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WHICH CIRCLE BUNDLES CAN BE TRIANGULATED OVER $\partial\Delta^3$?

Abstract. We prove that having the boundary of the standard three-dimensional simplex $\partial\Delta^3$ as the base of a triangulation, one can triangulate only trivial and Hopf circle bundles.

§1. Introduction

We can solve the difficult problem in the title posed long ago by nobody. It seems that it leads right into the simplicial wildness of classical combinatorial topology. But it happens that we can manage it. The answer is Theorem 4.3:

> *Having the boundary of the standard three-dimensional simplex $\partial\Delta^3$ as the base of a triangulation, one can triangulate only trivial and Hopf circle bundles.*

This statement is an immediate output of a really difficult bare hands construction of a minimal triangulation of the Hopf bundle from [5] and a very special probabilistic form of the *rational* local combinatorial formula[1] ${}^pC_1(-)$ for the Chern–Euler class of triangulated circle bundles [6]. The proof is as follows: oriented circle bundles over the sphere are classified up to isomorphism by their integer characteristic Chern–Euler number, and these numbers span the entire space $H^2(S^2;\mathbb{Z}) \approx \mathbb{Z}$ (see [2] and [1, Chap. 2] for a survey). The number measures a kind of winding of the total space over the base. Its sign depends only on the orientation, whereas its absolute value $\mathbf{c}$ is of interest. This is called the "Gauss–Bonnet–Chern" theorem. The formula ${}^pC_1(-)$ endows a triangulated bundle with a simplicial rational 2-cocycle on the base of the triangulation. The rational cocycle represents the integer Chern–Euler class in the ordered simplicial cohomology of the base. The value of the cochain on a 2-simplex is a function of the combinatorics of the stalk of the triangulation over that simplex. The integer Chern–Euler number of the bundle is the result of

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[1] G. Gangopadhyay [3] invented a nice name for it: the "counting triangles formula."

pairing this rational cochain with the fundamental class. The key point coming from the formula ${}^{p}C_1(-)$ is that a stalk of a triangulation over a simplex makes a contribution of less than $\frac{1}{2}$ into the rational simplicial Chern–Euler cochain on the base. Therefore, having only four 2-simplices in $\partial\Delta^3$, we can, for any triangulation over $\partial\Delta^3$, achieve $\mathbf{c} < 2$. For $\mathbf{c} = 0$ then, we have a trivial bundle, and we can certainly triangulate it taking the product of triangulations of the base and the fiber. When $\mathbf{c} = 1$, it is the Hopf bundle $S^1 \to S^3 \xrightarrow{h} \partial\Delta^3$, which was triangulated by Madahar and Sarkara. The other Chern–Euler numbers cannot be expected. It is interesting to know that Seifert fibrations $S^3 \to S^2$ with any Hopf invariant can be triangulated over $\partial\Delta^3$, see [4]. Here we recall a few points from [6] and add Lemma 4.1 providing a proof for the main statement.

§2. Elementary simplicial circle bundles over a k-simplex and $(k+1)$-colored necklaces

An *elementary simplicial circle bundle* (elementary s.c. bundle) [6, Sec. 3] over an ordered k-simplex $\langle k\rangle$ is a map $\mathfrak{R} \xrightarrow{e} \langle k\rangle$ from a simplicial complex $\mathfrak{R}$ onto $\langle k\rangle$ whose geometric realization $|\mathfrak{e}|$ is a trivial PL fiber bundle over the geometric simplex Δ^k with fiber S^1. We equip $\mathfrak{e}$ with an orientation, a fixed generator of the 1-dimensional integer homology of $\mathfrak{R}$. Figure 1 presents a picture of an elementary s.c. bundle over the 1-simplex $\langle 1\rangle$. We are especially interested in elementary s.c. bundles over the 2-simplex $\langle 2\rangle$. To an elementary simplicial circle bundle $\mathfrak{e}$ over $\langle k\rangle$ with n maximal $(k+1)$-dimensional simplices in the total space, we associate a $(k+1)$-necklace $\mathcal{N}(\mathfrak{e})$, i.e., a cyclic word of length n in the ordered alphabet of $k+1$ letters indexed by the vertices of the base simplex. Any $(k+1)$-dimensional simplex of $\mathfrak{R}$ has a unique edge that shrinks to a vertex i of the base simplex by the elementary simplicial degeneration $\langle k+1\rangle \xrightarrow{\langle\sigma_i\rangle} \langle k\rangle$, $i = 0, 1, 2, \ldots, k$. Take a general fiber of the projection $|\mathfrak{e}|$. It is a circle broken into n intervals oriented by the orientation of the bundle, every interval being an intersection with a maximal $(k+1)$-simplex. The maximal simplex is uniquely determined by the vertex of the base where its collapsing edge collapses. This creates a coloring of the n intervals by $k+1$ ordered vertices of the base simplex. Thus we obtain a necklace $\mathcal{N}(\mathfrak{e})$ out of the combinatorics of $\mathfrak{e}$ (see [6, Sec. 16]). The process is illustrated in Fig. 2.

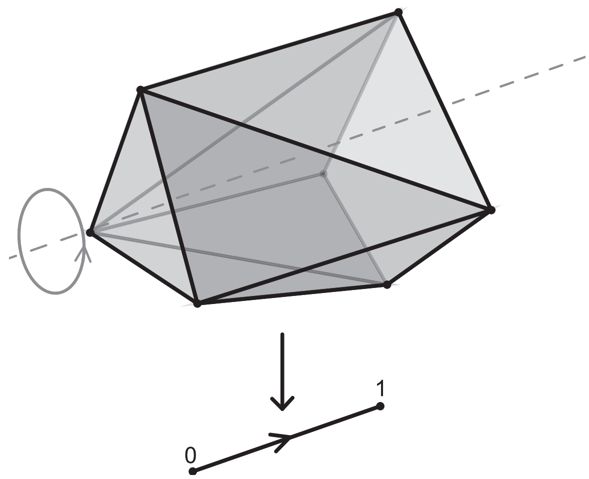

Figure 1. An elementary simplicial circle bundle over the interval.

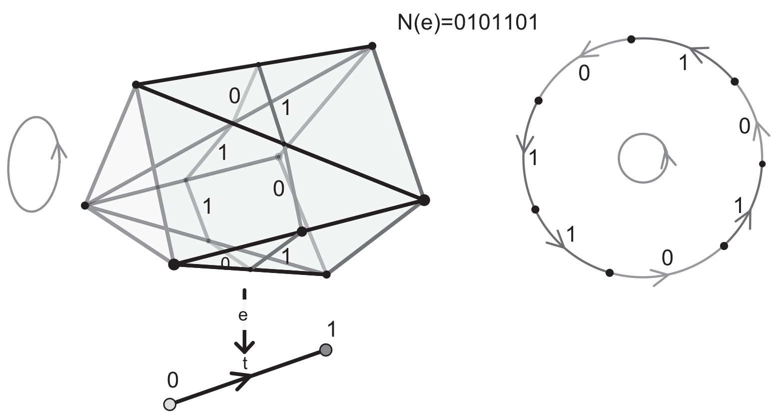

Figure 2.

§3. The parity local formula for the Chern–Euler class

Now we switch to the case of two-dimensional simplices in the base. We have the theorem [6, Theorem 4.1]. A word of length n in the alphabet

$[2] = \{0, 1, 2\}$ of three ordered letters can be viewed as a surjective map $[n] \xrightarrow{w} [2]$ (the map is not required to be monotone). Proper subwords of w are sections of this map, 3-letter subwords with all the letters different. A proper subword is a permutation of three elements, and it has a parity, even or odd. We define the rational parity of w as the expectation of the parities of all its proper subwords. Namely, we set

$$P(w) = \frac{\#(\text{even proper subwords}) - \#(\text{odd proper subwords})}{\#(\text{all proper subwords})}. \tag{1}$$

The parity of a permutation of three elements is invariant under cyclic shifts. Therefore, $P(w)$ is invariant under cyclic shifts of the word, and is an invariant of an oriented necklace with beads colored in tree colors $0, 1, 2$. We define a function of an elementary s.c. bundle over the 2-simplex by the formula

$${}^pC_1(\mathfrak{e}) = -\frac{1}{2}P(\mathcal{N}(\mathfrak{e})). \tag{2}$$

Theorem 4.1 of [6] states that the rational number ${}^pC_1(\mathfrak{e})$ is a local formula for the Chern–Euler class of a triangulated circle bundle.

It follows, by the Chern–Gauss–Bonnet theorem, that for any triangulated circle bundle $\mathfrak{E} \xrightarrow{\mathfrak{p}} \partial\Delta^3$ over $\partial\Delta^3$, the integer Chern–Euler number of $|\mathfrak{p}|$ can be computed as the value of the parity cocycle (2) on the fundamental class. This means the following. Let the simplicial complex $\partial\Delta^3$ be ordered, and let us fix an orientation of $\partial\Delta^3$, i.e., the fundamental class is fixed. Then the absolute Chern–Euler number of $|\mathfrak{p}|$ can be computed as the sum

$$\mathbf{c}(|\mathfrak{p}|) = \operatorname{abs}\left(\sum_{\sigma\in\partial\Delta^3(2)} (-1)^{\operatorname{or}\sigma}\left({}^pC_1(\mathfrak{p}_\sigma)\right)\right). \tag{3}$$

The sum is over all four two-dimensional ordered simplices of $\sigma \in \partial\Delta^3(2)$. It sums up the values of the formula ${}^pC_1(\mathfrak{p}_\sigma)$ on elementary subbundles of $\mathfrak{p}$ over simplices σ taken with the signs

$$(-1)^{\operatorname{or}\sigma} = \begin{cases} 1 & \text{if the orientation of } \sigma \text{ consides with the global} \\ & \text{orientaton of } \partial\Delta^3, \\ -1 & \text{otherwise.} \end{cases}$$

§4. Extremal parity expectation value

In addition to the existing knowledge, we need the following lemma.

Lemma 4.1. *For any elementary s.c. bundle $\mathfrak{e}$ over the 2-simplex,*

$$\operatorname{abs}\left({}^pC_1(\mathfrak{e})\right) < \frac{1}{2}.$$

Proof. Obviously, $\operatorname{abs}({}^pC_1(\mathfrak{e})) \leq \frac{1}{2}$. If $\operatorname{abs}({}^pC_1(\mathfrak{e})) = \frac{1}{2}$, then we look at the expectation value (1) and conclude that in the necklace $\mathcal{N}(\mathfrak{e})$, with probability one, all proper subwords are simultaneously even or simultaneously odd. Therefore, the necklace $\mathcal{N}(\mathfrak{e})$ looks like the word

$$\mathcal{N}(\mathfrak{e}) = 0000\ldots1111\ldots2222$$

up to cyclic shifts and reordering of letters. Namely, all three different letters $0, 1, 2$ occur in three solid blocks. Now let us see what it would mean for the bundle

$$\mathfrak{R} \xrightarrow{\mathfrak{e}} \langle 2\rangle$$

to have such a necklace. Consider the face of the base simplex $\langle 1\rangle \xrightarrow{\langle\delta_2\rangle} \langle 2\rangle$ and the induced subbundle

$$\delta_2^*\mathfrak{e} = \delta_2^*\mathfrak{R} \xrightarrow{\delta_2^*\mathfrak{e}} \langle 1\rangle$$

over the interval $\langle 1\rangle$. In this case (see [6, Secs. 16, 17]), the necklace $\mathcal{N}(\delta_2^*\mathfrak{e})$ of the subbundle $\delta_2^*\mathfrak{e}$ is the result of deleting the letter 2 from $\mathcal{N}(\mathfrak{e})$ and looks as follows:

$$\delta_2^*\mathcal{N}(\mathfrak{e}) = 0000011111 \sim 1100000111.$$

We may hope to draw the total complex $\delta_2^*\mathfrak{R}$ by reverse engineering the process of Sec. 2. It is a two-dimensional picture (see Fig. 3, where the upper and lower sides of the rectangle are identified). The edges of $\delta_2^*\mathfrak{R}$ that are projected surjectively onto the base interval correspond to pairs of adjacent triangles, which in turn correspond to pairs of cyclically adjacent letters in the cyclic word. In particular, there are two distinguished edges corresponding to two (cyclic) boundaries of two blocks: 1(10)000(01)11. What we see is that these boundary edges have both vertices in common. Therefore, $\delta_2^*\mathfrak{R}$ is not a simplicial complex, and hence $\mathfrak{R}$ is not a simplicial complex.

We have shown that for an elementary s.c. bundle, the value $|{}^pC_1(\mathfrak{e})|$ cannot be equal to $\frac{1}{2}$, and hence should be strictly smaller. $\square$

Corollary 4.2. *The Chern–Euler number of a bundle triangulated over $\partial\Delta^3$ is less than 2.*

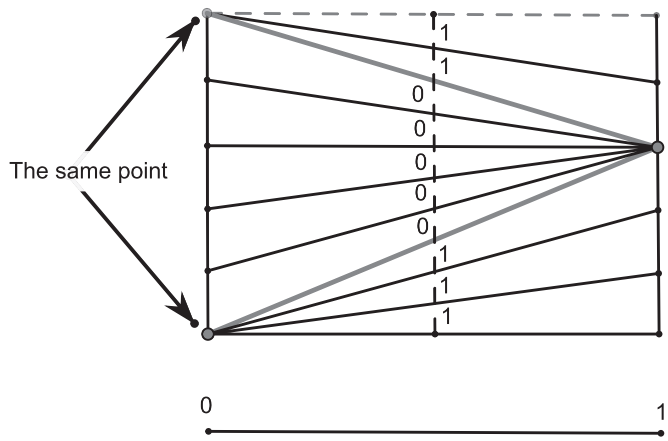

Figure 3.

Proof. This follows from Lemma 4.1 and our in-house Gauss–Bonnet–Chern formula (3). The sum of four terms each having absolute value smaller than $\frac{1}{2}$ has absolute value smaller than 2. $\square$

Theorem 4.3. *Having the boundary of the standard three-dimensional simplex $\partial\Delta^3$ as the base of a triangulation, one can triangulate only trivial and Hopf circle bundles.*

Proof. From Corollary 4.2 it follows that we may hope to triangulate only trivial and Hopf bundles. The trivial bundle can be triangulated by taking the product of triangulations of the base and the fiber; the Hopf bundle was triangulated by Madahar and Shakara [5]. $\square$

§5. Semi-simplicial boundary

The bundle triangulation in Fig. 3 is "semi-simplicial." Only semi-simplicial triangulations realize the extremal case $\mathbf{c} = 2$, the tangent bundle of S^2. We hope to cover these very interesting triangulations in more detail somewhere else. We consider this behavior of the bundle combinatorics in relation to the base combinatorics as a nice output of the local formula (2).

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