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THE BOUNDARY OF THE REFINED KINGMAN GRAPH

Abstract. We introduce the refined Kingman graph $\mathbb{D}$ whose vertices are indexed by the set of compositions of positive integers and multiplicity function reflects the Pieri rule for quasisymmetric monomial functions. We show that the Martin boundary of $\mathbb{D}$ can be identified with the space Ω of all sets of disjoint open subintervals of $[0,1]$ and coincides with the minimal boundary of $\mathbb{D}$.

§1. Introduction

The study of the set of nonnegative harmonic functions for particular examples of $\mathbb{Z}_+$-graded graphs (Bratteli diagrams) has a long and rich history, see [4–8] and references therein. Recall that a graph Δ with vertex set $\cup_{n=0}^{\infty}\Delta_n$ is a $\mathbb{Z}_+$*-graded graph* if for any two vertices $\mu \in \Delta_n$, $\boldsymbol{\mu} \in \Delta_N$ there is an edge from μ to $\boldsymbol{\mu}$ only if $N = n+1$. Such a graph Δ is fully determined by the set of vertices and a *multiplicity function* $\kappa : \left(\bigcup\limits_{n=0}^{\infty} \Delta_n\right)^2 \mapsto \mathbb{N}\cup\{0\}$. We consider graphs with $|\Delta_0| = 1, \Delta_0 = \{\varnothing\}$. A function $h : \Delta \mapsto \mathbb{R}_+$ is a *normalized harmonic* function if $h(\varnothing) = 1$ and

$$h(\mu) = \sum \kappa(\mu, \boldsymbol{\mu}) h(\boldsymbol{\mu}) \quad \text{for any} \quad \mu \in \bigcup_{n=0}^{\infty} \Delta_n.$$

Denote the space of such functions by $\mathcal{H}(\Delta)$. Recall that there is a natural correspondence $h \leftrightarrow \nu_h$ between the nonnegative harmonic functions on a $\mathbb{Z}_+$-graded graph and the central measures on it. The *absolute* of a graph Δ is the set of all ergodic probability central measures on Δ, see details in Sec. 3.

A natural example of a $\mathbb{Z}_+$-graded graph is the Kingman graph $\mathbb{K}$. Its vertices are indexed by partitions, $\mathbb{K}_n = \mathrm{Part}_n$, and the multiplicities of

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edges are defined as follows: for

$$\lambda = \left(1^{m_1}, 2^{m_2}, \dots, k^{m_k+1}, (k+1)^{m_{k+1}-1}, \dots\right)$$

and

$$\boldsymbol{\lambda} = \left(1^{m_1}, 2^{m_2}, \dots, k^{m_k}, (k+1)^{m_{k+1}}, \dots\right),$$

the multiplicity $\kappa_{\mathbb{K}}(\lambda, \boldsymbol{\lambda})$ equals m_{k+1}. Recall that the Kingman graph reflects the Pieri rule for monomial symmetric functions $m_\lambda(x_1, x_2, \dots)$ in the following way:

$$m_\lambda(x_1, x_2, \dots) \times \left(\sum x_i\right) = \sum \kappa_{\mathbb{K}}(\lambda, \boldsymbol{\lambda}) m_{\boldsymbol{\lambda}}(x_1, x_2, \dots).$$

(See [10] for the definition of monomial symmetric functions and their basic properties.)

Monomial symmetric functions provide examples of normalized harmonic functions on the Kingman graph, and they admit a continuous extension to a topological space that naturally contains the set of vertices of the Kingman graph. Namely, the so-called *extended* monomial symmetric functions are defined on the space

$$\Omega_{\mathbb{K}} = \{(\alpha_1 \geqslant \alpha_2 \geqslant \dots \geqslant 0) : \sum_i \alpha_i \leqslant 1\}$$

by the formula

$$\widetilde{m}_\lambda(\alpha_1, \alpha_2, \dots) = \widetilde{m}_\lambda(\alpha_1, \alpha_2, \dots; \gamma) := \sum_{j=0}^{k} \frac{\gamma^j}{j!} m_{1^{k-j}\lambda'}(\alpha_1, \alpha_2, \dots),$$

for $\lambda = 1^k \lambda'$, $\lambda' = 2^{k_2} 3^{k_3} \dots$, and $\gamma = 1 - \sum_i \alpha_i$.

In Sec. 3, we recall the definitions of the Martin boundary $E_{\mathrm{Mart}}(\Delta)$ and the minimal boundary $E_{\min}(\Delta)$ of a $\mathbb{Z}_+$-graded graph Δ. The extended monomial symmetric functions form the boundary of $\mathbb{K}$.

Theorem 1.1 ([3, 4, 9]).

(i) $\Omega_{\mathbb{K}} \cong E_{\mathrm{Mart}}(\mathbb{K}) = E_{\min}(\mathbb{K})$;

(ii) *The integral representation*

$$\phi(\lambda) = \int_{\Omega_{\mathbb{K}}} \widetilde{m}_\lambda(\omega) dP_\phi$$

gives a one-to-one correspondence between the space $\mathcal{H}(\mathbb{K})$ of normalized nonnegative harmonic functions on $\mathbb{K}$ and the space of probability measures on $\Omega_{\mathbb{K}}$.

It follows that *the absolute of* $\mathbb{K}$ *is* $\left\{\nu_{\widetilde{m}.(\omega)}\right\}_{\omega\in\Omega}$.

The main protagonist of this note is the *refined Kingman graph* $\mathbb{D}$. The vertices of the refined Kingman graph are indexed by the set Comp of all compositions. We consider the set of *monomial quasisymmetric functions* $\{M_\mu\}_{\mu\in\mathrm{Comp}}$ and define the multiplicity function $\kappa_{\mathbb{D}}(\mu,\boldsymbol{\mu})$ using the corresponding Pieri rule:

$$M_\mu(x_1,x_2,\dots)\times\left(\sum x_i\right)=\sum\kappa_{\mathbb{D}}(\mu,\boldsymbol{\mu})M_{\boldsymbol{\mu}}(x_1,x_2,\dots). \tag{1}$$

A general treatment of quasisymmetric functions can be found in [11, Sec. 7.19].

In Sec. 2, we consider the topological space $\Omega=\Omega_{\mathbb{D}}$ of all (finite or countable) sets of disjoint open subintervals of $[0,1]$ and define the *extended* monomial quasisymmetric functions $\widetilde{M_\mu}:\Omega\to\mathbb{R}_+$. There is a natural inclusion $\mathrm{Comp}\subset\Omega$, $\boldsymbol{\mu}\mapsto\omega_{\boldsymbol{\mu}}$. Then $\widetilde{M}_\mu(\omega)$, for any $\mu\in\mathrm{Comp}$, is the unique function such that $\widetilde{M}_\mu(\omega)$ is continuous on Ω and

$$\widetilde{M}_\mu(\omega_{\boldsymbol{\mu}})=M_\mu(\boldsymbol{\mu})\text{ for every }\boldsymbol{\mu}\in\mathrm{Comp}\,.$$

The space Ω is the *geometric boundary* of the graph $\mathbb{D}$, see Definition 2.1 for details. Our main result reads as follows: the Martin boundary $E_{\mathrm{Mart}}(\mathbb{D})$ of the graph $\mathbb{D}$ coincides with its minimal boundary $E_{\min}(\mathbb{D})$ and can be identified with its geometric boundary Ω.

Main theorem.

$$\Omega\cong E_{\mathrm{Mart}}(\mathbb{D})=E_{\min}(\mathbb{D}). \tag{2}$$

Proofs are given in Sec. 3.

A similar description of the minimal boundary of $\mathbb{D}$ was obtained by Gnedin in [2], using a different approach. We reprove Gnedin's theorem using the explicit description of the Martin boundary of $\mathbb{D}$.

Theorem 1.2 ([2]). *The integral representation*

$$\phi(\mu)=\int_\Omega\widetilde{M}_\mu(\omega)dP_\phi$$

gives a one-to-one correspondence between the space $\mathcal{H}(\mathbb{D})$ of normalized nonnegative harmonic functions on $\mathbb{D}$ and the space of probability measures on Ω.

It follows that *the absolute of $\mathbb{D}$ is* $\left\{\nu_{\widetilde{M.(\omega)}}\right\}_{\omega\in\Omega}$.

We want to emphasize that Eqs. (2) do not follow from Theorem 1.2. We also provide explicit formulas for quasisymmetric monomial functions in Sec. 4.

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§2. Definitions

Recall that a *composition* of a number $n \in \mathbb{N}$ is a sequence of positive integers $\mu = (\mu_1, \mu_2, \dots, \mu_{\ell(\mu)})$ such that $\sum\limits_i \mu_i = n$. The number n is referred to as the *weight* of μ. The number $\ell(\mu)$ is called the *length* of μ. The set of compositions of weight n is denoted by Comp_n, and we set $\mathrm{Comp} = \cup_{n=1}^{\infty} \mathrm{Comp}_n$.

For any composition μ and a linear order $\prec$ on a multiset $\{x_i\}$, we define a *quasisymmetric monomial function* as follows:

$$M_\mu(x_1, x_2, \dots) = \sum_{x_{i_1} \prec x_{i_2} \prec \dots \prec x_{i_\ell}} x_{i_1}^{\mu_1} \cdots x_{i_{\ell(\mu)}}^{\mu_{\ell(\mu)}}.$$

Quasisymmetric monomial functions can be regarded as a refined version of monomial symmetric functions. For any partition λ, we have

$$m_\lambda(x_1, x_2, \dots) = \sum_{\mu \in S_{\ell(\lambda)}\lambda} M_\mu(x_1, x_2, \dots),$$

where the symmetric group $S_{\ell(\lambda)}$ acts on the partition λ by permutations of its parts. For example,

$$\begin{aligned} m_{(2,1)}(x_1, x_2, \dots) &= \sum_{i \neq j} x_i^2 x_j = \sum_{x_i \prec x_j} x_i^2 x_j + \sum_{x_i \prec x_j} x_i x_j^2 \\ &= M_{(2,1)}(x_1, x_2, \dots) + M_{(1,2)}(x_1, x_2, \dots). \end{aligned}$$

Remark 2.1. It is not difficult to check that the space QSymm of quasisymmetric functions is a graded algebra: $\mathrm{QSymm} = \bigoplus \mathrm{QSymm}_n$, where QSymm_n is spanned by $\{M_\mu\}_{\mu \in \mathrm{Comp}_n}$. In other words, if $f \in \mathrm{QSymm}_m$ and $g \in \mathrm{QSymm}_n$, then $fg \in \mathrm{QSymm}_{m+n}$, see [11, Ex. 7.93].

The *refined Kingman graph* $\mathbb{D}$ is constructed as follows. The vertices of the nth level $\mathbb{D}_n$ of $\mathbb{D}$ are indexed by the compositions of weight n. There is an edge between the vertices indexed by $\mu \in D_n$ and $\boldsymbol{\mu} \in \mathbb{D}_{n+1}$ if one of the two situations occurs: for $\mu = (\mu_1, \dots, \mu_{\ell(\mu)})$,

- either there exists $j = 1, \dots, \ell(\mu)$ with
$$\boldsymbol{\mu} = \big(\mu_1, \dots, \mu_j + 1, \dots \mu_{\ell(\mu)}\big),$$
- or there exists $j = 1, \dots, \ell(\mu) + 1$ such that
$$\boldsymbol{\mu}_i = \begin{cases} \mu_i & \text{if } i < j, \\ 1 & \text{if } i = j, \\ \mu_{i-1} & \text{if } i > j. \end{cases}$$

In the second case, there can be several possible choices of j satisfying the condition. Then we say that μ and $\boldsymbol{\mu}$ are connected by multiple edges, the multiplicity being equal to the number of possible choices of j. If vertices $\mu \in \mathbb{D}_n$ and $\boldsymbol{\mu} \in \mathbb{D}_{n+1}$ are connected by an edge, then we write $\mu \nearrow \boldsymbol{\mu}$ and denote the corresponding multiplicity by $\kappa_{\mathbb{D}}(\mu, \boldsymbol{\mu})$. One can easily check that (1) holds.

The main problem addressed in the present paper is to describe all normalized nonnegative harmonic functions on $\mathbb{D}$. Let f be a harmonic function on $\mathbb{D}$, i.e., $f(\mu) = \sum_{\mu \nearrow \boldsymbol{\mu}} \kappa_{\mathbb{D}}(\mu, \boldsymbol{\mu}) f(\boldsymbol{\mu})$. The *trace* of f is a function defined on the set of partitions:
$$(\operatorname{tr} f)(\lambda) = \sum_{\mu \in S_{\ell(\lambda)}\lambda} f(\mu).$$

Since f is a harmonic function,
$$(\operatorname{tr} f)(\lambda) = \sum_{\mu \in S_{\ell(\lambda)}\lambda} \sum_{\mu \nearrow \boldsymbol{\mu}} \kappa_{\mathbb{D}}(\mu, \boldsymbol{\mu}) f(\boldsymbol{\mu}).$$

Note that any composition $\boldsymbol{\mu}$ such that $\sigma\lambda \nearrow \boldsymbol{\mu}$ can be obtained as a permutation of the parts of a partition $\boldsymbol{\lambda}$ with $\lambda \nearrow \boldsymbol{\lambda}$. Thus, changing the order of summation, we obtain
$$(\operatorname{tr} f)(\lambda) = \sum_{\lambda \nearrow \boldsymbol{\lambda}} \kappa_{\mathbb{K}}(\lambda, \boldsymbol{\lambda}) (\operatorname{tr} f)(\boldsymbol{\lambda}),$$

where $\kappa_{\mathbb{K}}(\lambda, \boldsymbol{\lambda})$ are multiplicities of the Kingman graph. In other words, if $f \in \mathcal{H}(\mathbb{D})$ then $\operatorname{tr}(f) \in \mathcal{H}(\mathbb{K})$, and, moreover, we can recover the graph $\mathbb{K}$ from $\mathbb{D}$ if for any partition λ we glue together the compositions of $S_{\ell(\lambda)}\lambda$ as well as the corresponding edges. This observation justifies the term "*refined* Kingman graph."

For a fixed positive integer $n \in \mathbb{N}$ there is an obvious bijection between the compositions $\mu = (\mu_1, \mu_2, \dots)$ with $|\mu| = n$ and the subdivisions of the unit interval into disjoint open subintervals with endpoints in $\{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$:

$$\mu = (\mu_1, \mu_2, \dots) \ \mapsto \ \Big]0, \frac{\mu_1}{n}\Big[\cup \Big]\frac{\mu_1}{n}, \frac{\mu_1+\mu_2}{n}\Big[\cup \dots \cup \Big]\frac{\sum\limits_{i=1}^{\ell(\mu)-1} \mu_i}{n}, 1\Big[.$$

Every open subinterval[1] $]a, b[\subset [0,1]$ is determined by its size and its distance to 0, that is, by the pair $(b-a, a)$. So, we rewrite the previous mapping as

$$\mu = (\mu_1, \mu_2, \dots) \mapsto \omega_\mu = \left\{ \left(\frac{\mu_1}{|\mu|}, 0\right), \left(\frac{\mu_2}{|\mu|}, \frac{\mu_1}{|\mu|}\right), \dots, \left(\frac{\mu_{\ell(\mu)}}{|\mu|}, \frac{\sum\limits_{i=1}^{\ell(\mu)-1} \mu_i}{|\mu|}\right) \right\}.$$

We identify the set Comp of all compositions with the set

$$\widetilde{\mathbb{D}} = \bigcup_{n=1}^{\infty} \bigcup_{\mu \in \mathbb{D}_n} \left(\frac{1}{n}, \omega_\mu\right).$$

Definition 2.1. *Denote by Ω the topological space of sets of ordered pairs $\{(\alpha_i, \Gamma_i)\}_{i=1}^N$, $N \in \mathbb{N} \cup \{\infty\}$, such that*

$$\alpha_i > 0, \quad \Gamma_i \geqslant 0 \ \ \text{and} \ \ 1 \geqslant \Gamma_i + \alpha_i \ \text{for any } i,$$
$$]\Gamma_i, \Gamma_i + \alpha_i[\ \cap \ ]\Gamma_j, \Gamma_j + \alpha_j[= \varnothing \ \text{for any } i \neq j,$$
$$\alpha_1 \geqslant \alpha_2 \geqslant \dots, \text{and if } \alpha_i = \alpha_{i+1} \text{ then } \Gamma_i > \Gamma_{i+1}.$$

We also introduce on $\{(\alpha_i, \Gamma_i)\}_{i=1}^N$ a linear order $\prec$ that compares the distances of subintervals to 0: namely, $(\alpha_i, \Gamma_i) \prec (\alpha_j, \Gamma_j)$ if and only if $\Gamma_i < \Gamma_j$. Sometimes, abusing notation, we denote a pair (α_i, Γ_i) by its first coordinate α_i. In particular, the notation $\alpha_i \prec \alpha_j$ means that $\Gamma_i < \Gamma_j$.

We say that a sequence $\omega_n = \{(\alpha_i(n), \Gamma_i(n))\}_{i=1}^{N(n)}$ converges to $\omega = \{(\alpha_i, \Gamma_i)\}_{i=1}^N$ if

[1] We denote open intervals by $]a, b[$ to distinguish them from ordered pairs (a, b).

(1) *for any finite* $i \leqslant N$*, the number of terms for which* $N(n) < i$ *is finite, and after omitting these terms we have* $\alpha_i(n) \longrightarrow \alpha_i$*,* $\Gamma_i(n) \longrightarrow \Gamma_i$*;*
(2) *for any finite* $i > N$*, either the number of terms with* $N(n) \geqslant i$ *is finite, or we have* $\alpha_i(n) \longrightarrow 0$ *for the subsequence consisting of the terms with* $N(n) \geqslant i$*.*

The topological space Ω is, obviously, sequentially compact.
Set

$$\widetilde{\Omega} = \widetilde{\mathbb{D}} \cup (\{0\} \times \Omega).$$

Given a sequence of compositions $(\mu(k))$, we say that

$$\Big(1/|\mu(k)|, \omega_{\mu(k)}\Big) \to (q, \omega) \in [0,1] \times \Omega \quad \text{as} \quad k \to \infty$$

if and only if $1/|\mu(k)| \to q$ in $[0,1]$ and $\omega_{\mu(k)} \to \omega$ in Ω. Note that the boundary of the subset $\widetilde{\mathbb{D}}$ in $[0,1] \times \Omega$ is $\{0\} \times \Omega \cong \Omega$; following [4], we call Ω the *geometric boundary* of the graph $\mathbb{D}$.

§3. Proofs

The standard graphic representation of a partition is a Young diagram, and we use a similar representation for compositions. The only difference is that now the rows of a diagram are not ordered by their length. Given any diagram $\mu \in \mathrm{Comp}(n)$, we define a (*row-strict*) *tableau* on μ as a numbering of the squares of μ with the numbers $1, \ldots, n$ such that the numbers increase along each row. Recall that for any vertex $\lambda(n) \in \mathbb{K}_n$ there is a bijection between the set of paths $\lambda(1) \nearrow \cdots \nearrow \lambda(n)$ in the graph $\mathbb{K}$ with $|\lambda(i)| = i$ and the set of row-strict tableaux on the diagram $\lambda(n)$. We have a similar description for paths in $\mathbb{D}$: for any vertex $\mu(n) \in \mathbb{D}_n$ there is a bijection between the set of paths $\mu(1) \nearrow \cdots \nearrow \mu(n)$ in the graph $\mathbb{D}$ with $|\mu(i)| = i$ and the set of row-strict tableaux on the diagram $\mu(n)$. Moreover, for any $\mu \in \mathbb{D}_n$, $\boldsymbol{\mu} \in \mathbb{D}_N$ with $n \leqslant N$, we have the following two-step description of the set of paths from μ to $\boldsymbol{\mu}$ in $\mathbb{D}$. Take any inclusion $\mu \subset \boldsymbol{\mu}$, and then take any row-strict tableau on the *skew diagram* $\boldsymbol{\mu}/\mu$. Denote by $\dim(\boldsymbol{\mu})$ the number of paths from $\varnothing \in \mathbb{D}_0$ to $\boldsymbol{\mu} \in \mathbb{D}$, and by $\dim(\mu, \boldsymbol{\mu})$ the number of paths from $\mu \in \mathbb{D}$ to $\boldsymbol{\mu} \in \mathbb{D}$. From the discussion above we obtain the following lemma.

Lemma 3.1. *For $\mu = (\mu_1, \mu_2, \dots, \mu_\ell) \in \mathbb{D}_n$, $\boldsymbol{\mu} = (\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dots, \boldsymbol{\mu}_m) \in \mathbb{D}_N$ with $n \leqslant N$, we have*

$$\dim(\boldsymbol{\mu}) = \frac{N!}{\boldsymbol{\mu}_1! \boldsymbol{\mu}_2! \dots} = \binom{N}{\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dots},$$
$$\frac{\dim(\mu, \boldsymbol{\mu})}{\dim \boldsymbol{\mu}} = \frac{(N-n)!}{N!} \sum_{1 \leqslant i_1 < \dots < i_\ell \leqslant m} (\boldsymbol{\mu}_{i_1})_{\mu_1} (\boldsymbol{\mu}_{i_2})_{\mu_2} \cdots ,$$

where $(x)_n$ denotes the Pochhammer symbol (falling factorial):

$$(x)_n = x(x-1)\dots(x-n+1).$$

Lemma 3.2. *In the notation of the previous lemma, we have*

$$\left| \frac{\dim(\mu, \boldsymbol{\mu})}{\dim \boldsymbol{\mu}} - \frac{M_\mu(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dots)}{N(N-1)\dots(N-n+1)} \right| \leqslant \frac{C(\mu)}{N},$$

where the constant $C(\mu)$ depends only on μ.

Proof. Set $\xi = \max(\mu_1, \dots, \mu_{\ell(\mu)})$. By the previous lemma, we have

$$\left| \frac{\dim(\mu, \boldsymbol{\mu})}{\dim \boldsymbol{\mu}} - \frac{M_\mu(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dots)}{N(N-1)\dots(N-n+1)} \right|$$
$$= \frac{(N-n)!}{N!} \left| \sum_{1 \leqslant i_1 < \dots < i_\ell \leqslant m} (\boldsymbol{\mu}_{i_1})_{\mu_1} (\boldsymbol{\mu}_{i_2})_{\mu_2} \dots - M_\mu(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dots) \right|$$
$$\leqslant \frac{(N-n)!}{N!} \left| \sum_{1^{\ell(\mu)} \subset \mu' \subsetneq \mu} \xi^{n-|\mu'|} c(\mu, \mu') M_{\mu'}(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dots) \right|$$
$$= \frac{(N-n)!}{N!} \left| \sum_{1^{\ell(\mu)} \subset \mu' \subsetneq \mu} \xi^{n-|\mu'|} c(\mu, \mu') N^{|\mu'|} M_{\mu'}\left(\frac{\boldsymbol{\mu}_1}{N}, \frac{\boldsymbol{\mu}_2}{N}, \dots\right) \right| \leqslant \frac{C(\mu)}{N},$$

where $c(\mu, \mu')$ is a combinatorial factor depending only on μ and μ'. Here we have used the estimate $|M_{\mu'}(x_1, x_2, \dots)| \leqslant 1$ valid for any $\mu' \in \mathrm{Comp}$ if $\sum x_i \leqslant 1$. □

For $\mu = (\mu_1, \dots, \mu_\ell) \in \mathrm{Comp}_n$, we define an extension of the corresponding quasisymmetric monomial function to Ω. If no part of μ is of length 1, then for $\omega = \{(\alpha_i, \Gamma_i)\}$ we set

$$\widetilde{M}_\mu(\omega) = \sum_{\alpha_1 \prec \dots \prec \alpha_\ell} \alpha_1^{\mu_1} \cdots \alpha_\ell^{\mu_\ell} = M_\mu(\alpha_1, \alpha_2, \dots).$$

However, it is not difficult to check that a naive extension of M_μ fails to be continuous if μ contains parts of length 1. The idea is to set, for any $\boldsymbol{\mu} \in \mathrm{Comp}$,

$$\widetilde{M}_\mu(\omega_{\boldsymbol{\mu}}) = M_\mu(\alpha_1, \alpha_2, \dots),$$

where $\omega_{\boldsymbol{\mu}} = \{(\alpha_i, \Gamma_i)\}_{i=1}^N$, and then to derive defining relations for $\widetilde{M}_\mu(\omega)$, $\omega \in \Omega$, from the required continuity on Ω.

Set $\widetilde{M}_{(1)}(\omega) \equiv 1$. We use the following equality, which holds for any pair α_i, α_j with $\alpha_i \prec \alpha_j$ in $\omega_{\boldsymbol{\mu}}$:

$$\begin{aligned}\frac{(\Gamma_j - \Gamma_i - \alpha_i)^p}{p!} &= \frac{\Big(\sum\limits_{\alpha_i \prec \alpha \prec \alpha_j} \alpha\Big)^p}{p!} \\ &= \sum_{\rho \in \mathrm{Comp}_p} \sum_{\alpha_i \prec \alpha_1 \prec \dots \prec \alpha_{\ell(\rho)} \prec \alpha_j} \frac{\alpha_1^{\rho_1}}{\rho_1!} \cdots \frac{\alpha_{\ell(\rho)}^{\rho_{\ell(\rho)}}}{\rho_{\ell(\rho)}!}.\end{aligned} \tag{3}$$

In particular, it follows that for $\mu_1 \neq 1$, $\mu_2 \neq 1$ we have

$$\sum_{\alpha_i \prec \alpha_j} (\alpha_i)^{\mu_1} \frac{\Big(\Gamma_j - \Gamma_i - \alpha_j\Big)^p}{p!} (\alpha_j)^{\mu_2} = \sum_{\rho \in \mathrm{Comp}(p)} \frac{\widetilde{M}_{(\mu_1, \rho, \mu_2)}(\omega_{\boldsymbol{\mu}})}{\rho_1! \rho_2! \dots \rho_{\ell(\rho)}!}. \tag{4}$$

This motivates us to consider the following object. For a point $\omega \in \Omega$, a fixed collection of compositions $\mu^{(1)}, \dots, \mu^{(k)}$ such that none of them contains parts of length 1, and a collection of numbers $p_0, \dots, p_k \in \mathbb{N} \cup \{0\}$, we define

$$\begin{aligned}&\widetilde{M}^{p_0, \dots, p_k}_{\mu^{(1)}, \dots, \mu^{(k)}}(\omega) \\ &\quad = \sum_{\substack{\alpha_{1;(1)} \prec \alpha_{2;(1)} \prec \dots \prec \alpha_{\ell(\mu^{(1)});(1)} \\ \dots \\ \prec \alpha_{1;(k)} \prec \alpha_{2;(k)} \prec \dots \prec \alpha_{\ell(\mu^{(k)});(k)}}} \Gamma_{1;(1)}^{p_0} \prod_{i=1}^k \alpha_{1;(i)}^{\mu_1^{(i)}} \alpha_{2;(i)}^{\mu_2^{(i)}} \cdots \alpha_{\ell(\mu^{(i)});(i)}^{\mu^{(i)}_{\ell(\mu^{(i)})}} \\ &\qquad\qquad \times \big(\Gamma_{1;(i+1)} - \Gamma_{\ell(\mu^{(i)});(i)} - \alpha_{\ell(\mu^{(i)});(i)}\big)^{p_i},\end{aligned}$$

where we assume that $\Gamma_{1;(k+1)} = 1$. It follows from (3) that for any composition $\boldsymbol{\mu} \in \mathrm{Comp}$ and $\omega = \omega_{\boldsymbol{\mu}}$, the following identity holds:

$$\frac{\widetilde{M}^{p_0,\dots,p_k}_{\mu^{(1)},\dots,\mu^{(k)}}(\omega)}{p_0!\cdots p_k!} = \sum_{\substack{\rho^{(0)}\in\mathrm{Comp}_{p_0}\\ \cdots \\ \rho^{(k)}\in\mathrm{Comp}_{p_k}}} \frac{\widetilde{M}_{\rho^{(0)}\mu^{(1)}\rho^{(1)}\cdots\mu^{(k)}\rho^{(k)}}(\omega)}{\prod\limits_{i=0}^{k}\prod\limits_{j=1}^{\ell(\rho^{(i)})}\rho^{(i)}_j!}. \tag{5}$$

We postulate that identity (5) holds for any $\omega \in \Omega$. This allows us to define inductively the extended monomial quasisymmetric function $\widetilde{M}_\mu(\omega)$ for an arbitrary composition μ. An explicit formula for $\widetilde{M}_\mu(\omega)$ is given in Sec. 4; we will not use it in what follows.

Lemma 3.3. *For any $\mu \in \mathrm{Comp}$, the extended quasisymmetric monomial function $\widetilde{M}_\mu$ is a continuous function on Ω.*

Proof. The base case. For $\mu = (\mu_1, \dots, \mu_\ell) \in \mathrm{Comp}$ with $\mu_i > 1$, we can write

$$\begin{aligned}\widetilde{M}_\mu(\omega) &= \sum_{i_1\prec\cdots\prec i_\ell} \alpha_{i_1}^{\mu_1}\cdots\alpha_{i_\ell}^{\mu_\ell}\\ &= \sum_{\substack{i_1\prec\cdots\prec i_\ell;\\ i_k<N,\forall k}} \alpha_{i_1}^{\mu_1}\cdots\alpha_{i_\ell}^{\mu_\ell} + \sum_{\substack{i_1\prec\cdots\prec i_\ell;\\ \exists k, i_k\geqslant N}} \alpha_{i_1}^{\mu_1}\cdots\alpha_{i_\ell}^{\mu_\ell}.\end{aligned} \tag{6}$$

For a sequence of points in Ω, we have

$$\lim_{n\to\infty} \sum_{\substack{i_1\prec\cdots\prec i_\ell;\\ i_k<N,\forall k}} \Big(\alpha_{i_1}(n)\Big)^{\mu_1}\cdots\Big(\alpha_{i_\ell}(n)\Big)^{\mu_\ell} = \sum_{\substack{i_1\prec\cdots\prec i_\ell;\\ i_k<N,\forall k}} \alpha_{i_1}^{\mu_1}\cdots\alpha_{i_\ell}^{\mu_\ell}$$

if $\lim_{n\to\infty}\alpha_i(n) = \alpha_i$. Moreover, the second term in (6) converges to zero uniformly, since $\alpha_k \leqslant \frac{1}{k}$ for all k and

$$\sum_{\substack{i_1\prec\cdots\prec i_\ell;\\ \exists k, i_k\geqslant N}} \alpha_{i_1}^{\mu_1}\cdots\alpha_{i_\ell}^{\mu_\ell} \leqslant \sum_{i=N}^{\infty}\frac{\ell}{i^2}$$

due to the estimate

$$\sum_{\substack{i_1\prec\cdots\prec i_\ell;\\ \max(i_k)=i}} \alpha_{i_1}^{\mu_1}\cdots\alpha_{i_\ell}^{\mu_\ell} \leqslant \sum_{j=1}^{\ell} \sum_{\substack{i_1\prec\cdots\prec i_\ell;\\ i_k\leqslant i_j=i\ \forall k}} \alpha_{i_1}^{\mu_1}\cdots\alpha_{i_\ell}^{\mu_\ell}$$

$$\leqslant \frac{1}{i^2}\sum_{j=1}^{\ell} \sum_{\substack{i_1\prec\cdots\prec i_\ell;\\ i_k\leqslant i_j=i\ \forall k}} \alpha_{i_1}^{\mu_1}\cdots\alpha_{i_j}^{\mu_j-2}\cdots\alpha_{i_\ell}^{\mu_\ell} \leqslant \frac{\ell}{i^2}.$$

The induction step. It suffices to check that the function $\widetilde{M}^{p_0,\ldots,p_k}_{\mu^{(1)},\ldots\mu^{(k)}}$ is continuous on Ω for arbitrary $\mu_1,\ldots,\mu_k\in\mathrm{Comp}$ such that none of them contains parts of length 1 and $p_0,\ldots,p_k\in\mathbb{N}\cup\{0\}$. The same reasoning as above applies to this function. We split it into two parts: the main part contains the contribution from α_k with $k\leqslant N$, and it converges nicely. The reminder can be estimated using the fact that

$$\left(\Gamma_{1;(i+1)}-\Gamma_{\ell(\mu^{(i)});(i)}-\alpha_{\ell(\mu^{(i)});(i)}\right)\leqslant 1$$

for all i. □

Lemma 3.4. *The linear space spanned by the extended functions*

$$\{\widetilde{M}_\mu(\omega)\}_{\mu\in\mathrm{Comp}}$$

is uniformly dense in the space of continuous functions on Ω.

Proof. The algebra QSymm of quasisymmetric functions has a basis consisting of the quasisymmetric monomial functions (see Remark 2.1). It follows that the linear span of the extended quasisymmetric monomial functions is closed under multiplication. The latter algebra contains the constant $\widetilde{M}_{(1)}(\omega)\equiv 1$; therefore, it suffices to check that it separates points and apply the Stone–Weierstrass theorem.

For an arbitrary point $\omega\in\Omega$, consider the series

$$\sum_{i=1}^{\infty}\frac{\Gamma_i\alpha_i^2}{z-\alpha_i} = \sum_{i=1}^{\infty}\frac{\Gamma_i\alpha_i^2}{z}\left(\sum_{s=2}^{\infty}\left(\frac{\alpha_i}{z}\right)^{s-2}\right) = \sum_{s=2}^{\infty}\widetilde{M}_{(1,s)}(\omega)z^{1-s}, \quad (7)$$

where in the last equality we have used (4). We see that in the case where $\alpha_1>\alpha_2>\ldots$, we can recover the sequence $\{(\alpha_i,\Gamma_i)\}_{i=1}^{\infty}$ from the information about the poles and residues of (7).

If $\alpha_{i-1}>\alpha_i=\cdots=\alpha_{i+j}>\alpha_{i+j+1}$, then we can recover from (7) only the sum $\Gamma_i+\cdots+\Gamma_{i+j}=p_1(\Gamma_i,\ldots,\Gamma_{i+j})$. For every $m\in\mathbb{N}$, consider the series

$$\sum_{i=1}^{\infty} \frac{(\Gamma_i)^m \alpha_i^2}{z-\alpha_i}.$$

By the same considerations, using (4) again, we can recover the sum $\Gamma_i^m + \cdots + \Gamma_{i+j}^m = p_m(\Gamma_i, \ldots, \Gamma_{i+j})$ from this series. Now we consider the series

$$\sum_{m=1}^{\infty} \frac{p_m(\Gamma_i, \ldots, \Gamma_{i+j})}{z^m} = \sum_{k=i}^{i+j} \frac{\Gamma_k}{z-\Gamma_k}$$

and recover the list $\{\Gamma_k\}_{k=i}^{i+j}$ as the corresponding poles. □

Proposition 3.5. *Let* $\left\{\left(|\boldsymbol{\mu}(k)|^{-1}, \omega_{\boldsymbol{\mu}(k)}\right)\right\}_{k=1}^{\infty}$ *be a sequence of elements of* $\widetilde{\Omega}$ *with* $\lim|\boldsymbol{\mu}(k)| = \infty$. *The following two conditions are equivalent*:

(1) *there exists* $(0,\omega) \in \widetilde{\Omega}$ *such that*

$$\left(\frac{1}{|\boldsymbol{\mu}(k)|}, \omega_{\boldsymbol{\mu}(k)}\right) \xrightarrow[k\to\infty]{} (0,\omega) \text{ in } \widetilde{\Omega}; \tag{8}$$

(2) *the limit*

$$\lim_{k\to\infty} \frac{\dim(\mu, \boldsymbol{\mu}(k))}{\dim \boldsymbol{\mu}(k)} \tag{9}$$

exists for every $\mu \in \mathrm{Comp}$.

The limit in (9) *equals* $\widetilde{M}_\mu(\omega)$.

Proof. If the limit (8) exists, then we combine Lemmas 3.2 and 3.3 to see that

$$\lim_{k\to\infty} \frac{\dim(\mu, \boldsymbol{\mu}(k))}{\dim \boldsymbol{\mu}(k)} = \widetilde{M}_\mu(\omega).$$

Conversely, assume that (9) holds, and assume that there are two subsequences in (8) with different limits $(0,\omega_1)$, $(0,\omega_2)$. We construct a function $f \in C(\Omega)$ with $f(\omega_1) \neq f(\omega_2)$ and use the density of the algebra spanned by the extended monomial functions in $C(\Omega)$ (Lemma 3.4) to see that $f(\omega_1) = f(\omega_2)$, a contradiction. □

Definition 3.1. *Consider the image* $\widetilde{\Delta}$ *of a* $\mathbb{Z}_+$*-graded graph* Δ *under the following mapping to* $\mathbb{R}_+^{\Delta}$:

$$B \mapsto \left(\beta \mapsto \frac{\dim(\beta, B)}{\dim B}\right),$$

where the space of functions is endowed with the topology of pointwise convergence. Let $\widetilde{E}$ be the closure of $\widetilde{\Delta}$, and denote by $E_{\mathrm{Mart}}(\Delta)$ the corresponding boundary: $E_{\mathrm{Mart}}(\Delta) = \widetilde{E}\backslash\widetilde{\Delta}$. It is called the Martin boundary *of the branching graph Δ.*

Every point $\omega \in E_{\mathrm{Mart}}$ of the Martin boundary corresponds to a normalized nonnegative harmonic function $K(\cdot,\omega)\colon \mu \mapsto K(\mu,\omega)$. Recall that such a function is said to be *indecomposable*, or *extremal*, if it cannot be written as a nontrivial convex combination $K(\mu,\omega) = ah_1(\mu)+(1-a)h_2(\mu)$, where $a \in]0,1[$, $h_1, h_2 \in E_{\mathrm{Mart}}$, $h_1 \neq h_2$. Denote by $E_{\min} \subset E_{\mathrm{Mart}}$ the subset of normalized nonnegative extremal harmonic functions on a $\mathbb{Z}_+$-graded graph Δ. We have the following integral representation.

Theorem 3.6 ([1]). *Every normalized nonnegative harmonic function ϕ admits a unique integral representation*

$$\phi(\mu) = \int\limits_{E_{\min}} K(\mu,\omega)dP_\phi,$$

where P_ϕ is a probability measure. Conversely, every probability measure P on $E_{\min}$ corresponds to a normalized nonnegative harmonic function.

Denote by T_Δ the space of paths in a graph Δ:

$$T_\Delta = \{(\boldsymbol{\mu}(0) \nearrow \boldsymbol{\mu}(1) \nearrow \dots) : \boldsymbol{\mu}(i) \in \Delta_i\}.$$

We say that a path $(\boldsymbol{\mu}(0) \nearrow \boldsymbol{\mu}(1) \nearrow \dots)$ in the graph $\mathbb{D}$ is *regular* if condition (8) holds.

Corollary 3.7. *For any regular path $(\boldsymbol{\mu}(0) \nearrow \boldsymbol{\mu}(1) \nearrow \dots)$, denote by $\omega \in \Omega$ the corresponding limiting point of Ω. For every $\mu \in$ Comp, we have*

$$\lim_{n\to\infty} \frac{\dim(\mu,\boldsymbol{\mu}(n))}{\dim \boldsymbol{\mu}(n)} = \widetilde{M}_\mu(\omega).$$

Recall that a measure P on T_Δ is said to be *central* if for any $n \in \mathbb{N}$, any $\mu \in \Delta_n$, and any two finite paths $(\mu(0) \nearrow \mu(1) \nearrow \dots \nearrow \mu(n-1) \nearrow \mu)$ and $(\mu'(0) \nearrow \mu'(1) \nearrow \dots \nearrow \mu'(n-1) \nearrow \mu)$, the following measures coincide:

$$\begin{aligned} &P(\{(\boldsymbol{\mu}(0) \nearrow \boldsymbol{\mu}(1) \nearrow \dots) : \boldsymbol{\mu}(i) = \mu(i), 1 \leqslant i \leqslant n-1; \boldsymbol{\mu}(n) = \mu\}) \\ &= P(\{(\boldsymbol{\mu}(0) \nearrow \boldsymbol{\mu}(1) \nearrow \dots) : \boldsymbol{\mu}(i) = \mu'(i), 1 \leqslant i \leqslant n-1; \boldsymbol{\mu}(n) = \mu\}). \end{aligned}$$

There is a natural isomorphism between the nonnegative harmonic functions and the central measures on Δ. If $h \in \mathcal{H}(\Delta)$, then we define the corresponding central measure ν_h as follows:

$$\nu_h(\{(\boldsymbol{\mu}(0) \nearrow \boldsymbol{\mu}(1) \nearrow \dots) : \boldsymbol{\mu}(i) = \mu(i), 1 \leqslant i \leqslant n-1; \boldsymbol{\mu}(n) = \mu\}) = h(\mu).$$

Normalized nonnegative extremal harmonic functions correspond to extremal probability central measures. Following [7], we define the *absolute* of a $\mathbb{Z}_+$-graded graph Δ as the set of all extremal probability central measures on the space of paths T_Δ. One can alternatively define the absolute as the set of all *ergodic* probability central measures with respect to the corresponding tail equivalence relation, see [5, 7].

Theorem 3.8 (Main theorem)**.**

$$\Omega \cong E_{\mathrm{Mart}}(\mathbb{D}) = E_{\min}(\mathbb{D}).$$

Proof. We see from Proposition 3.5 that $E_{\mathrm{Mart}}(\mathbb{D}) \cong \Omega$. Moreover, the extended quasisymmetric monomial functions $\widetilde{M}_{\cdot}(\omega)\colon \mu \mapsto \widetilde{M}_\mu(\omega)$ are normalized nonnegative harmonic functions on $\mathbb{D}$. Therefore, it suffices to check that these functions are extremal.

Assume that $\widetilde{M}_{\cdot}(\omega_0)$ is not extremal for $\omega_0 \in \Omega$. By Theorem 3.6, there exists a probability measure dP_{ω_0} such that

$$\widetilde{M}_\mu(\omega_0) = \int\limits_{E_{\min}(\mathbb{D})} \widetilde{M}_\mu(\omega) dP_{\omega_0}$$

for any $\mu \in \mathrm{Comp}$. By Lemma 3.4, the linear space spanned by the extended quasisymmetric functions is uniformly dense in $C(\Omega)$; therefore, the equality

$$f(\omega_0) = \int\limits_{E_{\min}(\mathbb{D})} f(\omega) dP_{\omega_0}$$

must hold for any $f \in C(\Omega)$. However, it is easy to construct a nonnegative function $f_0 \in C(\Omega)$ such that $f(\omega_0) = 1$ and $f(\omega) < 1$ for $\omega \neq \omega_0$. We have

$$1 = f_0(\omega_0) = \int\limits_{E_{\min}(\mathbb{D})} f_0(\omega) dP_{\omega_0} < \int\limits_{E_{\min}(\mathbb{D})} dP_{\omega_0} = 1,$$

a contradiction.

Thus, all the functions $\widetilde{M}_{\cdot}(\omega)$ are extremal, and $E_{\mathrm{Mart}}(\mathbb{D}) = E_{\min}(\mathbb{D})$. The theorem is proved. □

We can now combine Theorem 3.6 with Theorem 3.8 to obtain another proof of Theorem 1.2.

Corollary 3.9. *The absolute of* $\mathbb{D}$ *is* $\left\{\nu_{\widetilde{M.(\omega)}}\right\}_{\omega\in\Omega}$.

§4. Formulas for extended quasisymmetric monomial functions

We will construct another parametrization of the boundary Ω. Namely, for any union of disjoint open intervals

$$\bigcup_{i=1}^{N}]\Gamma_i, \Gamma_i + \alpha_i[\subset [0,1],$$

its complement is a union of disjoint closed intervals. Denote the set of these intervals by $\{[G_j, G_j+\gamma_j]\}_{j=1}^{P}$:

$$[0,1] \setminus \bigcup_{i=1}^{N}]\Gamma_i, \Gamma_i + \alpha_i[= \bigcup_{j=1}^{P} [G_j, G_j + \gamma_j],$$

where $P = N+1$ if N is finite and $P = \infty$ if $N = \infty$.

As before, the linear order $\prec$ compares the distances of subintervals to 0. Abusing the notation, we use the same symbol for an element γ_i and for the corresponding interval $[G_i, G_i+\gamma_i]$. In particular, $\gamma_i \prec \alpha_j$ means $G_i < \Gamma_j$.

Obviously, we can recover the sequence $\{\Gamma_i\}$ as

$$\Gamma_i = \sum_{\alpha_j \prec \alpha_i} \alpha_j + \sum_{\gamma_j \prec \alpha_i} \gamma_j;$$

therefore, we have constructed a bijection

$$\Big\{(\alpha_i, \Gamma_i)\Big\} \leftrightarrow \Big(\{\alpha_i\}, \{\gamma_i\}, \prec\Big).$$

We have the following restrictions for $(\{\alpha_i\}, \{\gamma_i\}, \prec)$:

$$\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant 0, \quad \gamma_i \geqslant 0, \quad \sum_i \alpha_i + \sum_i \gamma_i = 1;$$

$$\exists\, \gamma_{\min}, \gamma_{\max} \text{ such that } \forall\, \delta \in \{\alpha_i\} \cup \{\gamma_i\}\ \gamma_{\min} \prec \delta \prec \gamma_{\max};$$

$$\forall\, \alpha_1, \alpha_2, \alpha_1 \prec \alpha_2 \quad \exists\, \gamma \text{ such that } \alpha_1 \prec \gamma \prec \alpha_2;$$

$$\forall\, \gamma_1, \gamma_2, \gamma_1 \prec \gamma_2 \quad \exists\, \alpha \text{ such that } \gamma_1 \prec \alpha \prec \gamma_2.$$

Any composition $\mu \in \mathrm{Comp}$ has a unique decomposition

$$\mu = (1^{p_0})\mu^{(1)}(1^{p_1})\dots\mu^{(k)}(1^{p_k}),$$

where $\mu^{(1)}, \dots, \mu^{(k)}$ are compositions such that none of them contains parts of length 1 and $p_i \in \mathbb{N} \cup \{0\}$, $p_i > 0$ for $1 \leqslant i \leqslant k-1$.

We set

$$\begin{aligned}\widetilde{M}_\mu(\omega) = \sum_{\substack{\alpha_{1;(1)} \prec \alpha_{2;(1)} \prec \dots \prec \alpha_{\ell(\mu^{(1)});(1)} \prec \\ \dots \\ \prec \alpha_{1;(k)} \prec \alpha_{2;(k)} \prec \dots \prec \alpha_{\ell(\mu^{(k)});(k)}}} \left(\sum_{\rho^{(0)} \in \mathrm{Comp}(p_0)} \frac{\widetilde{\alpha}_{1;(0)}^{\rho_1^{(0)}} \dots \widetilde{\alpha}_{\ell(\rho^{(0)});(0)}^{\rho^{(0)}_{\ell(\rho^{(0)})}}}{\rho_1^{(0)}! \dots \rho^{(0)}_{\ell(\rho^{(0)})}!} \right) \\ \times \prod_{i=1}^{k} \alpha_{1;(i)}^{\mu_1^{(i)}} \alpha_{2;(i)}^{\mu_2^{(i)}} \cdots \alpha_{\ell(\mu^{(i)});(i)}^{\mu^{(i)}_{\ell(\mu^{(i)})}} \left(\sum_{\rho^{(i)} \in \mathrm{Comp}(p_i)} \frac{\widetilde{\alpha}_{1;(i)}^{\rho_1^{(i)}} \dots \widetilde{\alpha}_{\ell(\rho^{(i)});(i)}^{\rho^{(i)}_{\ell(\rho^{(i)})}}}{\rho_1^{(i)}! \dots \rho^{(i)}_{\ell(\rho^{(i)})}!} \right),\end{aligned} \tag{10}$$

where the inner sums are over

$$\begin{gathered}\widetilde{\alpha}_{1;(0)} \prec \dots \prec \widetilde{\alpha}_{\ell(\rho^{(0)});(0)} \prec \alpha_{1;(1)}, \\ \alpha_{\ell(\mu^{(1)});(1)} \prec \widetilde{\alpha}_{1;(1)} \prec \dots \prec \widetilde{\alpha}_{\ell(\rho^{(1)});(1)} \prec \alpha_{1;(2)}, \\ \dots \\ \alpha_{\ell(\mu^{(k)});(k)} \prec \widetilde{\alpha}_{1;(k)} \prec \dots \prec \widetilde{\alpha}_{\ell(\rho^{(k)});(k)},\end{gathered}$$

and where $\widetilde{\alpha}_{s;(t)} \in \{\gamma_i\}$ if $\rho_i > 1$, and $\widetilde{\alpha}_{s;(t)} \in \{\alpha_i\} \cup \{\gamma_i\}$ if $\rho_i = 1$.

For any pair α_i, α_j with $\alpha_i \prec \alpha_j$ in ω, we have

$$\Gamma_j - \Gamma_i - \alpha_i = \sum_{\alpha_i \prec \alpha_k \prec \alpha_j} \alpha_k + \sum_{\alpha_i \prec \gamma_k \prec \alpha_j} \gamma_k;$$

therefore,

$$\begin{aligned}\widetilde{M}^{p_0,\dots,p_k}_{\mu^{(1)},\dots,\mu^{(k)}}(\omega) &= \sum_{\substack{\alpha_{1;(1)} \prec \alpha_{2;(1)} \prec \dots \prec \alpha_{\ell(\mu^{(1)});(1)} \prec \\ \dots \\ \prec \alpha_{1;(k)} \prec \alpha_{2;(k)} \prec \dots \prec \alpha_{\ell(\mu^{(k)});(k)}}} \Gamma_{1;(1)}^{p_0} \\ &\times \prod_{i=1}^{k} \alpha_{1;(i)}^{\mu_1^{(i)}} \alpha_{2;(i)}^{\mu_2^{(i)}} \cdots \alpha_{\ell(\mu^{(i)});(i)}^{\mu^{(i)}_{\ell(\mu^{(i)})}} \left(\Gamma_{1;(i+1)} - \Gamma_{\ell(\mu^{(i)});(i)} - \alpha_{\ell(\mu^{(i)});(i)}\right)^{p_i} \\ &= \sum_{\substack{\alpha_{1;(1)} \prec \alpha_{2;(1)} \prec \dots \prec \alpha_{\ell(\mu^{(1)});(1)} \prec \\ \dots \\ \prec \alpha_{1;(k)} \prec \alpha_{2;(k)} \prec \dots \prec \alpha_{\ell(\mu^{(k)});(k)}}} \left(\sum_{\widetilde{\alpha}_{j;(0)} \prec \alpha_{1;(1)}} \widetilde{\alpha}_{j;(0)} \right)^{p_0}\end{aligned}$$

$$\times \prod_{i=1}^{k-1} \alpha_{1;(i)}^{\mu_1^{(i)}} \alpha_{2;(i)}^{\mu_2^{(i)}} \cdots \alpha_{\ell(\mu^{(i)});(i)}^{\mu^{(i)}_{\ell(\mu^{(i)})}} \Bigg(\sum_{\alpha_{\ell(\mu^{(i)});(i)} \prec \widetilde{\alpha}_{j;(i)} \prec \alpha_{1;(i+1)}} \widetilde{\alpha}_{j;(i)} \Bigg)^{p_i}$$

$$\times \alpha_{1;(k)}^{\mu_1^{(k)}} \alpha_{2;(k)}^{\mu_2^{(k)}} \cdots \alpha_{\ell(\mu^{(k)});(k)}^{\mu^{(k)}_{\ell(\mu^{(k)})}} \Bigg(\sum_{\alpha_{\ell(\mu^{(k)});(k)} \prec \widetilde{\alpha}_{j;(k)}} \widetilde{\alpha}_{j;(k)} \Bigg)^{p_k},$$

where $\widetilde{\alpha}_{s;(t)} \in \{\alpha_i\} \cup \{\gamma_i\}$.

Finally, we combine this equation with the definition (10) to obtain (5).

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