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ON THE GROUP OF INFINITE p -ADIC MATRICES WITH INTEGER ELEMENTS

ABSTRACT. Let G be an infinite-dimensional real classical group containing the complete unitary group (or the complete orthogonal group) as a subgroup. Then G generates a category of double cosets (train), and any unitary representation of G can be canonically extended to the train. We prove a technical lemma on the complete group GL of infinite p -adic matrices with integer coefficients; this lemma implies that the phenomenon of an automatic extension of unitary representations to a train is valid for infinite-dimensional p -adic groups.

§1. THE STATEMENT

1.1. Notation. Denote by \mathbb{Q}_p the field of p -adic numbers, by \mathbb{O}_p the ring of p -adic integers. We consider infinite matrices $g = \{g_{ij}\}$, where $i, j \in \mathbb{N}$, over \mathbb{O}_p . We define three versions of the group $\mathrm{GL}(\infty)$ over \mathbb{O}_p .

1) Our main object is the group $\mathrm{GL}(\infty, \mathbb{O}_p)$ which consists of invertible matrices satisfying two conditions:

(A^*) for each i we have $\lim_{j \rightarrow \infty} |g_{ij}| = 0$;

(B^*) for each j we have $\lim_{i \rightarrow \infty} |g_{ij}| = 0$.

2) We also consider the larger group $\overline{\mathrm{GL}}(\infty, \mathbb{O}_p)$ consisting of invertible matrices satisfying condition (A^*).

3) We regard compact groups $\mathrm{GL}(n, \mathbb{O}_p)$ as subgroups of $\mathrm{GL}(\infty, \mathbb{O}_p)$ consisting of $(n + \infty)$ -block matrices of the form $\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}$.

We say that an infinite matrix g is *finitary* if $g - 1$ has only finitely many nonzero matrix elements. Denote by $\mathrm{GL}_{\mathrm{fin}}(\infty, \mathbb{O}_p)$ the group of invertible finitary infinite matrices over \mathbb{O}_p ; this group is an inductive limit

$$\mathrm{GL}_{\mathrm{fin}}(\infty, \mathbb{O}_p) = \varinjlim \mathrm{GL}(n, \mathbb{O}_p)$$

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and is equipped with the inductive limit topology: a function on $\mathrm{GL}_{\mathrm{fin}}(\infty, \mathbb{O}_p)$ is continuous if and only if its restriction to every prelimit subgroup is continuous.

Remark. The group $\overline{\mathrm{GL}}(\infty, \mathbb{O}_p)$ appears in the context of [11]. However, $\mathrm{GL}(\infty, \mathbb{O}_p)$ is a more interesting object from the point of view of [12].

1.2. The result of the paper. Denote by θ_j the following matrix:

$$\theta_j = \begin{pmatrix} 0 & 1_j & 0 \\ 1_j & 0 & 0 \\ 0 & 0 & 1_\infty \end{pmatrix} \in \mathrm{GL}(\infty, \mathbb{O}_p),$$

where 1_j denotes the unit matrix of size j . The purpose of this note is to prove the following statement.

Lemma 1.1. *Consider a unitary representation ρ of the group $\mathrm{GL}(\infty, \mathbb{O}_p)$ in a Hilbert space H . Denote by $H^{\mathrm{GL}} \subset H$ the space of all vectors fixed by all operators $\rho(g)$. Then the sequence $\rho(\theta_j)$ weakly converges to the orthogonal projection to H^{GL} .*

Since $\mathrm{GL}(\infty, \mathbb{O}_p)$ is dense in $\overline{\mathrm{GL}}(\infty, \mathbb{O}_p)$, we get the following corollary.

Corollary 1.2. *The same statement holds for the group $\overline{\mathrm{GL}}(\infty, \mathbb{O}_p)$.*

1.3. Variations. Define the orthogonal group $\mathrm{O}(\infty, \mathbb{O}_p)$ as the subgroup in $\mathrm{GL}(\infty, \mathbb{O}_p)$ consisting of all matrices with $g^t g = 1$, where t denotes the transpose. Denote by J the 2×2 matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ over \mathbb{O}_p . Denote

$$I := \begin{pmatrix} J & 0 & \dots \\ 0 & J & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}. \quad (1.1)$$

Denote by $\mathrm{Sp}(\infty, \mathbb{O}_p)$ the subgroup in $\mathrm{GL}(\infty, \mathbb{O}_p)$ consisting of all matrices with $g^t I g = 1$.

Lemma 1.1 (with the same proof) holds for the groups $\mathrm{O}(\infty, \mathbb{O}_p)$ and $\mathrm{Sp}(\infty, \mathbb{O}_p)$; for $\mathrm{Sp}(\infty, \mathbb{O}_p)$, we must consider the sequence $\theta_{2m} \in \mathrm{Sp}(\infty, \mathbb{O}_p)$.

1.4. Admissibility in the sense of Olshanski. We also prove the following technical statement. Consider a unitary representation ρ of the group $\mathrm{GL}_{\mathrm{fin}}(\infty, \mathbb{O}_p)$ in a Hilbert space H . Denote by H_m the space of $\mathrm{GL}_{\mathrm{fin}}^{[m]}(\infty, \mathbb{O}_p)$ -invariant vectors. We say that a representation ρ is *admissible* (see [17]) if the subspace $\cup_{m=0}^\infty H_m$ is dense in H .

Lemma 1.3. *The following conditions for a representation ρ of the group $\mathrm{GL}_{\mathrm{fin}}(\infty, \mathbb{O}_p)$ are equivalent:*

- ρ admits a continuous extension to $\mathrm{GL}(\infty, \mathbb{O}_p)$;
- ρ is admissible.

1.5. The structure of the paper. Lemma 1.1 seems rather technical, however, it implies that $\mathrm{GL}(\infty, \mathbb{O}_p)$ is a heavy group in the sense of [8, Chap. VIII]. This implies numerous “multiplicativity theorems,” an example is discussed in the next section. Lemma 1.1 is proved in Sec. 3, and Lemma 1.3 is proved in Sec. 4.

§2. INTRODUCTION. AN EXAMPLE OF A MULTIPLICATIVITY THEOREM

2.1. Initial data. Denote by $S_{\mathrm{fin}}(\infty)$ the group of all finitely supported permutations of \mathbb{N} . Fix a ring R . Let G be a subgroup in $\mathrm{GL}_{\mathrm{fin}}(\infty, R)$ and K be a subgroup in G . Assume that K contains $S_{\mathrm{fin}}(\infty)$ embedded as the group of all 0–1 matrices.

Examples. a) $G = K = S_{\mathrm{fin}}(\infty)$.

b) $G = \mathrm{GL}_{\mathrm{fin}}(\infty, \mathbb{R})$, $K = \mathrm{O}_{\mathrm{fin}}(\infty)$.

c) Let R be the algebra of 2×2 real matrices. Let $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in R$.

We consider the group $G = \mathrm{Sp}_{\mathrm{fin}}(2\infty, \mathbb{R})$ of matrices over R preserving the skew-symmetric bilinear form with matrix I given by (1.1). The subgroup $K = \mathrm{U}_{\mathrm{fin}}(\infty)$ consists of the matrices whose entries have the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in R$.

d) $G = \mathrm{GL}_{\mathrm{fin}}(\infty, \mathbb{Q}_p)$, $K = \mathrm{GL}_{\mathrm{fin}}(\infty, \mathbb{O}_p)$.

Remark. Denote by $G(\alpha)$ (respectively, $K(\alpha)$) the subgroup in G (respectively, K) consisting of all $(\alpha + \infty)$ -block matrices of the form $\begin{pmatrix} w & 0 \\ 0 & 1 \end{pmatrix}$. These groups contain at least the finite symmetric group $S(\alpha)$. Then

$$G = \varinjlim G(\alpha), \quad K = \varinjlim K(\alpha). \quad (2.1)$$

2.2. The multiplication of double cosets. We fix n and consider the product \tilde{G} of n copies of the group G :

$$\tilde{G} = G \times G \times \cdots \times G.$$

We write elements of this product as

$$g = \{g^{(l)}\} := (g^{(1)}, \dots, g^{(n)}), \quad \text{where } g_j \in G. \quad (2.2)$$

Consider the diagonal subgroup $K \subset \tilde{G}$, i.e., the group whose elements are collections

$$(u, \dots, u), \quad \text{where } u \in K.$$

Let $\alpha = 0, 1, 2, \dots$. Denote by K^α the subgroup of K consisting of all matrices having the form $\begin{pmatrix} 1_\alpha & 0 \\ 0 & u \end{pmatrix} \in K$. Denote by

$$K^\alpha \backslash \tilde{G} / K^\alpha$$

the space of double cosets, i.e., the space of collections (2.2) defined up to the equivalence

$$(g^{(1)}, \dots, g^{(n)}) \sim \left(\begin{pmatrix} 1_\alpha & 0 \\ 0 & u \end{pmatrix} g^{(1)} \begin{pmatrix} 1_\beta & 0 \\ 0 & v \end{pmatrix}, \dots, \begin{pmatrix} 1_\alpha & 0 \\ 0 & u \end{pmatrix} g^{(n)} \begin{pmatrix} 1_\beta & 0 \\ 0 & v \end{pmatrix} \right),$$

where $u, v \in K$.

For every α , we define a sequence $\theta_j^{[\alpha]}$ by the formula

$$\theta_j^{[\alpha]} = \begin{pmatrix} 1_\alpha & 0 & 0 & 0 \\ 0 & 0 & 1_j & 0 \\ 0 & 1_j & 0 & 0 \\ 0 & 0 & 0 & 1_\infty \end{pmatrix} \in K^\alpha \cap S_{\text{fin}}(\infty).$$

The following statements (a)–(c) can be verified in a straightforward way (see [4] for a formal proof for $G = S_{\text{fin}}(\infty)$, which is valid in the general case).

a) Let $\mathfrak{g}_1 \in K^\alpha \backslash \tilde{G} / K^\beta$, $\mathfrak{g}_2 \in K^\beta \backslash \tilde{G} / K^\gamma$. Let $g_1, g_2 \in \tilde{G}$ be representatives of these double cosets. Then the sequence

$$K^\alpha g_1 \theta_j^{[\beta]} g_2 K^\gamma \in K^\alpha \backslash \tilde{G} / K^\gamma$$

of double cosets is eventually constant. Moreover, the limit does not depend on the choice of representatives $g_1 \in \mathfrak{g}_1$, $g_2 \in \mathfrak{g}_2$.

b) Thus we get a multiplication $(\mathfrak{g}_1, \mathfrak{g}_2) \mapsto \mathfrak{g}_1 \circ \mathfrak{g}_2$,

$$K^\alpha \backslash \tilde{G} / K^\beta \times K^\beta \backslash \tilde{G} / K^\gamma \rightarrow K^\alpha \backslash \tilde{G} / K^\gamma,$$

which can be described as follows. We write representatives $g_1 \in \mathfrak{g}_1$, $g_2 \in \mathfrak{g}_2$ as block $(\alpha + \infty) \times (\beta + \infty)$ matrices and collections of $(\beta + \infty) \times (\gamma + \infty)$

matrices

$$\{g_1^{(l)}\} = \left\{ \begin{pmatrix} a_1^{(l)} & b_1^{(l)} \\ c_1^{(l)} & d_1^{(l)} \end{pmatrix} \right\}, \quad g_2^{(l)} = \left\{ \begin{pmatrix} a_2^{(l)} & b_2^{(l)} \\ c_2^{(l)} & d_2^{(l)} \end{pmatrix} \right\};$$

then a representative of $\mathfrak{g}_1 \circ \mathfrak{g}_2$ is given by

$$\{(g_1 \odot g_2)^{(l)}\} := \left\{ \begin{pmatrix} a_1^{(l)} & b_1^{(l)} & 0 \\ c_1^{(l)} & d_1^{(l)} & 0 \\ 0 & 0 & 1_\infty \end{pmatrix} \begin{pmatrix} a_2^{(l)} & 0 & b_2^{(l)} \\ 0 & 1_\infty & 0 \\ c_2^{(l)} & 0 & d_2^{(l)} \end{pmatrix} \right\}. \quad (2.3)$$

The size of these matrices is

$$(\alpha + [\infty + \infty]) \times (\gamma + [\infty + \infty]) = (\alpha + \infty) \times (\gamma + \infty),$$

so this collection can be regarded as a representative of an element of the space $K^\alpha \setminus \tilde{G}/K^\gamma$. More precisely, we must choose arbitrary bijections σ_1, σ_2 between the disjoint union $\mathbb{N} \coprod \mathbb{N}$ and \mathbb{N} to get an element of the desired size:

$$\left\{ \begin{pmatrix} 1_\alpha & 0 \\ 0 & \sigma_1 \end{pmatrix} (g_1 \odot g_2)^{(l)} \begin{pmatrix} 1_\alpha & 0 \\ 0 & \sigma_2 \end{pmatrix}^{-1} \right\}.$$

The double coset containing this matrix does not depend on the choice of σ_1, σ_2 .

c) *The product of double cosets is associative, i.e., for*

$$\mathfrak{g}_1 \in K^\alpha \setminus \tilde{G}/K^\beta, \quad \mathfrak{g}_2 \in K^\beta \setminus \tilde{G}/K^\gamma, \quad \mathfrak{g}_3 \in K^\gamma \setminus \tilde{G}/K^\delta,$$

we have

$$(\mathfrak{g}_1 \circ \mathfrak{g}_2) \circ \mathfrak{g}_3 = \mathfrak{g}_1 \circ (\mathfrak{g}_2 \circ \mathfrak{g}_3).$$

Remark. The formula for the \odot -product

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \odot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} := \begin{pmatrix} a_1 a_2 & b_1 & a_1 b_2 \\ c_1 a_2 & d_1 & c_1 b_2 \\ c_2 & 0 & d_2 \end{pmatrix}$$

of matrices initially arose as a formula for a product of operator colligations, see [1, 2].

2.3. Multiplicativity theorems. Next, consider a unitary representation ρ of \tilde{G} in a Hilbert space H . Denote by $H_\alpha \subset H$ the subspace of all K^α -fixed vectors. Denote by P_α the orthogonal projection to H_α . For a double coset $\mathfrak{g} \in K^\alpha \backslash \tilde{G}/K^\beta$, we define an operator

$$\tilde{\rho}(\mathfrak{g}) : H_\beta \rightarrow H_\alpha$$

by the formula

$$\tilde{\rho}_{\alpha,\beta}(\mathfrak{g}) := P_\alpha \rho(g) \Big|_{H_\beta}.$$

Remark. The operator $\tilde{\rho}(g)$ actually depends only on the double coset containing g . Indeed, for $\xi \in H_\beta$, $\eta \in H_\alpha$ and $\kappa_1 \in K^\alpha$, $\kappa_2 \in K^\beta$, we have

$$\langle \rho(\kappa_1 g \kappa_2) \xi, \eta \rangle_{H_\alpha} = \langle \rho(g) \rho(\kappa_2) \xi, \rho(\kappa_1^{-1}) \eta \rangle_{H_\alpha} = \langle \rho(g) \xi, \eta \rangle_{H_\alpha}.$$

This expression does not depend on κ_1, κ_2 .

Remark. Apparently, at this place we must require that the prelimit groups $K(\alpha)$ in (2.1) are compact. Otherwise I see no reason to hope for the existence of nonzero fixed vectors.

Theorem 2.1. *Let $G = \mathrm{GL}_{\mathrm{fin}}(\infty, \mathbb{Q}_p)$ and $K = \mathrm{GL}_{\mathrm{fin}}(\infty, \mathbb{O}_p)$. For any α, β, γ and*

$$\mathfrak{g}_1 \in K^\alpha \backslash \tilde{G}/K^\beta, \quad \mathfrak{g}_2 \in K^\beta \backslash \tilde{G}/K^\gamma,$$

we have

$$\tilde{\rho}_{\alpha,\beta}(\mathfrak{g}_1) \tilde{\rho}_{\beta,\gamma}(\mathfrak{g}_2) = \tilde{\rho}_{\alpha,\gamma}(\mathfrak{g}_1 \circ \mathfrak{g}_2). \quad (2.4)$$

Proof. First, assume that the restriction of ρ to $K = \mathrm{GL}_{\mathrm{fin}}(\infty, \mathbb{O}_p)$ is continuous in the topology of $\mathrm{GL}(\infty, \mathbb{O}_p)$. Denote by $\bar{\rho}_{\alpha,\beta}(\mathfrak{g})$ the following operator in H :

$$\bar{\rho}_{\alpha,\beta}(\mathfrak{g}) := P_\alpha \rho(g) P_\beta, \quad \text{where } g \in \mathfrak{g}.$$

Representing it in a block form

$$H_\beta \oplus H_\beta^\perp \rightarrow H_\alpha \oplus H_\alpha^\perp,$$

we get the expression

$$\bar{\rho}_{\alpha,\beta}(\mathfrak{g}) := \begin{pmatrix} \tilde{\rho}_{\alpha,\beta}(\mathfrak{g}) & 0 \\ 0 & 0 \end{pmatrix}.$$

Relation (2.4) is equivalent to

$$\bar{\rho}_{\alpha,\beta}(\mathfrak{g}_1) \bar{\rho}_{\beta,\gamma}(\mathfrak{g}_2) = \bar{\rho}_{\alpha,\gamma}(\mathfrak{g}_1 \circ \mathfrak{g}_2). \quad (2.5)$$

We have

$$\begin{aligned}\bar{\rho}_{\alpha,\beta}(\mathfrak{g}_1)\bar{\rho}_{\beta,\gamma}(\mathfrak{g}_2) &= P_\alpha \rho(g_1) P_\beta \rho(g_2) P_\gamma = P_\alpha \rho(g_1) \left(\lim_{j \rightarrow \infty} \rho(\theta_j^{[\beta]}) \right) \rho(g_2) P_\gamma \\ &= \lim_{j \rightarrow \infty} P_\alpha \rho(g_1) \rho(\theta_j^{[\beta]}) \rho(g_2) P_\gamma = \lim_{j \rightarrow \infty} P_\alpha \rho(g_1 \theta_j^{[\beta]} g_2) P_\gamma\end{aligned}$$

(here $\lim_{j \rightarrow \infty}$ denotes the weak limit). The sequence $g_1 \theta_j^{[\beta]} g_2$ is eventually constant, and we get the desired expression

$$P_\alpha \rho(g_1 \odot g_2) P_\gamma = \bar{\rho}(\mathfrak{g}_1 \circ \mathfrak{g}_2).$$

Next, let ρ be arbitrary. The group $\mathrm{GL}(m, \mathbb{O}_p)$ centralizes $\mathrm{GL}^{[m]}(\infty, \mathbb{O}_p)$, hence H_m is $\mathrm{GL}(m, \mathbb{O}_p)$ -invariant. For $n > m$, the space H_n is invariant with respect to $\mathrm{GL}(n, \mathbb{O}_p)$ and, therefore, with respect to the smaller subgroup $\mathrm{GL}(m, \mathbb{O}_p)$. Hence $\cup_{j=0}^\infty H_j$ is invariant with respect to $\mathrm{GL}(m, \mathbb{O}_p)$. This is valid for all m , so the subspace is invariant with respect to the inductive limit $\mathrm{GL}_{\mathrm{fin}}(\infty, \mathbb{O}_p)$. Thus we get a unitary representation of $\mathrm{GL}_{\mathrm{fin}}(\infty, \mathbb{O}_p)$ in the closure H_* of $\cup_{j=0}^\infty H_j$. By Lemma 1.3, this representation is continuous in the topology of $\mathrm{GL}(\infty, \mathbb{O}_p)$, and we arrive at the previous case.

In H_*^\perp we have no $\mathrm{GL}(m, \mathbb{O}_p)$ -fixed vectors, and the statement is trivial. \square

The crucial point here is Lemma 1.1. This picture is parallel to real classical groups and symmetric groups [8–10, 13, 15–17]. A further discussion of the p -adic case is contained in [12].

Remark. It can be shown that in the p -adic case the functions $\tilde{\rho}_{\alpha,\beta}(\mathfrak{g})$ do not separate elements of $K^\alpha \setminus G/K^\beta$. A similar phenomenon is known for finite fields, see [17].

Remark. Lemma 1.1 was formulated in [12] as Corollary 6.4, but its proof there is incomplete due to an incorrect definition of a topology on $\mathrm{GL}(\infty, \mathbb{O}_p)$.

§3. PROOF OF LEMMA 1.1

3.1. The symmetric group. Denote by $S(\infty)$ the group of all permutations of the set \mathbb{N} of positive integers. It has a structure of a totally disconnected topological group defined by the following condition: the stabilizers of finite subsets in \mathbb{N} form a neighborhood basis of open subgroups in $S(\infty)$. Denote by $S^{[m]}(\infty)$ the group stabilizing the points $1, \dots, m$.

Clearly, the open subgroups $S^{[m]}(\infty)$ form a basis of neighborhoods of the identity in $S(\infty)$.

Remark. This is the unique separable topology on $S(\infty)$ compatible with the group structure. Recall that a *Polish group* is a topological group that is homeomorphic to a complete separable metric space. There is a collection of statements on the rigidity of choosing a Polish topology on a group, see, e.g., [5, Sec. 3.2]. For instance, if two Polish topologies on a group generate the same Borel structures, then the topologies coincide. Of course, additive groups of all separable Banach spaces (they are Polish groups) are isomorphic as abstract groups. But the existence of such isomorphisms requires an application of the choice axiom, and isomorphisms are not Borel.

For a countable set Ω , we denote by $S(\Omega)$ the group of all permutations of Ω ; of course, $S(\Omega) \simeq S(\infty)$.

3.2. Induced representations. Let G be a totally disconnected group acting transitively on a countable set X , let R be the stabilizer of a point x_0 , and let ν be a unitary representation of R in a Hilbert space H . Then we can define the *induced representation* $\text{Ind}_R^G(\nu)$ of the group G in the usual way (see, e.g., [6, Sec. 13]). Namely, consider the space $G \times H$ and denote by B its quotient with respect to the equivalence relation

$$(x, r) \simeq (xr, \rho(r^{-1})h), \quad \text{where } r \text{ ranges in } R.$$

Then we have a “fiber bundle” $B \rightarrow X = G/R$ whose fibers H_x are copies of the space H . Transformations $(x, h) \mapsto (gx, h)$ induce transformations of B . Now we consider the space of “sections” ψ that send each point x to a vector $\psi(x) \in H_x$. We define the inner product of sections by the formula

$$\langle \psi_1, \psi_2 \rangle = \sum_x \langle \psi_1(x), \psi_2(x) \rangle_{H_x}.$$

In this way we obtain a Hilbert space; the group G acts on B and, therefore, on the space of sections. This determines a unitary representation of G .

According to the Lieberman theorem [7] (see also expositions in [8, 17]), any irreducible unitary representation of $S(\infty)$ is induced from an irreducible representation of a subgroup of the type $S(m) \times S^{[m]}(\infty)$ trivial on the factor $S(\infty - m)$. We need the following fact (see [8, Corollary VIII.1.5]), which immediately follows from the Lieberman theorem.

Lemma 3.1. *For any unitary representation ρ of $S(\infty)$, the sequence $\rho(\theta_j)$ weakly converges to the orthogonal projection to the space of vectors fixed by all operators $\rho(g)$.*

3.3. Oligomorphic groups. Recall that a closed subgroup G in $S(\Omega)$ is called *oligomorphic* if it has finitely many orbits on each finite product $\Omega \times \cdots \times \Omega$. We need the following Tsankov theorem [18].

Theorem 3.2. *Any unitary representation of an oligomorphic group G is a (countable or finite) direct sum of irreducible representations. For any irreducible representation ρ of G there are open subgroups $R \subset \tilde{R}$ such that R is a normal subgroup of finite index in \tilde{R} and*

$$\rho = \text{Ind}_R^G(\nu), \quad (3.1)$$

where ν is an irreducible representation of \tilde{R} trivial on R .

Corollary 3.3. *Any irreducible representation of an oligomorphic group G is a subrepresentation of a quasiregular representation in ℓ^2 on some homogeneous space G/R , where R is an open subgroup in G .*

Proof. Let τ be a unitary representation of the group \tilde{R}/R . Denote by τ_\circ the same representation regarded as a representation of \tilde{R} trivial on R . Denote by Reg the regular representation of \tilde{R}/R . Denote by τ^0 the trivial (one-dimensional) representation of \tilde{R}/R .

It is easy to see that

$$\text{Ind}_{\tilde{R}}^{\tilde{R}}(\tau_\circ^0) = \text{Reg}_\circ.$$

Let ν be as above. Then ν is a subrepresentation of Reg_\circ ; therefore, the representation ρ given by (3.1) is a subrepresentation of

$$\text{Ind}_R^G(\text{Ind}_{\tilde{R}}^{\tilde{R}}(\tau_\circ^0)) = \text{Ind}_R^G(\tau_\circ^0).$$

The last representation is the quasiregular representation of the group G in $\ell^2(G/R)$. \square

3.4. Definitions. A) MODULES. Denote by \mathbb{Z}_{p^k} the residue rings $\mathbb{Z}/p^k\mathbb{Z}$. A module over \mathbb{Z}_{p^k} is nothing but an Abelian p -group whose elements have orders $\leq p^k$.

The ring of p -adic integers \mathbb{O}_p is the inverse limit

$$\mathbb{O}_p = \varprojlim \mathbb{Z}_{p^k}. \quad (3.2)$$

The reduction of a p -adic integer x modulo p^k is denoted by

$$((x))_{p^k} \in \mathbb{Z}_{p^k}.$$

We will use the same notation for reductions of vectors and matrices.

For each k define a \mathbb{Z}_{p^k} -module $V(\mathbb{Z}_{p^k})$ as the space of all sequences $z = (z_1, z_2, \dots)$ where $z_j \in \mathbb{Z}_{p^k}$ and $z_l = 0$ for sufficiently large l . We equip this space with the discrete topology.

Next, we define an \mathbb{O}_p -module $V(\mathbb{O}_p)$ as the space of all sequences $z = (z_1, z_2, \dots)$ where $z_j \in \mathbb{O}_p$ and $|z_j| \rightarrow \infty$ as $j \rightarrow \infty$; in other words,

$$V(\mathbb{O}_p) = \varprojlim V(\mathbb{Z}_{p^k}).$$

We equip this space with the projective limit topology. Thus, a sequence $z^{(l)} \in V(\mathbb{O}_p)$ converges if all reductions $((z^{(l)}))_{p^k} \in V(\mathbb{Z}_{p^k})$ are eventually constant. The same topology is induced by the norm

$$\|z\| := \max_j |z_j|.$$

We also consider the “dual” modules $V^\circ(\mathbb{Z}_{p^k})$, $V^\circ(\mathbb{O}_p)$ consisting of vector-columns satisfying the same properties.

B) THE GROUPS $\mathrm{GL}(\infty, \mathbb{Z}_{p^k})$. We define $\mathrm{GL}(\infty, \mathbb{Z}_{p^k})$ as the group of all infinite matrices over \mathbb{Z}_{p^k} such that each row and each column contains only finitely many nonzero elements. The group $\mathrm{GL}(\infty, \mathbb{Z}_{p^k})$ acts by automorphisms on the module $V(\mathbb{Z}_{p^k}) \oplus V^\circ(\mathbb{Z}_{p^k})$:

$$g : (v, w^\circ) \rightarrow (vg, g^{-1}w^\circ).$$

Thus we have an embedding into a symmetric group:

$$\mathrm{GL}(\infty, \mathbb{Z}_{p^k}) \rightarrow S(V(\mathbb{Z}_{p^k}) \oplus V^\circ(\mathbb{Z}_{p^k})).$$

We equip $\mathrm{GL}(\infty, \mathbb{Z}_{p^k})$ with the induced topology. For any collection of vectors $v_1, \dots, v_l \in V$ and covectors $w_1^\circ, \dots, w_n^\circ$, its stabilizer

$$G(v_1, \dots, v_l; w_1^\circ, \dots, w_n^\circ) \tag{3.3}$$

is an open subgroup in $\mathrm{GL}(\infty, \mathbb{Z}_{p^k})$. By definition, such subgroups form a basis of neighborhoods of the identity in our group.

Next, for each m consider the subgroup $\mathrm{GL}^{[m]}(\infty, \mathbb{Z}_{p^k}) \subset \mathrm{GL}(\infty, \mathbb{Z}_{p^k})$ consisting of all matrices of the form $\begin{pmatrix} 1_m & 0 \\ 0 & * \end{pmatrix}$. This group has the form $G(e_1, \dots, e_m; f_1^\circ, \dots, f_m^\circ)$, where e_j is the standard basis in V and f_j° is the standard basis in V° . Since the vectors and covectors v_i and w_j° in (3.3)

actually have only finitely many nonzero coordinates, each stabilizer $G(\dots)$ contains some group $\mathrm{GL}^{[m]}(\infty, \mathbb{Z}_{p^k})$.

Thus, the subgroups $\mathrm{GL}^{[m]}(\infty, \mathbb{Z}_{p^k})$ form a basis of neighborhoods of the identity in our group.

We can also define the topology in the following way. A sequence g_l converges to g if for every i the sequence of the i th rows (respectively, columns) of g_l coincides with the i th row (respectively, column) of g for sufficiently large l .

C) THE GROUP $\mathrm{GL}(\infty, \mathbb{O}_p)$. We have natural homomorphisms of rings $\mathbb{Z}_{p^k} \rightarrow \mathbb{Z}_{p^{k-1}}$ and, therefore, homomorphisms of groups

$$\mathrm{GL}(\infty, \mathbb{Z}_{p^k}) \rightarrow \mathrm{GL}(\infty, \mathbb{Z}_{p^{k-1}}).$$

We define the group $\mathrm{GL}(\infty, \mathbb{O}_p)$ as the projective limit

$$\mathrm{GL}(\infty, \mathbb{O}_p) := \varprojlim \mathrm{GL}(\infty, \mathbb{Z}_{p^k}).$$

In other words, this group consists of all infinite matrices g over \mathbb{O}_p such that $((g))_{p^k} \in \mathrm{GL}(\infty, \mathbb{Z}_{p^k})$ for all k .

We equip $\mathrm{GL}(\infty, \mathbb{O}_p)$ with the projective limit topology. A sequence $g^{(j)}$ converges to g if $((g^{(j)}))_{p^k} \in \mathrm{GL}(\infty, \mathbb{Z}_{p^k})$ converges to $((g))_{p^k}$ for all k .

D) OPEN SUBGROUPS IN $\mathrm{GL}(\infty, \mathbb{O}_p)$. For nonnegative integers m, k , we introduce the subgroups $\mathrm{GL}_k^{[m]}(\infty, \mathbb{O}_p)$ consisting of $(m + \infty)$ -block matrices of the form

$$\begin{pmatrix} 1 + p^k A & p^k B \\ p^k C & D \end{pmatrix},$$

where A, B, C, D are matrices over \mathbb{O}_p . These subgroups are open and form a basis of neighborhoods of the identity.

We define the *congruence subgroup* $\mathrm{GL}_k(\infty, \mathbb{O}_p)$ in $\mathrm{GL}(\infty, \mathbb{O}_p)$ as the subgroup consisting of matrices of the form $1 + p^k Q$ where Q is a matrix over \mathbb{O}_p (the congruence subgroups are not open).

3.5. Lemmas. Next, we apply the following general statement, see [8, Proposition VII.1.3].

Proposition 3.4. *Let G be a topological group and $G_1 \supset G_2 \supset \dots$ be a sequence of subgroups such that any neighborhood of the identity in G contains a subgroup G_j . Let ρ be a unitary representation of G in a Hilbert space H . Denote by H_k the space of vectors invariant with respect to G_k . Then $\cup H_k$ is dense in H .*

Corollary 3.5. *Any unitary representation ρ_j of $\mathrm{GL}(\infty, \mathbb{O}_p)$ can be decomposed into a direct sum $\bigoplus_{k=1}^{\infty} \rho_k$ where ρ_k is trivial on the congruence subgroup $\mathrm{GL}_k(\infty, \mathbb{O}_p)$.*

Proof. We apply Proposition 3.4 to the group $\mathrm{GL}(\infty, \mathbb{O}_p)$ and the sequence of congruence subgroups $\mathrm{GL}_k(\infty, \mathbb{O}_p)$. Since $\mathrm{GL}_k(\infty, \mathbb{O}_p)$ is a normal subgroup, for $h \in H_k$, $g \in \mathrm{GL}(\infty, \mathbb{O}_p)$, $r \in \mathrm{GL}_k(\infty, \mathbb{O}_p)$ we have

$$\rho(r) \rho(g) h = \rho(g) \rho(g^{-1} r g) h.$$

Since the congruence subgroup is normal, $g^{-1} r g \in \mathrm{GL}_k(\infty, \mathbb{O}_p)$, whence

$$\rho(g^{-1} r g) h = h,$$

i.e., $h \in H_k$. Therefore, the subspace H_k is invariant with respect to the whole group $\mathrm{GL}(\infty, \mathbb{O}_p)$, and the congruence subgroup acts in H_k trivially. Thus,

$$H = \bigoplus_{k=1}^{\infty} (H_k \ominus H_{k-1}).$$

In each space $H_k \ominus H_{k-1}$ we have an action of $\mathrm{GL}(\infty, \mathbb{Z}_{p^k})$. \square

Thus, it suffices to prove Lemma 1.1 for the groups $\mathrm{GL}(\infty, \mathbb{Z}_{p^k})$.

Recall that we can regard $S(\infty)$ as a group of 0-1 matrices.

Lemma 3.6. *For any m , the group $\mathrm{GL}(\infty, \mathbb{Z}_{p^k})$ is generated by the subgroups $S(\infty)$ and $\mathrm{GL}^{[m]}(\infty, \mathbb{Z}_{p^k})$.*

Proof. Consider the subgroup G generated by these subgroups. Clearly, G contains all groups $\mathrm{GL}(n, \mathbb{O}_p)$. Indeed, for $y \in \mathrm{GL}(n, \mathbb{O}_p)$, we have $\theta_N y \theta_N^{-1} \in \mathrm{GL}^{[m]}(\infty, \mathbb{Z}_{p^k})$ for $N > \max(m, n)$.

Fix $g \in \mathrm{GL}(\infty, \mathbb{Z}_{p^k})$. For sufficiently large β , the expression for g as an $(m + \beta + \infty)$ -block matrix has the form

$$g = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & g_{23} \\ 0 & g_{32} & g_{33} \end{pmatrix}.$$

Multiplying this matrix from the right by an appropriate matrix of the form

$$\begin{pmatrix} r_{11} & r_{12} & 0 \\ r_{12} & r_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (3.4)$$

we can obtain a matrix of the form

$$g' = \begin{pmatrix} 1 & 0 & 0 \\ g'_{21} & g'_{22} & g'_{23} \\ 0 & g'_{32} & g'_{33} \end{pmatrix}. \quad (3.5)$$

Indeed, we can regard the rows u_1, \dots, u_m of the matrix $(g_{11} \ g_{12})$ as elements of the module $\mathbb{Z}_{p^k}^{m+\beta}$. Since the matrix g is invertible, the matrix $((g))_p$ over the finite field \mathbb{Z}_p is invertible. Therefore, the matrix $((g_{11} \ g_{12}))_p$ is nondegenerate. This implies that the rows u_1, \dots, u_m generate a submodule isomorphic to $\mathbb{Z}_{p^k}^m$. Adding an appropriate collection v_1, \dots, v_β , we can obtain a basis of the module $\mathbb{Z}_{p^k}^{m+\beta}$. The matrices (3.4) determine automorphisms of $\mathbb{Z}_{p^k}^{m+\beta}$. We send $u_1, \dots, u_m, v_1, \dots, v_\beta$ to the standard basis in $\mathbb{Z}_{p^k}^{m+\beta}$.

Thus we arrive at a matrix g' of the form (3.5). Multiplying g' from the left by

$$\begin{pmatrix} 1 & 0 & 0 \\ -g'_{21} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we obtain

$$g'' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & g'_{22} & g'_{23} \\ 0 & g'_{32} & g'_{33} \end{pmatrix} \in \mathrm{GL}^{[m]}(\infty, \mathbb{Z}_{p^k}). \quad \square$$

Lemma 3.7. *The groups $\mathrm{GL}(\infty, \mathbb{Z}_{p^k})$ are oligomorphic.*

Proof. We have an action of $\mathrm{GL}_\infty(\infty, \mathbb{Z}_{p^k})$ on $V(\mathbb{Z}_{p^k}) \oplus V^\circ(\mathbb{Z}_{p^k})^n$, i.e., on collections $(v_1, \dots, v_n; w_1^\circ, \dots, w_n^\circ)$ of vectors and covectors. We must show that there is a finite set containing representatives of all $\mathrm{GL}_\infty(\infty, \mathbb{Z}_{p^k})$ -orbits. Denote by V_N the submodule in $V(\mathbb{Z}_{p^k})$ consisting of the vectors whose coordinates with indices $> N$ vanish, $V_N = \mathbb{Z}_{p^k}^N$. An $(N + \infty)$ -block

matrix of the form $g = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ induces an automorphism of V_N . We can send v_1, \dots, v_n to the submodule $V_n \subset V_N$.

Next, consider the action of the group $\mathrm{GL}^{[n]}(\infty, \mathbb{Z}_{p^k})$ on the collections of vectors and covectors. It does not change vectors and the first n coordinates of covectors. The same argument as above shows that we can send all covectors to the module V_{2n}° .

Thus, any orbit intersects the finite set $V(\mathbb{Z}_{p^k})^n \oplus V^\circ(\mathbb{Z}_{p^k})^{2n}$. \square

3.6. Proof of Lemma 1.1. By Corollary 3.3, it suffices to prove the statement for a quasiregular representation ρ of $\mathrm{GL}(\infty, \mathbb{Z}_{p^k})$ in the space $\ell^2(X)$, where $X = \mathrm{GL}(\infty, \mathbb{Z}_{p^k})/P$ is the coset space with respect to an open subgroup P . For $x \in X$, denote by δ_x the element of ℓ^2 that is equal to 1 at x and 0 at the other points.

By Lemma 3.1, the weak limit of $\rho(\theta_j)$ exists and coincides with the orthogonal projection to the space of $S(\infty)$ -fixed vectors. Let

$$\psi = \sum_{x \in X} c_x \delta_x \neq 0$$

be such a vector. For $\sigma \in S(\infty)$, we have $c_{\sigma x} = c_x$. If x is not fixed by $S(\infty)$, then its orbit is infinite. Since $\psi \in \ell^2$, we have $c_x = 0$. Thus, ψ has the form

$$\psi = \sum_{x: \sigma x = x \text{ for all } \sigma \in S(\infty)} c_x \delta_x.$$

The stabilizer of x is an open subgroup in $\mathrm{GL}(\infty, \mathbb{O}_p)$. It contains some subgroup of the form $\mathrm{GL}^{[m]}(\infty, \mathbb{O}_p)$. On the other hand, it contains $S(\infty)$. By Lemma 3.6, the stabilizer of x is the whole group $\mathrm{GL}(\infty, \mathbb{O}_p)$. Thus, the space X consists of one point. This completes the proof. \square

§4. ADMISSIBILITY

Here we prove Lemma 1.3. It is sufficient to prove the implication \Leftarrow , since the implication \Rightarrow immediately follows from Proposition 3.4.

4.1. A normal form for double cosets.

Lemma 4.1. a) Any double coset of the group $\mathrm{GL}_{\mathrm{fin}}(\infty, \mathbb{O}_p)$ with respect to $\mathrm{GL}_{\mathrm{fin}}^{[m]}(\infty, \mathbb{O}_p)$ contains an element of $\mathrm{GL}(3m, \mathbb{O}_p)$.

b) The same statement holds for double cosets of $\mathrm{GL}(\infty, \mathbb{O}_p)$ with respect to $\mathrm{GL}^{[m]}(\infty, \mathbb{O}_p)$.

c) The natural map

$$\begin{aligned} \mathrm{GL}_{\mathrm{fin}}^{[m]}(\infty, \mathbb{O}_p) \setminus \mathrm{GL}_{\mathrm{fin}}(\infty, \mathbb{O}_p) / \mathrm{GL}_{\mathrm{fin}}^{[m]}(\infty, \mathbb{O}_p) \\ \longrightarrow \mathrm{GL}^{[m]}(\infty, \mathbb{O}_p) \setminus \mathrm{GL}(\infty, \mathbb{O}_p) / \mathrm{GL}^{[m]}(\infty, \mathbb{O}_p) \end{aligned} \quad (4.1)$$

is a bijection.

d) Let $M \geq 3m$. Assume that for $g_1, g_2 \in \mathrm{GL}(M, \mathbb{O}_p)$ there are elements $q, r \in \mathrm{GL}^{[m]}(\infty, \mathbb{O}_p)$ such that $g_1 = qg_2r$. Then for sufficiently large N

depending only on M there are elements

$$q', r' \in \mathrm{GL}(N, \mathbb{O}_p) \cap \mathrm{GL}^{[m]}(\infty, \mathbb{O}_p)$$

such that $g_1 = q' g_2 r'$.

Proof. a), b). In both cases, we can apply the following reduction. Represent an element $g \in \mathrm{GL}_{\mathrm{fin}}(\infty, \mathbb{O}_p)$ as an $(m + \infty)$ -block matrix

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}. \text{ Multiplying it from the right by matrices of the form } \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \in \mathrm{GL}^{[m]}, \text{ we can reduce it to the } (m + m + \infty)\text{-block form}$$

$$g' = \begin{pmatrix} g'_{11} & g'_{12} & 0 \\ g'_{21} & g'_{22} & g'_{23} \\ g'_{31} & g'_{32} & g'_{33} \end{pmatrix}$$

(in fact, g'_{12} can be made lower triangular). Applying a similar left multiplication, we can ensure that $g'_{31} = 0$.

Next, we multiply g' from the left and from the right by elements of $\mathrm{GL}_{\mathrm{fin}}^{[2m]}$ to simplify g'_{33} (such multiplications do not change the blocks g'_{11} , g'_{12} , g'_{21} , g'_{22}). If the reduction $((g'_{33}))_p$ is nondegenerate, we can ensure that $((g'_{33}))_p = 1$ and $g'_{33} = 1$.

However, $((g'_{33}))_p$ can be degenerate, with

$$\dim \ker g'_{33} = \dim \mathrm{coker} g'_{33} := \gamma \leq m.$$

In this case, we can transform $((g'_{33}))_p$ into a matrix of the form $\begin{pmatrix} 0_\gamma & 0 \\ 0 & 1_\infty \end{pmatrix}$

and reduce g'_{33} to the form $\begin{pmatrix} pA & pB \\ pC & 1_\infty + pD \end{pmatrix}$, where A, B, C, D are matrices over \mathbb{O}_p .

Applying the right multiplication by $\begin{pmatrix} 1_{2m+\gamma} & 0 \\ 0 & 1 + pD \end{pmatrix}^{-1} \in \mathrm{GL}^{[2m+\gamma]} \subset \mathrm{GL}^{[2m]}$, we “kill” pD and arrive at an $(m + m + \gamma + \infty)$ -block matrix of the form

$$g'' := \begin{pmatrix} g''_{11} & g''_{12} & 0 & 0 \\ g''_{21} & g''_{22} & g''_{23} & g''_{24} \\ 0 & g''_{32} & g''_{33} & g''_{34} \\ 0 & g''_{42} & g''_{43} & 1 \end{pmatrix}.$$

Multiplying by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -g''_{24} \\ 0 & 0 & 1 & -g''_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

from the left, we kill g''_{24} , g''_{34} (and change only g''_{22} , g''_{23} , g''_{33} , g''_{34}). In the same way (by a multiplication from the right) we kill g''_{42} , g''_{43} .

d) Denote $l := M - m$. We wish to verify the following statement: if for given $g_1, g_2 \in \mathrm{GL}(m + l, \mathbb{O}_p)$ there exist $\xi, \eta \in \mathrm{GL}^{[m]}(\infty, \mathbb{O}_p)$ satisfying the equation

$$g_1 \xi = \eta g_2, \quad (4.2)$$

then there exist $\xi', \eta' \in \mathrm{GL}_{\mathrm{fin}}^{[m]}(\infty, \mathbb{O}_p)$ satisfying the same equation. Let us write (4.2) as a condition on $(m + l + \infty)$ -block matrices:

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1_\infty \end{pmatrix} \begin{pmatrix} 1_m & 0 & 0 \\ 0 & x & y \\ 0 & z & u \end{pmatrix} = \begin{pmatrix} 1_m & 0 & 0 \\ 0 & X & Y \\ 0 & Z & U \end{pmatrix} \begin{pmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1_\infty \end{pmatrix} \quad (4.3)$$

(the matrices in the left-hand side stand for g_1 and ξ , the matrices in the right-hand side, for η and g_2), or

$$\begin{pmatrix} a & bx & by \\ c & dx & dy \\ 0 & z & u \end{pmatrix} = \begin{pmatrix} A & B & 0 \\ XC & XD & Y \\ ZC & ZD & U \end{pmatrix}. \quad (4.4)$$

Let \varkappa be an infinite invertible matrix over \mathbb{O}_p . Then the transformations

$$Z \mapsto \varkappa Z, \quad U \mapsto \varkappa U, \quad z \mapsto \varkappa z, \quad u \mapsto \varkappa u$$

send a solution of the system of equations (4.4) to a solution. We can find a new solution, with Z of the form $\begin{pmatrix} Z' \\ 0 \end{pmatrix}$, the size of this matrix being

$(l + \infty) \times l$. By (4.4), the new matrix z is $ZD = \begin{pmatrix} Z'D \\ 0 \end{pmatrix}$. Applying a similar transformation

$$y \mapsto y\lambda, \quad u \mapsto u\lambda, \quad Y \mapsto Y\lambda, \quad U \mapsto u\lambda,$$

we can get a solution of (4.4) with y and Y of the form $\begin{pmatrix} * & 0 \end{pmatrix}$. Thus, we have a solution of (4.3) with finitary indeterminates z, Z, y, Y . Now, the

indeterminant factor in the left-hand side of (4.3) can be written in the $(m + l + l + \infty)$ -block form

$$\begin{pmatrix} 1_m & 0 & 0 & 0 \\ 0 & x' & y' & 0 \\ 0 & z' & u'_{11} & u'_{12} \\ 0 & 0 & u'_{21} & u'_{22} \end{pmatrix}. \quad (4.5)$$

The only equation in (4.4) containing u is $u = U$. Since the matrices $((x' \ y'))_p$ and $((\begin{smallmatrix} x' \\ z' \end{smallmatrix}))_p$ are nondegenerate, we can choose u'_{11} such that the matrix $((\begin{smallmatrix} x' & y' \\ z' & u'_{11} \end{smallmatrix}))_p$ is also nondegenerate. We set $u'_{12} = 0$, $u'_{21} = 0$, $u'_{22} = 1_\infty$. Then the matrix (4.5) is invertible.

Finally, we find the new U from the equation $u = U$. Then three factors in (4.3) are invertible, and, therefore, the fourth factor is also invertible. A finitary solution of (4.3) is obtained. Actually,

$$\xi', \eta' \in \mathrm{GL}^{[m]}(\infty, \mathbb{O}_p) \cap \mathrm{GL}(m + 2l, \mathbb{O}_p).$$

c) The surjectivity follows from a) and b), while the injectivity follows from d). \square

4.2. The metric on the space of double cosets. Here we prove the following lemma.

Lemma 4.2. *The maps (4.1) are homeomorphisms.*

Fix m . For every $M \geq 3m$, we have a natural partition of the group $\mathrm{GL}(M, \mathbb{O}_p)$ into the subsets $\mathrm{GL}(M, \mathbb{O}_p) \cap \mathfrak{g}$ where \mathfrak{g} are double cosets of $\mathrm{GL}(\infty, \mathbb{O}_p)$ with respect to $\mathrm{GL}^{[m]}(\infty, \mathbb{O}_p)$. Denote by \mathcal{K}_M the quotient space. According to Lemma 4.1(d), elements of these partitions are compact; therefore, the quotients are compact. For $M < M'$, the natural map $\mathcal{K}_M \rightarrow \mathcal{K}_{M'}$ is continuous. By Lemma 4.1(a), it is a bijection, hence it is a homeomorphism. This also implies that the bijections

$$\mathcal{K}_M \longleftrightarrow \mathrm{GL}_{\mathrm{fin}}^{[m]}(\infty, \mathbb{O}_p) \setminus \mathrm{GL}_{\mathrm{fin}}(\infty, \mathbb{O}_p) / \mathrm{GL}_{\mathrm{fin}}^{[m]}(\infty, \mathbb{O}_p)$$

are homeomorphisms. Also, it is clear that the maps

$$\mathcal{K}_M \longrightarrow \mathrm{GL}^{[m]}(\infty, \mathbb{O}_p) \setminus \mathrm{GL}(\infty, \mathbb{O}_p) / \mathrm{GL}^{[m]}(\infty, \mathbb{O}_p)$$

are continuous. We must prove the continuity of the inverse map.

We define a left-right-invariant metric on $\mathrm{GL}(\infty, \mathbb{O}_p)$ by the formula

$$d(z, u) = \max_{i,j} |z_{ij} - u_{ij}|.$$

Remark. This metric determines the standard topology on each group $\mathrm{GL}(n, \mathbb{O}_p)$. On the whole group $\mathrm{GL}(\infty, \mathbb{O}_p)$, it determines a nonseparable topology which is stronger than the natural topology. The restriction of the metric to $\mathrm{GL}_{\mathrm{fin}}(\infty, \mathbb{O}_p)$ induces a topology which is weaker than the natural topology.

Recall that the Hausdorff metric on the space of compact subsets of a metric space is given by the formula

$$\mathrm{dist}_H(A, B) := \max \left[\max_{x \in A} \min_{y \in B} d(x, y), \max_{y \in B} \min_{x \in A} d(x, y) \right].$$

Restricting this metric to elements of the partition of $\mathrm{GL}(M, \mathbb{O}_p)$, we get a metric on the double coset space

$$\mathrm{dist}_H^M(\mathfrak{g}_1, \mathfrak{g}_2) = \mathrm{dist}_H(\mathfrak{g}_1 \cap \mathrm{GL}(M, \mathbb{O}_p), \mathfrak{g}_2 \cap \mathrm{GL}(M, \mathbb{O}_p)) \quad (4.6)$$

compatible with the topology on \mathcal{K}_M .

Next, we define another metric on

$$\mathrm{GL}^{[m]}(\infty, \mathbb{O}_p) / \mathrm{GL}(\infty, \mathbb{O}_p) / \mathrm{GL}^{[m]}(\infty, \mathbb{O}_p).$$

Let $\mathfrak{g}_1, \mathfrak{g}_2$ be double cosets. Fix $g \in \mathfrak{g}_1$. Then

$$\mathrm{dist}(\mathfrak{g}_1, \mathfrak{g}_2) = \inf_{z \in \mathfrak{g}_2} d(g, z) \quad (4.7)$$

(the result does not depend on g).

Lemma 4.3. *These metrics coincide.*

We have the obvious inequality

$$\mathrm{dist}_H^{3m}(\mathfrak{g}_1, \mathfrak{g}_2) \geq \mathrm{dist}(\mathfrak{g}_1, \mathfrak{g}_2).$$

The inverse inequality follows from the following lemma.

Lemma 4.4. *Let $\mathfrak{g}_1, \mathfrak{g}_2$ be double cosets. Let $g_1 \in \mathfrak{g}_1, g_2 \in \mathfrak{g}_2$. Let $u \in \mathfrak{g}_1 \cap \mathrm{GL}(3m, \mathbb{O}_p)$. Then there exists $w \in \mathfrak{g}_2 \cap \mathrm{GL}(3m, \mathbb{O}_p)$ such that*

$$d(u, w) \leq d(g_1, g_2).$$

Proof. Let $d(g_1, g_2) = p^{-k}$. Let $u = qg_1r$ where $q, r \in \mathrm{GL}^{[m]}(\infty, \mathbb{O}_p)$. We take $v := qg_2r$. Then we can make a reduction as in the proof of Lemma 4.1 using only elements of the congruence subgroup, i.e., we find $w = tvs \in \mathrm{GL}(3m, \mathbb{O}_p)$ with

$$t, s \in \mathrm{GL}^{[m]}(\infty, \mathbb{O}_p) \cap \mathrm{GL}_k(\infty, \mathbb{O}_p).$$

Then we have $d(v, u) = p^{-k}$, $d(u, w) \leq p^{-k}$. \square

4.3. Proof of Lemma 4.2. It suffices to show that for any \mathfrak{g} for any k there is a neighborhood \mathcal{N} of \mathfrak{g} in the sense of

$$\mathrm{GL}^{[m]}(\infty, \mathbb{O}_p) \setminus \mathrm{GL}(\infty, \mathbb{O}_p) / \mathrm{GL}^{[m]}(\infty, \mathbb{O}_p)$$

such that for every $\mathfrak{h} \in \mathcal{N}$ we have $\mathrm{dist}(\mathfrak{g}, \mathfrak{h}) \leq p^{-k}$.

Choose $g \in \mathfrak{g}$ and choose N such that the matrix $((g))_{p^k}$ has the following $(m + N + \infty)$ -block form:

$$((g))_{p^k} = \begin{pmatrix} u_{11} & u_{12} & 0 \\ u_{21} & u_{22} & u_{23} \\ 0 & u_{32} & u_{33} \end{pmatrix}.$$

Next, consider the open subgroup $\mathrm{GL}_k^{[m+N]}(\infty, \mathbb{O}_p)$ and the neighborhood

$$\mathcal{O} := g \mathrm{GL}_k^{[m+N]}(\infty, \mathbb{O}_p)$$

of g . Let $r \in \mathrm{GL}_k^{[m+N]}(\infty, \mathbb{O}_p)$. Then $h = gr \in \mathcal{O}$. Consider the matrix

$$((r))_{p^k} \in \mathrm{GL}^{[m+N]}(\infty, \mathbb{Z}_{p^k}).$$

Let us regard it as a matrix $\tilde{r} \in \mathrm{GL}^{[m+N]}(\infty, \mathbb{O}_p)$ composed of p -adic integers contained in the set $0, 1, \dots, p^{k-1}$. Consider the matrix $gr\tilde{r}^{-1}$, which is contained in the same double coset. Then $((gr\tilde{r}^{-1}))_{p^k} = ((g))_{p^k}$. Thus

$$|g - gr\tilde{r}^{-1}| \leq p^{-k}.$$

We apply Lemma 4.4, and this completes the proof.

4.4. End of the proof of Lemma 1.3. It suffices to show that matrix elements of the form

$$\langle \rho(g)\xi, \eta \rangle_H, \quad \text{where } \xi, \eta \in \cup H_j,$$

have continuous extensions to the whole group $\mathrm{GL}(\infty, \mathbb{O}_p)$. We may assume that $\xi, \eta \in H_m$. Such matrix elements are continuous functions on the inductive limit $\mathrm{GL}_{\mathrm{fin}}(\infty, \mathbb{O}_p)$ that are constant on double cosets with respect to $\mathrm{GL}_{\mathrm{fin}}^{[m]}(\infty, \mathbb{O}_p)$. By Lemma 4.2, they are continuous on $\mathrm{GL}(\infty, \mathbb{O}_p)$.

REFERENCES

1. M. S. Brodskii, *Unitary operator colligations and their characteristic functions*. — Russian Math. Surveys **33**, No. 4 (1978), 159–191.
2. V. M. Brodskii, *On operator colligations and their characteristic functions*. — Sov. Math. Dokl. **12** (1971), 696–700.
3. P. J. Cameron, *Oligomorphic Permutation Groups*, Cambridge Univ. Press, Cambridge, 1990.
4. A. A. Gaifullin, Yu. A. Neretin, *Infinite symmetric group, pseudomanifolds, and combinatorial cobordism-like structures*, to appear in J. Topol. Anal., DOI: 10.1142/S179352531850022X.
5. S. Gao, *Invariant Descriptive Set Theory*, CRC Press, Boca Raton, 2009.
6. A. A. Kirillov, *Elements of the Theory of Representations*, Springer-Verlag, Berlin–New York, 1976.
7. A. Lieberman, *The structure of certain unitary representations of infinite symmetric groups*. — Trans. Amer. Math. Soc. **164** (1972), 189–198.
8. Yu. A. Neretin, *Categories of Symmetries and Infinite-Dimensional Groups*, Oxford Univ. Press, New York, 1996.
9. Yu. A. Neretin, *Multi-operator colligations and multivariate characteristic functions*. — Anal. Math. Phys. **1**, No. 2–3 (2011), 121–138.
10. Yu. A. Neretin, *Sphericity and multiplication of double cosets for infinite-dimensional classical groups*. — Funct. Anal. Appl. **45**, No. 3 (2011), 225–239.
11. Yu. A. Neretin, *Hua measures on the space of p -adic matrices and inverse limits of Grassmannians*. — Izv. Math. **77**, No. 5 (2013), 941–953.
12. Yu. A. Neretin, *Infinite-dimensional p -adic groups, semigroups of double cosets, and inner functions on Bruhat–Tits buildings*. — Izv. Math. **79**, No. 3 (2015), 512–553.
13. Yu. A. Neretin, *Infinite symmetric groups and combinatorial constructions of topological field theory type*. — Russian Math. Surveys **70**, No. 4 (2015), 715–773.
14. G. I. Olshanski, *Unitary representations of the infinite symmetric group: a semi-group approach*. — In: *Representations of Lie Groups and Lie Algebras*, Akad. Kiadó, Budapest, 1985, pp. 181–197.
15. G. I. Olshanski, *Unitary representations of (G, K) -pairs that are connected with the infinite symmetric group $S(\infty)$* . — Leningrad Math. J. **1**, No. 4 (1990), 983–1014.
16. G. I. Olshanski, *Unitary representations of infinite-dimensional pairs (G, K) and the formalism of R. Howe*. — In: *Representation of Lie Groups and Related Topics*, Gordon and Breach, New York, 1990, pp. 269–463.

-
17. G. I. Olshanski, *On semigroups related to infinite-dimensional groups*. — In: *Topics in Representation Theory*, Adv. Sov. Math. **2**, Amer. Math. Soc., Providence, RI, 1991, pp. 67–101.
 18. T. Tsankov, *Unitary representations of oligomorphic groups*. — *Geom. Funct. Anal.* **22**, No. 2 (2012), 528–555.

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