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THE LIMIT SHAPE OF A PROBABILITY MEASURE ON A TENSOR PRODUCT OF MODULES OF THE B_n ALGEBRA

Abstract. We study a probability measure on the integral dominant weights in the decomposition of the Nth tensor power of the spinor representation of the Lie algebra $\mathrm{so}(2n+1)$. The probability of a dominant weight λ is defined as the dimension of the irreducible component of λ divided by the total dimension 2^{nN} of the tensor power. We prove that as $N \to \infty$, the measure weakly converges to the radial part of the $\mathrm{SO}(2n+1)$-invariant measure on $\mathrm{so}(2n+1)$ induced by the Killing form. Thus, we generalize Kerov's theorem for $\mathrm{su}(n)$ to $\mathrm{so}(2n+1)$.

Introduction

Let $V_1, V_2, \dots, V_k$ be finite-dimensional representations of a simple Lie algebra $\mathfrak{g}$. The tensor product of these representations is isomorphic to a direct sum of irreducible highest weight representations:

$$V_1 \otimes V_2 \otimes \dots \otimes V_k \cong \bigoplus_{\lambda} W_{\lambda}^{1,\dots,k} \otimes L^{\lambda}, \tag{1}$$

where L_λ is the irreducible representation with highest weight λ and $W_{\lambda}^{1,\dots,k} \simeq \mathrm{Hom}_{\mathfrak{g}}(L^{\lambda}, V_1 \otimes \dots \otimes V_k)$. The dimension $M_{\lambda}^{1,\dots,k} = \mathrm{dim} W_{\lambda}^{1,\dots,k}$ is the multiplicity of L^{λ} in the tensor product decomposition. Rewriting this decomposition equation in terms of dimensions of representations and dividing by $\dim \cdot V_1 \cdot \mathrm{dim} V_2 \cdot \ldots \cdot \mathrm{dim} V_k$, we get

$$\sum_{\lambda} \frac{M_{\lambda}^{1,\dots,k} \mathrm{dim} L^{\lambda}}{\mathrm{dim} V_1 \dots \mathrm{dim} V_k} = 1. \tag{2}$$

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Thus, we have a discrete probability measure on the dominant integral weights λ that appear in this decomposition:

$$\mu_N(\lambda) = \frac{M_\lambda^{1,\dots,k} \mathrm{dim} L^\lambda}{\mathrm{dim} V_1 \dots \mathrm{dim} V_k}. \tag{3}$$

A particularly interesting case of such a measure appears when we take the Nth tensor power of the vector representation of $\mathrm{su}(n+1)$. Because of the Schur–Weyl duality, in this case M_λ is the dimension of an irreducible representation of the symmetric group S_N. Kerov [1] discovered that as $N \to \infty$, the discrete measures (3) in the weight space $\mathbb{R}^n$ of $\mathrm{su}(n+1)$ converge weakly to some continuous measure on the main Weyl chamber.

In this paper, we consider the algebra $\mathfrak{g} = B_n$ and tensor powers of the spinor representation L^{ω_n}. We prove the weak convergence of the measures (3) on the main Weyl chamber of the weight space for B_n to the measure induced by the G-invariant Euclidean measure on $\mathfrak{g}$.

In Theorem 1, we consider a small domain $U(N) \subset \mathbb{R}^n$ whose volume goes to zero as $N \to \infty$. We show that the probability density function of the limit measure is given by the formula

$$\phi\left(\{x_i\}\right) = \prod_{i<j}(x_i^2 - x_j^2)^2 \prod_{l=1}^{n} x_l^2 \exp\left(-\frac{1}{2}\sum_k x_k^2 \right) \cdot \frac{2^{2n} n!}{(2n)!(2n-2)!\dots 2!},$$

where $x_i = \frac{1}{\sqrt{N}} a_i$ and $a_i = \lambda_i + \rho_i$ are the shifted Euclidean coordinates on the weight space of B_n, with ρ being the Weyl vector.

In Theorem 2, we show that the same holds for every n-orthotope in the main Weyl chamber.

In Theorem 3, we show that the measures converge weakly on the entire main Weyl chamber.

Kerov's proof for the A_n case was based upon hook-length formulas and Young diagrams for multiplicities of tensor product decompositions and dimensions of representations. In this, it was similar to the famous Vershik–Kerov [2, 3] and Logan–Shepp [4] result on the limit shape of Young diagrams. A relation of the limit shape of Young diagrams to random matrices was established in [5, 6]. The weak convergence was established by comparing the discrete probability measure with the multinomial distribution and using the de Moivre–Laplace theorem.

Unfortunately, there are no analogs of hook-length formulas for the other Lie algebras. Thus, we use a combinatorial formula for the coefficients of tensor product decompositions [7] and the Weyl dimension formula to prove an analog of the de Moivre–Laplace theorem [8,9] for an arbitrary n-orthotope in the main Weyl chamber, and then apply a weak convergence criterion for probability measures [10].

The paper is organized as follows. In Sec. 1, we fix notation and give the definition of probability measures on subsets of dominant integral weights. In Sec. 2, we present a formula for the multiplicities of the tensor product decomposition for tensor powers of the spinor representation of the algebra B_n. In Sec. 3, we pass to the limit of the infinite tensor power and prove the convergence of the discrete probability measures on subsets of dominant integral weights to a continuous probability measure on the main Weyl chamber. In the conclusion section, we discuss the connection to the invariant measure and further work.

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§1. The tensor product decomposition and the definition of the probability measure

Consider the simple Lie algebra $\mathfrak{g} = B_n$ of rank n. Denote the simple roots of $\mathfrak{g}$ by $\alpha_1, \dots, \alpha_n$ and the fundamental weights by $\omega_1, \dots \omega_n$, with $(\alpha_i, \omega_j) = \delta_{ij}$. We denote the irreducible highest weight representation of $\mathfrak{g}$ with highest weight λ by L^λ.

The root system is denoted by $\Delta = \{\pm e_i \pm e_j|_{i\neq j}\} \cup \{\pm e_i\}$, while $\Delta^+ = \{e_i + e_j|_{i\leqslant j}\} \cup \{e_i\} \cup \{e_j - e_i|_{j\leqslant i}\}$ is the subset of positive roots, and $\rho = \omega_1 + \dots + \omega_n$ is the Weyl vector.

We consider the multiplicity function for a tensor power of the last fundamental module $\omega_n = \frac{1}{2}(e_1 + \dots + e_n)$. We decompose the Nth tensor power of L^{ω_n} into a direct sum of irreducible representations:

$$(L^{\omega_n})^{\otimes N} = \bigoplus_{\lambda \in P^+(\omega_n, N)} M_\lambda^{\omega_n, N} L^\lambda, \tag{4}$$

where $P^+(\omega_n, N)$ is the subset of dominant integral weights of the (reducible) representation $(L^{\omega_n})^{\otimes N}$.

Relation (3) gives us a probability measure on $P^+(\omega_n, N)$ with the probability density function

$$\mu_N(\lambda) = \frac{M_\lambda^{\omega_n, N} \dim L^\lambda}{(\dim L^{\omega_n})^N}. \tag{5}$$

It is easy to see that $\sum\limits_{\lambda\in P(\omega_n,N)} \mu_N(\lambda) = 1$, since the dimension of the left-hand side of (4) is equal to $(\dim L^{\omega_n})^N$.

In the present paper, we will study this measure in the $N \to \infty$ limit for n fixed.

§2. The multiplicity formula for the tensor product decomposition

The papers [7,11–13] suggested a method for decomposing tensor powers of the fundamental module of the smallest dimension for the simple Lie algebras of type A_n and B_n.

Instead of focusing on the multiplicity function $M_\lambda^{\omega,N}$, which is defined on the set of dominant weights P^+, this method is aimed at finding an expression for the extended multiplicity function $\widetilde{M}_\lambda^{\omega,N}$ defined on the whole weight lattice P as

$$\widetilde{M}^{\omega,N}_{w(\lambda+\rho)-\rho}|_{w\in W} = \epsilon\,(w)\,M_\lambda^{\omega,N}. \tag{6}$$

If an expression for $\widetilde{M}_\lambda^{\omega,N}$ is obtained, we can find $M_\lambda^{\omega,N}$ as its restriction to the set of dominant weights P^+ lying in the main Weyl chamber.

It was shown in [12, 13] that the extended multiplicity function $\widetilde{M}_\lambda^{\omega,N}$ is a solution of the set of recurrence relations

$$\sum_{\xi\in P} \widetilde{M}_\xi^{\omega,N} e^\xi = \mathcal{N}\left(L_{\mathfrak{g}}^{(\omega)}\right) \sum_{\gamma\in P} \widetilde{M}_\gamma^{\omega,N-1} e^\gamma, \tag{7}$$

where $\mathcal{N}\left(L_{\mathfrak{g}}^{(\omega)}\right)$ is the weight diagram of the module $L_{\mathfrak{g}}^{(\omega)}$.

It was also proved that in the case of fundamental modules of the smallest dimension, a solution of (7) is uniquely determined by the requirement of antiinvariance with respect to the Weyl group transformations and the boundary conditions.

In the case of the B_n algebra, this method allowed us to obtain the multiplicity function for the tensor power of the last fundamental module $\omega_n = \frac{1}{2}(e_1 + \cdots + e_n)$. This expression has an explicit dependence on N:

$$\begin{aligned}&\widetilde{M}^{\omega_n,N}_{\lambda(a_1\ldots a_n)}\\ &= \prod_{k=0}^{n-1} \frac{(N+2k)!}{2^{2k}\left(\frac{N+a_{k+1}+2n-1}{2}\right)!\left(\frac{N-a_{k+1}+2n-1}{2}\right)!} \prod_{l=1}^{n} a_l \prod_{i<j}\left(a_i^2 - a_j^2\right);\end{aligned} \tag{8}$$

here $\{a_i\}$ are the coordinates of λ in the basis $\left\{\{\vec{\widetilde{e_i}}\} : \vec{\widetilde{e_i}} \parallel \vec{e_i}, |\widetilde{e_i}| = |\frac{e_i}{2}|\right\}$ with the origin shifted to $-\rho = -\omega_1 \cdots - \omega_n$. The factors in the numerator vanish at the boundaries of the shifted Weyl chambers, and the denominator ensures that $\widetilde{M}_{\lambda}^{\omega_1, N}$ satisfies the boundary conditions and also that the whole expression is antiinvariant with respect to the Weyl group transformations.

Note that there are two congruence classes of weights, one is parametrized by even values of a_i and the other one, by odd values. A class is determined by the parity of N. For even N, we get odd a_i, and vice versa.

The expression after $(N+2k)!$ in the numerator is related to the Weyl dimension formula for the irreducible module L^{λ}:

$$\dim L^{\lambda} = \prod_{\alpha \in \Delta^+} \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)}, \tag{9}$$

which in the case of a B_n module has the form

$$\dim L^{\lambda} = \frac{\prod_{i<j} (a_i^2 - a_j^2) \prod_{l=1}^{n} a_l}{(2n)!(2n-2)! \dots 2!} \cdot 2^{-n^2+2n} n!. \tag{10}$$

Thus we obtain the discrete probability measure with density function (or probability mass function)

$$\begin{aligned} \mu_N(\lambda) = \mu_N(\{a_i\}) &= \frac{\widetilde{M}_{\lambda(a_1 \dots a_n)}^{\omega_n, N} \dim L^{\lambda}}{(2^n)^N} \\ &= \prod_{k=0}^{n-1} \frac{(N+2k)!}{2^{2k} \left(\frac{N + a_{k+1} + 2n - 1}{2}\right)! \left(\frac{N - a_{k+1} + 2n - 1}{2}\right)!} \\ &\quad \times \prod_{i<j} (a_i^2 - a_j^2)^2 \prod_{l=1}^{n} a_l^2 \cdot \frac{2^{-n^2+2n-nN} n!}{(2n)!(2n-2)! \dots 2!}. \end{aligned} \tag{11}$$

§3. The infinite tensor product limit for a finite-rank algebra

Now let X be the n-dimensional random vector distributed according to the discrete measure with density function (or, more correctly, probability mass function) (11), i.e., $X \sim \mu_N(\{a_i\}) : U \subset P^+ \to \mathbb{R}$. Here P^+ is the dominant weight lattice,

$$\begin{cases} 0 \leqslant \mu_N(U) \leqslant 1, \\ \mu_N(P^+) = 1. \end{cases}$$

We fix n in formula (11) and study the $N \to \infty$ limit to see that this probability mass function converges to a continuous probability density function $\phi\{x_i\} : U \subset \mathbb{R}^n \to \mathbb{R}$.

First, we embed P^+ into $\mathbb{R}^n$ in the following way. To a weight λ in P^+ with coordinates $\{a_i\}$ we associate the domain

$$U_a = \cup_i [a_i - 1, a_i + 1) \in \mathbb{R}^n.$$

There is only one weight inside this domain. Consider the probability mass function of the vector X:

$$\begin{aligned} \mu_N(\lambda) = p_X(\lambda) &= \mathbf{P}\{X = \lambda\} = \mathbf{P}\{X \in U_a\} \\ &= \mathbf{P}\{a_i - 1 \leqslant X_i < a_i + 1\} \\ &= \mathbf{P}\left\{\frac{1}{\sqrt{N}}(a_i - 1) \leqslant \frac{1}{\sqrt{N}} X_i < \frac{1}{\sqrt{N}}(a_i + 1)\right\} \\ &= \mathbf{P}\left\{\frac{1}{\sqrt{N}} X \in U_a(N)\right\}. \end{aligned} \tag{12}$$

The volume of the rescaled domain $U_a(N)$ goes to zero as $N \to \infty$.

Then, as $N \to \infty$ we would expect on this domain the convergence

$$\left| p_X(\lambda) \cdot \left(\frac{\sqrt{N}}{2}\right)^n - \phi\left(\left\{\frac{1}{\sqrt{N}} a_i\right\}\right) \right| \longrightarrow 0. \tag{13}$$

Theorem 1. *Let $X \sim \mu_N(\{a_i\})$, and let C_N be a nondecreasing sequence such that $\lim\limits_{N \longrightarrow \infty} C_N / N^{\frac{1}{6}} = 0$. Then*

$$\max_{|a_i + 2n - 1| < \sqrt{N} \cdot C_N} \left| \frac{p_X(\lambda)}{\phi(\{x_i\})} \left(\frac{\sqrt{N}}{2}\right)^n - 1 \right| = \mathcal{O}\left(\frac{C_N^3}{\sqrt{N}}\right), \tag{14}$$

where $x_i = \frac{1}{\sqrt{N}} a_i$ and

$$\phi(\{x_i\}) = \prod_{i<j} (x_i^2 - x_j^2)^2 \prod_{l=1}^{n} x_l^2 \exp\left(-\frac{1}{2} \sum_k x_k^2\right) \cdot \frac{2^{2n} n!}{(2n)!(2n-2)! \dots 2!}. \tag{15}$$

Proof. Consider the factor in (11) that depends on N:

$$I_{N,n} = \prod_{k=0}^{n-1} \frac{(N+2k)!}{2^{2k}\left(\frac{N+a_{k+1}+2n-1}{2}\right)!\left(\frac{N-a_{k+1}+2n-1}{2}\right)!}. \tag{16}$$

We will treat the numerator of $I_{N,n}$ and the denominator

$$D_{N,n} = \prod_{k=0}^{n-1} 2^{2k}\left(\frac{N+a_{k+1}+2n-1}{2}\right)!\left(\frac{N-a_{k+1}+2n-1}{2}\right)! \tag{17}$$

separately.

To obtain the asymptotics of (16), we will use Stirling's formula for factorials

$$N! \approx \sqrt{2\pi}\exp\left(N\ln N - N + \frac{1}{2}\ln N\right)\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right), \tag{18}$$

and, assuming that $n \ll N$, the following expansion of the logarithm to the order of $\frac{1}{N^2}$:

$$\ln(N+i) = \ln N\left(1+\frac{i}{N}\right) = \ln N + \frac{i}{N} - \frac{i^2}{2N^2} + \mathcal{O}\bigg(\frac{1}{N^3}\bigg). \tag{19}$$

We have

$$\begin{aligned} S &= \sum_{i=1}^{n}(N+i)\ln(N+i) \\ &= \sum_{i=1}^{n}\frac{1}{N}\left(-\frac{i^3}{2N}+\frac{i^2}{2}+N(1+\ln N)i\right) + Nn\ln N + \mathcal{O}\left(\frac{1}{N^2}\right). \end{aligned} \tag{20}$$

Since the first term of the sum is of order $\frac{1}{N^2}$, we can include it into the error term:

$$S = \sum_{i=1}^{n}\frac{1}{N}\left(\frac{i^2}{2}+N(1+\ln N)i\right) + Nn\ln N + \mathcal{O}\left(\frac{1}{N^2}\right). \tag{21}$$

Then, as $N \longrightarrow \infty$, we apply Stirling's formula to the numerator of $I_{N,n}$:

$$\begin{aligned}
\prod_{k=0}^{n-1}(N+2k)! &= \prod_{k=1}^{n}(N+2k-2)! \\
&\simeq \prod_{k=1}^{n}\sqrt{2\pi}\exp\left[(N+2k-2)\ln(N+2k-2)-(N+2k-2)\right] \\
&\quad\times \prod_{k=1}^{n}(N+2k-2)^{\frac{1}{2}}\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right) \\
&= (\sqrt{2\pi})^n \exp\left[\sum_{k=1}^{n}(N+2k-2)\ln(N+2k-2)-(N+2k-2)\right] \\
&\quad\times \prod_{k=1}^{n}(N+2k-2)^{\frac{1}{2}}\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right).
\end{aligned} \tag{22}$$

Using (21), we now expand the sum under the exponent:

$$\begin{aligned}
\prod_{k=0}^{n-1}(N+2k)! &\simeq (\sqrt{2\pi})^n \exp\Bigg[\sum_{k=1}^{n}\Bigg(\frac{(2k-2)^2}{2N}+(1+\ln N)(2k-2) \\
&\qquad + N\ln N-(N+2k-2)\Bigg)+\mathcal{O}\left(\frac{1}{N^2}\right)\Bigg] \\
&\qquad \times \prod_{k=1}^{n}(N+2k-2)^{\frac{1}{2}}\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right) \\
&= (\sqrt{2\pi})^n \exp\Big[\frac{1}{N}\frac{4n^3-6n^2+2n}{6} \\
&\qquad + n(n-1)-n(N+n-1)\Big] N^{n(n-1)}N^{nN} \\
&\qquad \times \prod_{k=1}^{n}(N+2k-2)^{\frac{1}{2}}\left(1+\mathcal{O}\left(\frac{1}{N^2}\right)\right)\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right) \\
&= (\sqrt{2\pi})^n \exp\left[-nN+\frac{1}{N}\frac{4n^3-6n^2+2n}{6}\right] N^{n(n-1)}N^{nN} \\
&\qquad \times \prod_{k=1}^{n}(N+2k-2)^{\frac{1}{2}}\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right).
\end{aligned} \tag{23}$$

Next, we expand the denominator $D_{N,n}$ of $I_{N,n}$. We can expand logarithms such as $\ln(1+\frac{a_k+2n-1}{N})$ in the interval $\frac{|a_k+2n-1|}{N} < \frac{C_k}{\sqrt{N}} \ll 1$. Choose C_N that satisfies this condition for the largest a_k. Then the value $\frac{|a_k+2n-1|}{N}$ will be bounded for all a_k, as well as the value $\frac{|-a_k+2n-1|}{N}$.

Applying Stirling's formula to the denominator of $I_{N,n}$, we get

$$
\begin{aligned}
D_{N,n} &= \prod_{k=1}^{n} 2^{2(k-1)} \left(\frac{N+a_k+2n-1}{2}\right)! \left(\frac{N-a_k+2n-1}{2}\right)! \\
&\simeq 2^{n(n-1)} (\sqrt{2\pi})^{2n} \\
&\quad \times \prod_{k=1}^{n} \exp\left[\frac{N+a_k+2n-1}{2} \ln\left(\frac{N+a_k+2n-1}{2}\right)\right. \\
&\qquad\qquad \left. - \frac{N+a_k+2n-1}{2}\right] \prod_{k=1}^{n} \left(\frac{N+a_k+2n-1}{2}\right)^{\frac{1}{2}} \\
&\quad \times \prod_{k=1}^{n} \exp\left[\frac{N-a_k+2n-1}{2} \ln\left(\frac{N-a_k+2n-1}{2}\right)\right. \\
&\qquad\qquad \left. - \frac{N-a_k+2n-1}{2}\right] \prod_{k=1}^{n} \left(\frac{N-a_k+2n-1}{2}\right)^{\frac{1}{2}}.
\end{aligned}
\tag{24}
$$

This expansion has an error term of order $\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right)$. To simplify calculations, we combine some factors into one factor $Z(n,N)$. Thus we get

$$
\begin{aligned}
&D_{N,n} \\
&= \underbrace{2^{n(n-1)} (\sqrt{2\pi})^{2n} \exp\Big(-n(N+2n-1)\Big) \prod_{k=1}^{n} \left(\frac{N-a_k+2n-1}{2}\right)^{\frac{1}{2}} \left(\frac{N+a_k+2n-1}{2}\right)^{\frac{1}{2}}}_{Z(n,N)} \\
&\qquad \cdot \exp\left[\sum_{k=1}^{n} \frac{N-a_k+2n-1}{2} \ln\left(\frac{N-a_k+2n-1}{2}\right)\right. \\
&\qquad \left. + \frac{N+a_k+2n-1}{2} \ln\left(\frac{N+a_k+2n-1}{2}\right)\right] \cdot \left(1+\mathcal{O}\Big(\frac{1}{N}\Big)\right) \\
&\qquad = Z(n,N) \cdot \exp\left[\frac{1}{2} \sum_{k=1}^{n} (N+a_k+2n-1) \ln(N+a_k+2n-1)\right.
\end{aligned}
$$

$$+ (N - a_k + 2n - 1)\ln(N - a_k + 2n - 1)\Bigg]$$

$$\cdot \exp\left[-\sum_{k=1}^{n}(N + 2n - 1)\ln 2\right]\left(1 + \mathcal{O}\Big(\frac{1}{N}\Big)\right). \quad (25)$$

Using (21), we expand the sum under the exponent and get an approximate expression for the denominator:

$$\begin{aligned} D_{N,n} \simeq{}& 2^{-n(N+2n-1)} Z(n,N) \\ &\cdot \exp\Bigg[\frac{1}{2N}\sum_{k=1}^{n}\Bigg(\frac{(a_k + 2n - 1)^2}{2} \\ &+ N(1 + \ln N)(a_k + 2n - 1) + N^2 \ln N\Bigg) + \mathcal{O}\Big(\frac{C_N^3}{\sqrt{N}}\Big)\Bigg] \\ &\cdot \exp\Bigg[\frac{1}{2N}\sum_{k=1}^{n}\Bigg(\frac{(-a_k + 2n - 1)^2}{2} \\ &+ N(1 + \ln N)(-a_k + 2n - 1) + N^2 \ln N\Bigg) + \mathcal{O}\Big(\frac{C_N^3}{\sqrt{N}}\Big)\Bigg] \cdot \left(1 + \mathcal{O}\Big(\frac{1}{N}\Big)\right) \\ ={}& 2^{-n(N+2n-1)} Z(n,N) \exp\left[\sum_{k=1}^{n}\frac{a_k^2}{2N}\right] \\ &\times \exp\left[\frac{4n^3 - 4n^2 + n}{2N} + n(2n-1) + n(2n-1)\ln N + nN\ln N\right] \cdot \\ &\cdot \left(1 + \mathcal{O}\Big(\frac{C_N^3}{\sqrt{N}}\Big)\right)\left(1 + \mathcal{O}\Big(\frac{1}{N}\Big)\right). \end{aligned} \quad (26)$$

Extracting the factors from $Z(n,N)$, we get

$$\begin{aligned} D_{N,n} \simeq{}& 2^{-nN-n^2}(\sqrt{2\pi})^{2n} N^{nN+n(2n-1)} \\ &\times \exp\left[\sum_{k=1}^{n}\frac{a_k^2}{2N}\right]\exp\left[-nN + \frac{4n^3 - 4n^2 + n}{2N}\right] \\ &\cdot \prod_{k=1}^{n}\left(\frac{N - a_k + 2n - 1}{2}\right)^{\frac{1}{2}}\left(\frac{N + a_k + 2n - 1}{2}\right)^{\frac{1}{2}} \\ &\left(1 + \mathcal{O}\left(\frac{C_N^3}{\sqrt{N}}\right)\right)\left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right). \end{aligned} \quad (27)$$

Then, dividing (23) by (27), we obtain

$$
\begin{aligned}
I_{N,n} &\simeq \frac{2^{nN+n^2}}{(\sqrt{2\pi})^n} N^{-n^2} \exp\left[-\sum_{k=1}^{n} \frac{a_k^2}{N}\right] \\
&\quad \times \exp\left[\frac{-8n^3-6n^2-n}{3N}\right] \underbrace{\prod_{k=1}^{n} \frac{(N+2k-2)^{\frac{1}{2}}}{\left(\frac{N-a_k+2n-1}{2}\right)^{\frac{1}{2}}\left(\frac{N+a_k+2n-1}{2}\right)^{\frac{1}{2}}}}_{d_{N,n}} \\
&\quad \cdot \left(1+\mathcal{O}\left(\frac{1}{N}\right)\right)\left(1+\mathcal{O}\left(\frac{C_N^3}{\sqrt{N}}\right)\right)\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right). \qquad (28)
\end{aligned}
$$

We denote the product over k by $d_{N,n}$ and expand $d_{N,n}$ separately:

$$
\begin{aligned}
d_{N,n} &= \exp\left[\frac{1}{2}\sum_{k=1}^{n}\left\{\ln\left(N\left(1+\frac{2k-2}{N}\right)\right) - \ln\left(N\left(\frac{1}{2}+\frac{a_k+2n-1}{2N}\right)\right)\right.\right. \\
&\qquad \left.\left. - \ln\left(N\left(\frac{1}{2}+\frac{-a_k+2n-1}{2N}\right)\right)\right\}\right] \\
&\simeq \exp\left[\frac{1}{2N}\sum_{k=1}^{n}\Big(2(k-1)-(a_k+2n-1)\right. \\
&\quad \left. -(-a_k+2n-1) - N\ln N - 2N\ln\frac{1}{2}\Big) + \mathcal{O}\left(\frac{C_N^2}{N^2}\right)\right] \cdot \left(1+\mathcal{O}\left(\frac{1}{N}\right)\right) \\
&\simeq 2^n N^{-\frac{n}{2}} \exp\left[\frac{-n^2-n}{N}\right]\left(1+\mathcal{O}\left(\frac{C_N^2}{N^2}\right)\right)\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right). \qquad (29)
\end{aligned}
$$

Choosing the largest of the error bounds, we obtain

$$
\begin{aligned}
I_{N,n} &\simeq \frac{2^{nN+n^2+n}}{(\sqrt{2\pi})^n} N^{-n^2-\frac{n}{2}} \exp\left[-\sum_{k=1}^{n}\frac{a_k^2}{2N}\right] \\
&\qquad \times \exp\left[\frac{-8n^3-6n^2-n}{3N}\right]\left(1+\mathcal{O}\left(\frac{C_N^3}{\sqrt{N}}\right)\right). \qquad (30)
\end{aligned}
$$

Finally, for the measure density (11) we get

$$
\lim_{N\longrightarrow\infty} p_X(\lambda) = \left(\frac{1}{N}\right)^{n^2+n/2} \frac{2^{3n}}{(\sqrt{2\pi})^n} \frac{n!}{(2n)!(2n-2)!\dots 2!}
$$

$$\times \prod_{i<j}(a_i^2 - a_j^2)^2 \prod_{l=1}^{n} a_l^2 \exp\left[-\sum_{k=1}^{n} \frac{a_k^2}{N}\right]$$
$$= \left(\frac{2}{\sqrt{N}}\right)^n \phi(\{x_i\}) \left(1 + \mathcal{O}\left(\frac{C_N^3}{\sqrt{N}}\right)\right), \quad (31)$$

where $x_i = \frac{1}{\sqrt{N}} a_i$ and

$$\phi(\{x_i\}) = \frac{2^{2n} n!}{(\sqrt{2\pi})^n (2n)!(2n-2)! \dots 2!} \quad (32)$$
$$\times \prod_{i<j}(x_i^2 - x_j^2)^2 \prod_{l=1}^{n} x_l^2 \exp\left(-\frac{1}{2}\sum_k x_k^2\right). \qquad \square$$

Above we have proved the local theorem for a small domain $U_a(N)$ whose volume goes to zero as $N \longrightarrow \infty$. We will now prove the global theorem.

Theorem 2. *Let $X \sim \mu_N(\{a_i\})$. Then for every n-orthotope*

$$H_n = \{c_1, d_1\} \times \{c_2, d_2\} \times \dots \times \{c_n, d_n\} \subset P^+,$$

where c_i and d_i are fixed real numbers with $\{c_i\} < \{d_i\}$,

$$\lim_{N \longrightarrow \infty} \mathbf{P}\left\{c_i \leqslant \frac{1}{\sqrt{N}} X_i < d_i\right\} = \int\limits_{H_n} \phi(\{x_i\}) dx_1 \dots dx_n. \quad (33)$$

Proof. First, we use the triangle inequality to obtain

$$\left|\mathbf{P}\{c_i \leqslant \frac{1}{\sqrt{N}} X_i < d_i\} - \int\limits_{H_n} \phi(\{x_i\}) dx_1 \dots dx_n\right|$$
$$\leqslant \left| \sum_{a_i = \lceil c_i \sqrt{N} \rceil}^{\lceil d_i \sqrt{N} - 1 \rceil} p_X(\lambda) - \int\limits_{H_n} \phi(\{x_i\}) dx_1 \dots dx_n \right|$$
$$\leqslant \left| \sum_{a_i = \lceil c_i \sqrt{N} \rceil}^{\lceil d_i \sqrt{N} - 1 \rceil} \left(p_X(\lambda) - \left(\frac{2}{\sqrt{N}}\right)^n \phi\left(\frac{\{a_i\}}{\sqrt{N}}\right)\right) \right|$$
$$+ \left| \sum_{a_i = \lceil c_i \sqrt{N} \rceil}^{\lceil d_i \sqrt{N} - 1 \rceil} \left(\frac{2}{\sqrt{N}}\right)^n \phi\left(\frac{\{a_i\}}{\sqrt{N}}\right) - \int\limits_{H_n} \phi(\{x_i\}) dx_1 \dots dx_n. \right| \quad (34)$$

The second term is the difference between an integral and a Riemann sum for this integral, hence it goes to zero. For the first term, let $c > \max(c_i, d_i)$; then

$$\begin{aligned}\max_{|a_i+2n-1|<\sqrt{N}\cdot c}&\left|p_X(\lambda) - \left(\frac{2}{\sqrt{N}}\right)^n \phi\left(\frac{\{a_i\}}{\sqrt{N}}\right)\right| \\ &= \max_{|a_i+2n-1|<\sqrt{N}\cdot c}\left|\frac{p_X(\lambda)\left(\frac{\sqrt{N}}{2}\right)^n}{\phi\left(\frac{\{a_i\}}{\sqrt{N}}\right)} - 1\right| \cdot \left(\frac{2}{\sqrt{N}}\right)^n \phi\left(\frac{\{a_i\}}{\sqrt{N}}\right) \\ &\leqslant \mathcal{O}\left(\frac{1}{N}\right)\cdot \mathcal{O}\left(\frac{1}{N^{n/2}}\right) = \mathcal{O}\left(\frac{1}{N^{(n+1)/2}}\right). \end{aligned} \tag{35}$$

There are n sums of these expressions, and each has $\mathcal{O}(\sqrt{N})$ summands. Therefore, there are $\mathcal{O}\left(N^{n/2}\right)$ summands in total, and the sum is of order $\mathcal{O}\left(\frac{1}{\sqrt{N}}\right)$. □

In fact, we can check that the obtained density function (32) defines a probability measure by integrating it over the main Weyl chamber. We will use the following integral, studied by Macdonald in [14]:

$$\begin{aligned}\frac{1}{(2\pi)^{n/2}}\int\cdots\int \prod_{i=1}^{n}(2|x_i|^2)^\gamma \prod_{1\leqslant i<j\leqslant n}\left|x_i^2 - x_j^2\right|^{2\gamma}& \\ \times \exp\left(-\sum_{k=1}^{n}\frac{|x_k|^2}{2}\right)dx_1\dots dx_n &= \prod_{j=1}^{n}\frac{\Gamma(1+2j\gamma)}{\Gamma(1+\gamma)}.\end{aligned} \tag{36}$$

It is easy to see that if we integrate (32) over the whole space following Macdonald, then we will obtain the factor $2^n n!$. But $2^n n!$ is the order of the Weyl group of B_n, so integrating over the main Weyl chamber will give us exactly 1.

Theorem 3. *The sequence of discrete probability measures with densities $\mu_N(\lambda)$ on the main Weyl chamber converges weakly to the continuous measure μ with density $\phi(\{x_i\})$.*

Proof. We will use the following weak convergence criterion for measures [10]:

Let $\mathcal{E}$ be a class of open sets in a metric space X that is closed under finite intersections, and such that every open set can be represented as a *countable or finite union of sets from $\mathcal{E}$. Let μ_N, μ be probability Borel measures such that $\mu_N(E) \longrightarrow \mu(E)$ for all $E \in \mathcal{E}$. Then the sequence μ_N converges weakly to μ.*

Since $\mathcal{E}$ can be comprised of certain n-orthotopes, and for every n-orthotope $\mu_N(H_n) \longrightarrow \mu(H_n)$ by Theorem 2, the weak convergence $\mu_N \Rightarrow \mu$ is proved. $\square$

Example 1. We illustrate the result with a simple example. In Fig. 1, we plot values of the probability density function $\phi(\{x_i\})$ for the algebra $B_2(\mathrm{so}(5))$ and the probability mass function $p_X(\lambda)$ (indicated by dots) for the $(N = 100)$th power of the second fundamental representation L^{ω_2}.

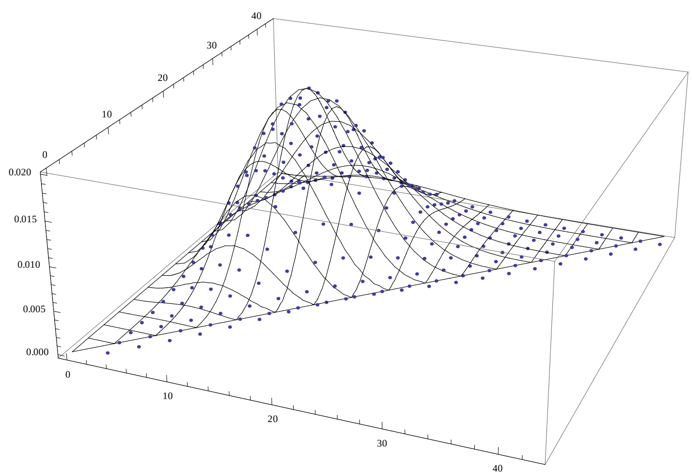

Figure 1. Values of the probability mass function $p_X(\lambda)$ (indicated by dots) and the probability density function $\left(\frac{2}{\sqrt{N}}\right)^n \cdot \phi(\{x_i\})$ for $n = 2$ and $N = 100$ in the main Weyl chamber in the rescaled axes $x_i = \frac{a_i}{\sqrt{N}}$.

Conclusion and outlook

The Lie algebra so$(2n+1)$ is, naturally, the Lie algebra of orthogonal $(2n+1)\times(2n+1)$ matrices, i.e., matrices A such that $A^t = A^{-1}$. The Killing form on so$(2n+1)$ is proportional to the bilinear form $\mathrm{tr}(A^tB)$, which is symmetric, positive definite, and determines a Riemannian metric on so$(2n+1)$.

The corresponding Riemannian integration measure is SO$(2n+1)$-invariant (with respect to the transformations $A \mapsto gAg^{-1}$). It is well known that the integral of an SO$(2n+1)$-invariant function over so$(2n+1)$ can be written in terms of its radial part (integration over eigenvalues):

$$\int f(A)dA = \int_{\mathbb{R}^n} f\left(\mathrm{diag}(a_1,\dots a_n,-a_n,\dots,-a_1)\right)\cdot\frac{2^{n^2}}{\pi^n n!} \times \prod_{1\leqslant i<j\leqslant n}\left(a_i^2-a_j^2\right)^2\prod_{k=1}^{n}a_k^2\,da_1\dots da_n. \tag{37}$$

Remarkably, the radial part of this measure gives precisely the non-Gaussian factors in the limit of the Plancherel measure, exactly as in Kerov's work [1] on su$(n+1)$. We expect this to hold for other simple Lie algebras.

Remark. There is a simple relation between the invariant measure on so$(2n+1)$ and the Haar measure on SO$(2n+1)$. If dg is the Haar measure normalized as $\int_G dg = 1$ and f is a function on G invariant with respect to conjugations, we have

$$\int f(g)\mathrm{dg} = \frac{2^{n^2}}{\pi^n n!}\int_{[0,\pi]^n} f(\Theta_1,\dots,\Theta_n) \times \prod_{1\leqslant i<j\leqslant n}(\cos(\Theta_i)-\cos(\Theta_j))^2\prod_{k=1}^{n}\sin^2\frac{\Theta_k}{2}d\Theta_1\dots d\Theta_n, \tag{38}$$

where $\lambda_j = e^{i\Theta_j}$ are distinct eigenvalues of a generic element $g\in G$.

Let $f_\epsilon(g)$ be a family of such functions supported by neighborhoods U_ϵ of $\Theta = 0$ such that $\frac{1}{\epsilon}U_\epsilon \to W \subset \mathbb{R}^n_+$ as $\epsilon \to 0$. Then

$$\int_{[0,\pi]^n} f(\Theta)d\mu(\Theta) = \int_W \epsilon^{2n^2+n} f(\epsilon a)(1+\mathcal{O}(\epsilon))d\mu_0(a), \tag{39}$$

where $d\mu(\Theta)$ is the radial part of the Haar measure as in (38) and $d\mu_0$ is the radial part of the measure on the Lie algebra so$(2n+1)$.

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