1-\sqrt{5}\right)\right)$.

(3) SW kernels with a triple degeneracy:

(a) SW kernel of 1|2|34 type:

$$\operatorname{spec}\left(\Delta_{(1|2|34)}\right)=\left\{\frac{1+\sqrt{5}}{4},\frac{1+\sqrt{5}}{4},\frac{1+\sqrt{5}}{4},\frac{1-3\sqrt{5}}{4}\right\}; \tag{58}$$

(b) SW kernel of 12|3|4 type:

$$\operatorname{spec}\left(\Delta_{(12|3|4)}\right)=\left\{\frac{1+3\sqrt{5}}{4},\frac{1-\sqrt{5}}{4},\frac{1-\sqrt{5}}{4},\frac{1-\sqrt{5}}{4}\right\}. \tag{59}$$

(4) SW kernel with two double degeneracy:

SW kernel of 1|23|4 type:

$$\operatorname{spec}\left(\Delta_{(1|23|4)}\right)=\left\{\frac{1+\sqrt{15}}{4},\frac{1+\sqrt{15}}{4},\frac{1-\sqrt{15}}{4},\frac{1-\sqrt{15}}{4}\right\}. \tag{60}$$

All the above categories of SW kernels of a quatrit are depicted in Fig.2. The interior of a curvilinear triangle ABC on (ν_1,ν_2)-plane corresponds to the regular SW kernels. The boundary lines of the domain describe the double degeneracy cases:

(a) SW kernel of type 12|34–side $AB/\{A,B\}$ with both end points A and B excluded:

$$AB/\{A,B\}:\ \nu_2=\frac{1}{2}-\nu_1-\frac{1}{2}\sqrt{7+4\nu_1-8\nu_1^2},\quad \nu_1\in\left(\frac{1-\sqrt{5}}{4},\frac{1+\sqrt{5}}{4}\right);$$

(b) SW kernel of type 1|234-side $CB/\{C,B\}$ without end points:

$$CB/\{C,B\}:\ \nu_2=\frac{1}{3}-\frac{1}{3}\nu_1-\frac{1}{3}\sqrt{22+4\nu_1-8\nu_1^2},\quad \nu_1\in\left(\frac{1-\sqrt{15}}{4},\frac{1+\sqrt{5}}{4}\right);$$

(c) SW kernel of type 123|4-side $AC/\{A,C\}$ without end points:

$$\nu_2 = \nu_1, \quad \nu_1 \in \left(\frac{1-\sqrt{15}}{4}, \frac{1-\sqrt{5}}{4}\right).$$

The vertexes A and B describe a quatrit kernels with a triple degeneracy:

(a) SW kernel of 12|3|4 type – point A: $\nu_1 = \frac{1-\sqrt{5}}{4}, \nu_2 = \frac{1-\sqrt{5}}{4}$;

(b) SW kernel of 1|2|34 type – point B: $\nu_1 = \frac{1+\sqrt{5}}{4}, \nu_2 = \frac{1-3\sqrt{5}}{4}$,

while the vertex C corresponds to a quatrit kernel with two double degeneracy of 1|23|4 type: $\nu_1 = \nu_2 = \frac{1-\sqrt{15}}{4}$.

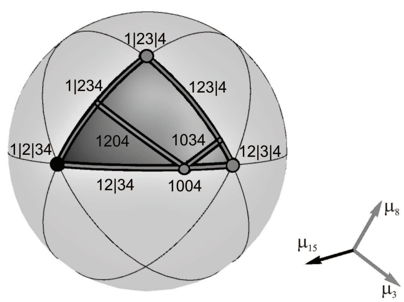

Figure 3. A quatrit moduli space represented by the Möbius spherical triangle $(2,3,3)$ on a unit sphere.

• **The singular SW kernels of a quatrit** • Among the above described SW kernels one can distinguish a set of special elements with a vanishing determinant. These singular quatrit kernels of are listed below in the accordance with the increasing singularity of the determinant:

- SW kernels with a simple root of the determinant:
  (a) 1-parameter family of 1204 type, $\frac{1}{3}(1-\sqrt{22}) \leqslant \nu < \frac{1}{2}(1-\sqrt{7})$,

$$\operatorname{spec}(\Delta_{(1204)}) = \left\{\frac{1-\nu+\sqrt{7+2\nu-3\nu^2}}{2}, \frac{1-\nu-\sqrt{7+2\nu-3\nu^2}}{2}, 0, \nu\right\}, \quad (61)$$

(b) 1-parameter family of 1034 type, $\frac{1}{6}\left(2-\sqrt{22}\right) \leqslant \nu < 0$,

$$\operatorname{spec}\left(\Delta_{(1034)}\right)=\left\{\frac{1-\nu+\sqrt{7+2\nu-3\nu^2}}{2}, 0, \nu, \frac{1-\nu-\sqrt{7+2\nu-3\nu^2}}{2}\right\}, \quad (62)$$

- SW kernel with double zero of determinant:

$$\operatorname{spec}\left(\Delta_{(1004)}\right)=\left\{\frac{1+\sqrt{7}}{2}, 0, 0, \frac{1-\sqrt{7}}{2}\right\}. \quad (63)$$

• **A quatrit moduli space as the Möbius spherical triangle** • As it was mentioned before, the spectrum of $\Delta^{(4)}(\nu_1, \nu_2)$ is in correspondence with points on a unit 2-sphere associated with expansion coefficients μ_3, μ_8 and μ_{15}:

$$\mu_3^2(\boldsymbol{\nu}) + \mu_8^2(\boldsymbol{\nu}) + \mu_{15}^2(\boldsymbol{\nu}) = 1,$$

which satisfy the inequalities:

$$\mu_3 \geqslant 0, \quad \mu_8 \geqslant \frac{\mu_3}{\sqrt{3}}, \quad \mu_{15} \geqslant \frac{\mu_8}{\sqrt{2}}.$$

Geometrically these constraints define one out of 24 possible spherical triangles with angles $(\pi/2, \pi/3, \pi/3)$ that tessellate a unit sphere. Repeated reflections in the sides of the triangles will tile a sphere exactly once. In accordance with Girard's theorem, a spherical excess of a triangle determines a solid angle: $\pi/2 + \pi/3 + \pi/3 - \pi = 4\pi/24$. Relation between "flat" representation of a quatrit moduli space (Fig. 2) and its spherical realization (Fig. 3) is demonstrated by the projection pattern in Fig. 4.

Concluding remark

The master equations (25) for kernels of the Wigner functions determine the first and second degrees polynomial $SU(N)$ invariants of N-dimensional system. The remaining $N-2$ algebraically independent invariants parametrize the moduli space of SW kernels. In the present article we establish relation between this moduli space and the orbit space of $SU(N)$ group. Next important issue is to clarify the role these unitary invariant moduli parameters play in dynamics of classical and quantum systems. With this aim in the forthcoming publication, a detailed analysis of the Kirillov–Kostant–Souriau symplectic 2-form for the whole family of the Wigner functions will be given.

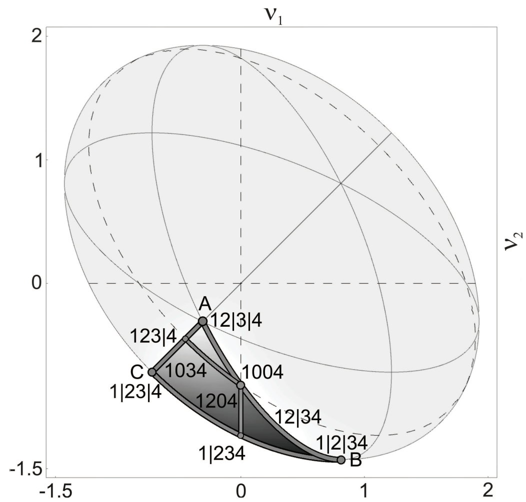

Figure 4. Mapping of the tiling of $\mathbb{S}_2(1)$ sphere by the Möbius triangles $(2, 3, 3)$ onto a subset of the plane (ν_1, ν_2). The dashed lines represent the degeneracies of the spectrum.

Appendix §A. Parametrization of the moduli space $\mathcal{P}_N(\boldsymbol{\nu})$

As it was mentioned in the main text, the Stratonovich–Weyl kernel can be parameterized by $N-2$ spherical angles. Each member of the Wigner functions family can be associated with a point of subspace $\mathcal{P}_N(\boldsymbol{\nu}) \subset \mathbb{S}_{N-2}(1)$, which is determined by the ordering of the eigenvalues of the Stratonovich–Weyl kernel. In order to define the $\mathcal{P}_N(\boldsymbol{\nu})$ corresponding to the descending ordering and by means of using kernel decomposition in Gell-Mann bases, let us represent the spectrum of the Stratonovich–Weyl kernel in the following form:

$$\pi_1 = \frac{1}{N}\left(1 + \sqrt{2}\,\kappa \sum_{s=2}^{N} \frac{\mu_{s^2-1}}{\sqrt{s\,(s-1)}}\right),$$

$$\vdots$$

$$\pi_i = \frac{1}{N}\left(1+\sqrt{2}\,\kappa\sum_{s=i+1}^{N}\frac{\mu_{s^2-1}}{\sqrt{s\,(s-1)}} - \kappa\sqrt{\frac{2\,(i-1)}{i}}\mu_{i^2-1}\right),$$
$$\vdots$$
$$\pi_N = \frac{1}{N}\left(1-\frac{N^2-1}{\sqrt{N+1}}\mu_{N^2-1}\right).$$

Introducing the conventional parametrization for a unit sphere $\mathbb{S}_{N-2}(1)$ in terms of spherical $N-2$ angles:

$$\begin{aligned} \mu_3 &= \sin\psi_1\cdots\sin\psi_{N-2}, \\ \mu_8 &= \sin\psi_1\cdots\sin\psi_{N-3}\cos\psi_{N-2}, \\ &\vdots \\ \mu_{i^2-1} &= \sin\psi_1\cdots\sin\psi_{N-i}\cos\psi_{N-i+1}, \\ &\vdots \\ \mu_{N^2-1} &= \cos\psi_1, \end{aligned} \tag{64}$$

$$\text{with}\quad \psi_i\in[0,\pi],\ i=\overline{1,N-3}\quad\text{and}\quad \psi_{N-2}\in[0,2\pi),$$

and demanding the descending order of the eigenvalues, we obtain the following constraints on the coefficients μ_i:

$$\mu_3 \geqslant 0, \tag{65}$$
$$\mu_{(i+1)^2-1} \geqslant \sqrt{\frac{i-1}{i+1}}\,\mu_{i^2-1},\quad i=\overline{2,N-1}. \tag{66}$$

Let us introduce the following notations:

$$\mathcal{P}_1 = \{\psi_1 = 0\},$$
$$\mathcal{P}_2^{(k)} = \begin{cases} \sin\psi_{N-k} = 0 \\ \sin\psi_{N-(k+1)}\cos\psi_{N-k} > 0 \\ \cot\psi_{N-i} \geqslant \sqrt{\frac{i-1}{i+1}}\,\cos\psi_{N-i+1} \\ 0<\psi_{i-k}<\pi,\quad i=\overline{k+1,N-1}, \end{cases} \tag{67}$$

$$\mathcal{P}_3 = \begin{cases} \sin\psi_{N-2} > 0 \\ \cos\psi_{N-2} \geqslant \frac{1}{\sqrt{3}}\sin\psi_{N-2} \\ \cot\psi_{N-i} \geqslant \sqrt{\frac{i-1}{i+1}}\,\cos\psi_{N-i+1} \\ 0 < \psi_{i-2} < \pi, \quad i = \overline{3, N-1} \\ 0 < \psi_{N-2} < 2\pi. \end{cases}$$

In the introduced notations substitution of expressions for μ_i in terms of the spherical angles ψ_i into (65) and (66) shows: if $k = 2, \cdots N-2$ is the biggest natural number for which $\sin\psi_{N-k} = 0$, if there is any, then the simplex is described by the restrictions $\mathcal{P}_2^{(k)} \subset \mathbb{S}_{N-(k+1)}(1)$ (these are some of $(N-(k+1))$-dimensional boundaries of the simplex); otherwise, if there is no such k, then the restrictions are $\mathcal{P}_3$. Hence, the simplex will be completely defined by

$$\mathcal{P} = \mathcal{P}_1 \cup \left(\bigcup_{k=2}^{N-2} \mathcal{P}_2^{(k)} \right) \cup \mathcal{P}_3. \tag{68}$$

Partially reducing the set of inequalities for $\mathcal{P}_3$, we get:

$$\mathcal{P}_3 = \begin{cases} 0 < \psi_{N-2} \leqslant \frac{\pi}{3} \\ 0 < \psi_{i-2} < \pi, \quad i = \overline{3, N-1} \\ \cot\psi_{N-i} \geqslant \sqrt{\frac{i-1}{i+1}}\cos\psi_{N-i+1}. \end{cases} \tag{69}$$

References

1. Johann v. Neumann, *Mathematische Grundlagen der Quantenmechanik*, Verlang von Julius Springer, Berlin, 1932.
2. H. Weyl, *Gruppentheorie und Quantenmechanik*, Hirzel-Verlag, Leipzig, 1928.
3. E. P. Wigner, *On the quantum correction for thermodynamic equilibrium.* — Phys. Rev. **40**, 749–759 (1932).
4. K. Husimi, *Some formal properties of the density matrix.* — Proc. Phys. Math. Soc. Japan, **22**, 264–314 (1940).
5. H. J. Groenewold *On the principles of elementary quantum mechanics.* — Physica **12**, 405–460 (1946).
6. J. E. Moyal, *Quantum mechanics as a statistical theory.* — Mathematical Proceedings of the Cambridge Philosophical Society **45**, 99–124 (1949).
7. G. Dito and D. Sternheimer, *Deformation quantization: genesis, developments and metamorphoses*, `arXiv: https://arxiv.org/abs/math/0201168` (2002).
8. E. C. G. Sudarshan, *Equivalence of semiclassical and quantum mechanical descriptions of statistical light beams.* — Phys. Rev. Lett. **10**, 277–279 (1963).

9. R.J.Glauber, *Coherent and incoherent states of the radiation field.* — Phys. Rev. **131**, 2766–2788 (1963).
10. M. Hillery, R. F. O'Connell, M. O. Scully, E. P. Wigner, *Distribution functions in physics: Fundamentals.* — Phys. Rep. **106**, 121–167 (1984).
11. D. J. Rowe, B. C. Sanders and H. de Guise, *Representations of the Weyl group and Wigner functions for SU(3).* — J. Math. Phys. **40**, 3604 (1999).
12. S. Chumakov, A. Klimov, K. B. Wolf, *Connection between two Wigner functions for spin systems.* — Phys. Rev. **A 61**, 034101 (2000).
13. M. A. Alonso, G. S. Pogosyan, K.B.Wolf, *Wigner functions for curved spaces.* — J. Math. Phys. **43**, 5857 (2002).
14. A. I. Lvovsky, M. G. Raymer, *Continuous-variable optical quantum-state tomography.* — Rev. Mod. Phys. **81**, 299–332 (2009).
15. A.B. Klimov, H. de Guise, *General approach to* $\mathfrak{SU}(n)$ *quasi-distribution functions.* — J. Phys. A. **43**, 402001 (2010).
16. I. Rigas, L. L. Sánchez-Soto, A. B. Klimov, J. Rehacek, Z. Hradil, *Orbital angular momentum in phase space.* — Ann. Phys. **326**, 426–439 (2011).
17. T. Tilma, M. J. Everitt, J. H. Samson, W. J. Munro, K. Nemoto, *Wigner functions for arbitrary quantum systems.* — Phys. Rev. Lett. **117**, 180401 (2016).
18. A. B. Klimov, J. L. Romero, H. de Guise, *Generalized SU(2) covariant Wigner functions and some of their applications.* — J. Phys. A **50**, 323001 (2017).
19. M. O. Scully, M. S. Zubairy, *Quantum Optics*, Cambridge University Press, Cambridge, 2006.
20. I. Bengtsson, K. E. Życzkowski, *Geometry of Quantum States. An Introduction to Quantum Entanglement*, Cambridge University Press, Cambridge, 2nd Edition, 2017.
21. R. P. Feynman, *Negative Probaility*, In Basil J. Hiley & D. Peat (eds.), Quantum Implications: Essays in Honour of David Bohm. Methuen., 235–248 (1987).
22. R.L. Stratonovich, *On distributions in representation space.* — Sov. Phys. JETP **4**, No. 6, 891–898 (1957).
23. C. Briff, A. Mann, *Phase-space formulation of quantum mechanics and quantum-state reconstruction for physical systems with Lie-group symmetries.* — Phys. Rev. **59**, 971 (1999).
24. A. Khvedelidze and V. Abgaryan, *On the Family of Wigner Functions for* $\boldsymbol{N}$*-Level Quantum System*, `arXiv: https://arxiv.org/abs/1708.05981`, 2018.
25. A. A. Kirillov, *Lectures on the Orbit Method*, Graduate Studies in Mathematics, **64**, Amer. Math. Soc. 2004.
26. D. Weingarten, *Asymptotic behavior of group integrals in the limit of infinite rank.* — J. Mat. Phys. **19**, 999 (1978).
27. B. Collins, *Moments and cumulants of polynomial random variables on unitary groups, the Itzykson-Zuber integral, and free probability.* — Int. Math. Res. Not. **17**, 953 (2003).
28. B. Collins and P. Sniady, *Integration with respect to the Haar measure on unitary, orthogonal and symplectic group.* — Comm. Math. Phys. **264**, 3, 773 (2006).
29. A. Luis, *A SU(3) Wigner function for three-dimensional systems.* — J. Phys. A **41**, 495302 (2008).

30. V. I. Arnold, *Remarks on eigenvalues and eigenvectors of Hermitian matrices, Berry phase, adiabatic connections and quantum Hall effect.* — Selecta Mathematica, New Series **1**, No. 1 (1995).

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Laboratory of Information Technologies
Joint Institute for Nuclear Research
141980 Dubna, Russia
E-mail: vahagnab@googlemail.com

A Razmadze Mathematical Institute
Iv. Javakhishvili, Tbilisi State University
Tbilisi, Georgia
Institute of Quantum Physics and Engineering Technologies
Georgian Technical University
Tbilisi, Georgia
Laboratory of Information Technologies
Joint Institute for Nuclear Research
141980 Dubna, Russia
E-mail: akhved@jinr.ru

Laboratory of Information Technologies
Joint Institute for Nuclear Research
141980 Dubna, Russia
E-mail: astghik@jinr.ru