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AN ANNOUNCE OF RESULTS LINKING KOLMOGOROV COMPLEXITY TO ENTROPY FOR AMENABLE GROUP ACTIONS

Abstract. We announce a generalization of Brudno's results on the relation between the Kolmogorov complexity and the entropy of a subshift for actions of computable amenable groups.

§1. Introduction

In [9], A. N. Kolmogorov defined the notion of Kolmogorov complexity, which measures the information content of an individual combinatorial object. It turned out that this notion has an intimate relation with another his famous concept, that of the entropy of a dynamical system. This was first shown by Brudno in [4,5] and [6]. He proved that almost every point of an ergodic $\mathbb{Z}$-action has asymptotic complexity equal to the entropy of the system. He also proved that every point of a symbolic action of the group $\mathbb{Z}$ has asymptotic complexity bounded from above by the topological entropy of the action. There also exists a point whose asymptotic complexity is equal to the topological entropy of the action. It is natural to ask about generalizations of these results.

Simpson in [12] proved that the topological part of Brudno's results holds for actions of $\mathbb{Z}^d$. In my unpublished thesis [2], I proved that the topological part can be extended to the case of an arbitrary computable group action. This result is presented here as Theorem 2. A minor but important detail is the notion of a modest Følner sequence. A Følner sequence of a computable amenable group is called modest if the complexity of its elements is negligible relative to their sizes. This implies that the complexity of the Følner set does not interfere with various asymptotic computations. It is easy to show that any computable amenable group

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has a modest Følner sequence, see [1]. In the thesis, I also proved a nice geometric criterion, presented as Theorem 1 in this announce. Afterwards, N. Moriakov independently proved that Brudno's results (both topological and measure-theoretic) hold for actions of a certain class of computable amenable groups. Namely, he required that the group should have a computable Følner monotiling. The latter means that right shifts of every element of the Følner sequence can tile the whole group without intersections and uncovered places, and that this tiling can be obtained in a computable manner. At this point, it is not known whether every amenable group has a Følner monotiling, let alone a computable one. Nonetheless, Moriakov showed that some well-known classes of groups satisfy these requirements. Motivated by Moriakov's results, I have proved that the measure-theoretic Brudno theorem holds for actions of general computable amenable groups, see Theorem 3. Note that a related result was considered in [3]. Namely, in that paper, an integrated inequality between Kolmogorov complexity and entropy was proved.

§2. Preliminaries

2.1. Kolmogorov complexity. Let 2^* be the set of all binary strings (the empty one included). The length $|x|$ for any $x \in 2^*$ is defined naturally. Let $f : 2^* \to 2^*$ be any function, we will call it a *decompressor*. For any $x \in 2^*$, the *Kolmogorov complexity* $C_f(x)$ of x relative to f is defined to be the infimum of $|y|$ where $f(y) = x$ (and $+\infty$ if x has no preimages). We will define an order on the set of decompressors; we will say that $f' \preceq f''$ for two decompressors f' and f'' if there is a constant K such that $C_{F'}(x) \leqslant C_{f''}(x) + K$ for every $x \in 2^*$. Now we will restrict our attention to the set of computable decompressors. By the Kolmogorov theorem (see [9]), there is a minimal, in the order defined above, decompressor among the computable ones. We will call it a *universal decompressor.* Fix one such decompressor f. The *Kolmogorov complexity* of a string $x \in 2^*$ is defined now as $C_f(x)$ and is denoted by $C(x)$. Note that the Kolmogorov complexity is unique up to an additive constant. We will routinely abbreviate "Kolmogorov complexity" to "complexity." Let us list some nice properties of Kolmogorov complexity.

Lemma 1. *The folowing properties hold for Kolmogorov complexity:*

(1) *there is a number K such that $C(x) \leqslant |x| + K$ for every $x \in 2^*$;*

(2) *for every computable function f there is a constant K such that* $C(f(x)) \leqslant C(x) + K$ *for every* $x \in 2^*$.
(3) *for any* $n \in \mathbb{N}$ *there are at most* 2^n *strings* $x \in 2^*$ *such that* $C(x) < n$.

Note that it makes sense to consider Kolmogorov complexity for various finite combinatorial objects from a given class; indeed, it suffices to fix a Gödel enumeration for this class. What is a Gödel enumeration? It means that all objects have unique numbers, all numbers are used (or form a decidable subset of integers, which is equivalent), and all necessary constructive operations and necessary distinguishing queries are computable. For example, we may consider the set of finite subsets of the set of positive integers. We require that given a number i of a subset $B_i \subset \mathbb{N}$ and an element x, we have that $x \in B_i$ is a computable predicate of (i, x). The size function should also be computable. We also require that there is a computable function f such that $B_{f(i,x)} = B_i \cup \{x\}$. It is not hard to see that there is a computable bijection between any two Gödel enumerations. Note that we can construct a Gödel enumeration of the set of finite subsets of any constructible family. We also can construct such an enumeration for the set of all finitely supported maps from a constructible family to a finite set.

2.2. Groups. Let G be a countable group. A *Følner sequence* for G is a sequence (F_i) of finite subsets of the group such that

$$\lim_{i\to\infty} \frac{|gF_i \setminus F_i|}{|F_i|} = 0$$

for any $g \in G$. A group is called *amenable* if it has a Følner sequence.

Consider a group whose set of elements is $\mathbb{N}$ and 0 is the neutral element. We say that the group is *computable* if the composition rule $(g, h) \mapsto gh$ is computable as a function $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. It is easy to see that this implies also the computability of the inversion.

Let G be a computable amenable group. A Følner sequence (F_i) is called *modest* if it satisfies the estimate

$$C(F_i) = o(|F_i|).$$

It is easy to see that for any computable amenable group there is at least one modest Følner sequence, see [1]. The following geometric criterion also holds.

Theorem 1. *Assume that G is a computable amenable group that is finitely generated. Fix a finite symmetric generating set S. This determines the structure of the Cayley graph on G. Assume that (F_i) is a Følner sequence such that $1_G \in F_i$ for all i and such that all F_i are edge-connected subsets of the Cayley graph. Then (F_i) is a modest Følner sequence.*

A Følner sequence is called *tempered* if there is a constant C such that for any $i > 0$

$$\left| \bigcup_{j<i} F_j^{-1} F_i \right| \leqslant C \left| F_i \right|.$$

Note that one can extract a tempered subsequence from any Følner sequence.

2.3. Actions and entropy. Let G be a countable amenable group. Let A be a finite set. We endow the set A^G with the product topology, assuming the discrete topology on A. We define the *shift action* of G on A^G by the formula

$$(gx)(h) = x(hg)$$

for $g, h \in G$ and $x \in A^G$. This is an action by homeomorphisms.

For any $x \in A^G$ and $D \subset G$, we denote by $\mathrm{pr}_D(x)$ the restriction of x to the subset D.

A *subshift* is any closed subset X invariant under the shift action. Let (F_i) be a Følner sequence. The *topological entropy* $\mathrm{h}_G^{top}(X)$ of a subshift X is defined by the formula

$$\mathrm{h}_G^{top}(X) = \frac{\log \left| \mathrm{pr}_{F_i}(X) \right|}{|F_i|}.$$

This definition is a special case of a general definition for arbitrary topological actions of amenable groups. It is well known that the limit in the formula above exists and does not depend on the Følner sequence chosen.

Consider a measure-preserving action of a countable amenable group G on a standard probability space (X, μ). Let α be a partition, that is, a finite or countable collection $\{B_1, B_2, \ldots\}$ of measurable subsets whose union has measure 1. For $g \in G$, we denote by α^g the partition $\{g^{-1}(B_1), g^{-1}(B_2), \ldots\}$. For two partitions $\alpha' = \{B'_1, B'_2, \ldots\}$ and $\alpha'' = \{B''_1, B''_2, \ldots\}$, we denote by $\alpha' \vee \alpha''$ the partition $\{B' \cap B'' | B' \in \alpha', B'' \in \alpha''\}$. For a finite subset D of G, we denote by α^D the partition $\bigvee_{g \in D} \alpha^g$. Let (F_i) be a Følner sequence for the group G. The entropy of the action is defined by the following formula:

$$\mathrm{h}_G(X,\mu) = \sup_{\alpha} \lim_{i\to\infty} \frac{H(\alpha^{F_i})}{|F_i|},$$

where the supremum is taken over the set of all partitions of finite Shannon entropy. It is known that the limit above does not depend on the choice of a Følner sequence.

2.4. Asymptotic Kolmogorov complexity. Let $\mathcal{F} = (F_i)$ be a modest Følner sequence. For a point $x \in A^G$, we define its *upper asymptotic complexity* $\overline{\mathrm{AC}}_{\mathcal{F}}(x)$ relative to the Følner sequence $\mathcal{F}$ by the formula

$$\overline{\mathrm{AC}}_{\mathcal{F}}(x) = \limsup_{i\to\infty} \frac{C(\mathrm{pr}_{F_i}(x))}{|F_i|}.$$

The lower asymptotic complexity is defined in a similar manner:

$$\underline{\mathrm{AC}}_{\mathcal{F}}(x) = \liminf_{i\to\infty} \frac{C(\mathrm{pr}_{F_i}(x))}{|F_i|}.$$

§3. Main results

Theorem 2. *Let $X \subset A^G$ be a subshift over a computable amenable group. Let $\mathcal{F}$ be a modest Følner sequence. For every $x \in X$,*

$$\overline{\mathrm{AC}}_{\mathcal{F}}(x) \leqslant \mathrm{h}_G^{top}(X).$$

If $\mathcal{F}$ is also tempered, then there is a point $y \in X$ such that

$$\overline{\mathrm{AC}}_{\mathcal{F}}(y) = \underline{\mathrm{AC}}_{\mathcal{F}}(y) = \mathrm{h}_G^{top}(X).$$

If, in addition, X has the cardinality of the continuum, then there is a continuum of such points y.

Theorem 3. *Let G be a computable amenable group. Let $\mathcal{F}$ be a modest tempered Følner sequence. Let μ be an ergodic invariant measure on A^G. Then for μ-a.e. $x \in A^G$,*

$$\overline{\mathrm{AC}}_{\mathcal{F}}(x) = \underline{\mathrm{AC}}_{\mathcal{F}}(x) = \mathrm{h}_G(A^G,\mu).$$

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