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THE STIELTJES INTEGRALS IN THE THEORY OF HARMONIC FUNCTIONS

Abstract. We study various Stieltjes integrals, such as Poisson–Stieltjes, conjugate Poisson–Stieltjes, Schwartz–Stieltjes and Cauchy–Stieltjes, and prove theorems on the existence of their finite angular limits a.e. in terms of the Hilbert–Stieltjes integral. These results are valid for arbitrary bounded integrands that are differentiable a.e. and, in particular, for integrands of the class $\mathcal{CBV}$ (countably bounded variation).

§1. Introduction

Recall that a path in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ terminating at $\zeta \in \partial\mathbb{D}$ is called **nontangential** if its part in a neighborhood of ζ lies inside of an angle in $\mathbb{D}$ with vertex at ζ. Hence the limit along all nontangential paths at $\zeta \in \partial\mathbb{D}$ is also said to be **angular** at the point. This is a traditional tool of the geometric function theory, see, e.g., the monographs [4, 12, 15, 20], and [23].

It was proved in the previous paper [27], see also [28], that a harmonic function u given in the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$ has angular limits at a.e. point $\zeta \in \partial\mathbb{D}$ if and only if its conjugate harmonic function v in $\mathbb{D}$ has the same property, see Problem 4 in Section III of [12]. This is the key fact together with Lemma 2 in Section 5 below to establish the existence of the so-called Hilbert–Stieltjes integral for a.e. $\zeta \in \partial\mathbb{D}$ and the corresponding result on the angular limits of Cauchy–Stieltjes integrals under fairly general assumptions on integrands, cf., e.g., [9, 21] and [31]; see also [5, 11, 12], and [23].

We recall a subtle statement due to Lusin that a harmonic function in the unit disk with continuous (even absolutely continuous!) boundary data may have conjugate harmonic function whose boundary data are not continuous, furthermore, they may even be not essentially bounded near each point of the unit circle, see, e.g., Theorem VIII.13.1 in [1]. Thus, a

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correlation between boundary data of conjugate harmonic functions is not a simple matter, see also I.E in [12].

Denote by h^p, $p \in (0, \infty)$, the class of all harmonic functions u in $\mathbb{D}$ with bounded L^p-norms over the circles $|z| = r \in (0, 1)$. It is clear that $h^p \subseteq h^{p'}$ for all $p > p'$ and, in particular, $h^p \subseteq h^1$ for all $p > 1$. It is important that every function in the class h^1 has nontangential boundary limits a.e., see, e.g., Corollary IX.2.2 in [8]. It is also known that a harmonic function u in $\mathbb{D}$ can be represented as the Poisson integral

$$u(re^{i\vartheta}) \;=\; \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r\cos(\vartheta - t) + r^2}\ \varphi(t)\ dt \tag{1.1}$$

with a function $\varphi \in L^p(-\pi, \pi)$, $p > 1$, if and only if $u \in h^p$, see, e.g., Theorem IX.2.3 in [8]. Thus, $u(z) \to \varphi(\vartheta)$ as $z \to e^{i\vartheta}$ along any nontangential path for a.e. ϑ, see, e.g., Corollary IX.1.1 in [8]. Moreover, $u(z) \to \varphi(\vartheta_0)$ as $z \to e^{i\vartheta_0}$ at the points ϑ_0 of continuity of φ, see, e.g., Theorem IX.1.1 in [8].

Note also that $v \in h^p$ whenever $u \in h^p$ for all $p > 1$ by the M. Riesz theorem, see [24], see also Theorem IX.2.4 in [8]. Generally speaking, this fact is not trivial but it follows immediately for $p = 2$ from the Parseval identity, see, e.g., the proof of Theorem IX.2.4 in [8]. The case of $u \in h^1$ is more complicated.

We remind the reader that a harmonic function u in $\mathbb{D}$ belongs to h^1 if and only if it can be represented as the Poisson–Stieltjes integral

$$u(re^{i\vartheta}) \;=\; \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r\cos(\vartheta - t) + r^2}\ d\Phi(t) \tag{1.2}$$

with $\Phi : [-\pi, \pi] \to \mathbb{R}$ of bounded variation, see, e.g., Theorem IX.2.2 in [8]. Moreover, if the function Φ has finite derivative at a point $\vartheta_0 \in (-\pi, \pi)$, then $u(z) \to \Phi'(\vartheta_0)$ as $z := re^{i\vartheta} \to \zeta_0 := e^{i\vartheta_0}$ along all nontangential paths in $\mathbb{D}$ to the point ζ_0, see, e.g., Theorem IX.1.4 in [8].

The present paper is devoted to the study of the corresponding Stieltjes integrals in the case when Φ is, generally speaking, not of bounded variation. The emphasis here is on the method. Namely, a basic fact in Section 2 is that the formula of integration by parts remains true with no regularity conditions on the functions involved if at least one of the integrals exists, see Lemma 1, and that the latter is true if one of the functions is absolutely continuous and the second one is bounded and its set of points of discontinuity has measure zero, see Proposition 1. Moreover, it is demonstrated by Example 1 that the boundedness condition is essential. Finally, by Remark 1 measure zero for the set of points of discontinuity is also necessary. Precisely on this basis, it has become possible to extend all results on various Stieltjes integrals to the case of the absence of bounded variation.

§2. Expansion of the Riemann–Stieltjes integral

First of all, recall a classical definition of the Riemann–Stieltjes integral. Namely, let $\mathrm{I} = [a, b]$ be a compact interval in $\mathbb{R}$. A *partition* P of I is a collection of points $t_0, t_1, \ldots, t_p \in \mathrm{I}$ such that $a = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_p = b$. Now, let $g : \mathrm{I} \to \mathbb{R}$ and $f : \mathrm{I} \to \mathbb{R}$ be bounded functions, and, moreover, let f be monotone nondecreasing. The *Riemann–Stieltjes integral* of g with respect to f is a real number A, written $A = \int\limits_{\mathrm{I}} g\, df$, if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\Big|\sum_{k=1}^{p} g(\tau_k)[f(t_k) - f(t_{k-1})] \ - \ A\Big| < \ \varepsilon$$

for every partition $\mathrm{P} = \{t_0, t_1, \ldots, t_p\}$ of I with $|t_k - t_{k-1}| \leqslant \delta$, $k = 1, \ldots, p$, and every collection $\tau_k \in [t_{k-1}, t_k]$, $k = 1, \ldots, p$. In other words and in a different notation,

$$\int\limits_a^b g\, df \ := \ \lim_{\delta\to 0} \sum_{k=1}^{p} g(\tau_k)\cdot\Delta_k f \quad \text{as} \quad \delta := \max_{k=1,\ldots,p} |t_k - t_{k-1}| \to 0, \tag{2.1}$$

where $\Delta_k f := f(t_k) - f(t_{k-1})$, $k = 1, \ldots, p$, if a finite limit in (2.1) exists and it is uniform with respect to partitions $\{t_k\}$ and intermediate points $\{\tau_k\}$.

We extend the definition of the Riemann–Stieltjes integral to arbitrary functions g and f for which the limit (2.1) exists. Let us start with the following general fact, cf., e.g., [2, 10, 19], and [30].

Lemma 1. *Let* $\mathrm{I} = [a, b]$, $g : \mathrm{I} \to \mathbb{R}$, *and* $f : \mathrm{I} \to \mathbb{R}$ *be arbitrary functions. If one of the integrals* $\int g\, df$ *and* $\int f\, dg$ *exists, then the second one also exists and*

$$\int\limits_{\mathrm{I}} g\, df \ + \ \int\limits_{\mathrm{I}} f\, dg \ = \ g(b)\cdot f(b) \ - \ g(a)\cdot f(a). \tag{2.2}$$

Proof. For definiteness, we assume that the integral $\int f\, dg$ exists. Then for an arbitrary integral sum of the integral $\int g\, df$, we have

$$\sum_{k=1}^{p} g(\tau_k)\cdot(f(t_k)-f(t_{k-1})) = -g(t_0)f(t_0)$$
$$-\sum_{k=1}^{p+1} f(t_k)\cdot(g(\tau_k)-g(\tau_{k-1})) + g(t_p)f(t_p),$$

where $\tau_0 := t_0$ and $\tau_{p+1} = t_p$, which implies the desired conclusions. □

By Theorem 13.1.b in [10], we have an important consequence of Lemma 1.

Proposition 1. *Let* $\mathrm{I} = [a, b]$*, let a function* $f : \mathrm{I} \to \mathbb{R}$ *be absolutely continuous, and let* $g : \mathrm{I} \to \mathbb{R}$ *be a bounded function whose set of points of discontinuity is of measure zero. Then the two integrals* $\int\limits_{\mathrm{I}} g\, df$ *and* $\int\limits_{\mathrm{I}} f\, dg$ *exist and relation* (2.2) *holds true.*

Example 1. The condition that the function g is bounded in Proposition 1 is essential. This is clear from the simplest example of the pair of the following functions on $[0, 1]$: $f(t) = t$ and $g(t) = 0$ except the points $t_n = 1/n$ where $g(t_n) := n^2$, $n = 1, 2, \ldots$. Indeed, we see that the lower limit of the integral sums is 0 and the upper limit is ∞.

Remark 1. The condition of measure zero for the set of points of discontinuity of g in Proposition 1 is also necessary. Indeed, take the case of $f(t) \equiv t$ and consider a function g with its set S of points of discontinuity of a length $l > 0$. By subadditivity $l \leqslant \sum l_n$ where l_n, $n = 1, 2, \ldots$, is the length of the set S_n of points in S with jumps that are greater than or equal to $1/n$. Hence $l_N > 0$ for some $N = 1, 2, \ldots$. We cover every point $t \in S_N \setminus \{a, b\}$ by all intervals in (a, b) centered at t whose lengths are less than an arbitrary prescribed $\delta > 0$. By the Vitali theorem there is a countable subcollection of mutually disjoint intervals covering almost every point of S_N, see, e.g. Theorem IV.3.1 in [30]. The sum of their lengths is at least l_N and all such intervals contain jumps of g that are at least $1/N$. Thus, the difference between the lower and upper limits of integral sums is at least $N^{-1}l_N$, i.e., it is not zero.

It is clear that formula (2.2) is also valid for complex-valued functions of the natural parameter on rectifiable Jordan curves because their real and imaginary parts can be regarded as real-valued functions on segments of $\mathbb{R}$. Moreover, the corresponding statements hold true on closed Jordan curves J with the relation

$$\int_{\mathrm{J}} g\,df \;=\; -\int_{\mathrm{J}} f\,dg, \tag{2.3}$$

where we should apply *cyclic partitions* P of J by collections of cyclically ordered points $\zeta_0, \zeta_1, \ldots, \zeta_p$ on J with $\zeta_0 = \zeta_p$.

§3. On the Poisson–Stieltjes integrals

Recall that the *Poisson kernel* is the 2π-periodic function

$$P_r(\Theta) \;=\; \frac{1-r^2}{1-2r\cos\Theta + r^2}\;,\; r<1\;,\; \Theta\in\mathbb{R}. \tag{3.1}$$

By Proposition 1, the *Poisson–Stieltjes integral*

$$\mathbb{U}(z) = \mathbb{U}_{\Phi}(z) := \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\vartheta - t)\; d\Phi(t), \quad z = re^{i\vartheta},\; r<1\;,\; \vartheta\in\mathbb{R} \tag{3.2}$$

is well defined for 2π-periodic continuous functions, furthermore, for bounded functions $\Phi : \mathbb{R}\to\mathbb{R}$ whose set of points of discontinuity is of measure zero because the function $P_r(\Theta)$ is continuously differentiable and hence it is absolutely continuous.

Moreover, directly by the definition of the Riemann–Stieltjes integral and the Weierstrass type theorem for harmonic functions, see, e.g., Theorem I.3.1 in [8], $\mathbb{U}$ is a harmonic function in the unit disk $\mathbb{D} := \{z\in\mathbb{C} : |z|<1\}$ because the function $P_r(\vartheta - t)$ is the real part of the analytic function

$$\mathcal{A}_{\zeta}(z) := \frac{\zeta + z}{\zeta - z}, \quad \zeta = e^{it}, \quad z = re^{i\vartheta}, \quad r<1, \quad \vartheta \quad \text{and} \quad t\in\mathbb{R}. \tag{3.3}$$

Theorem 1. *Let $\Phi : \mathbb{R}\to\mathbb{R}$ be a 2π-periodic bounded function whose set of points of discontinuity has measure zero. Suppose that Φ is differentiable at a point $t_0\in\mathbb{R}$. Then*

$$\lim_{z\to\zeta_0} \mathbb{U}_{\Phi}(z) \;=\; \Phi'(t_0) \tag{3.4}$$

along all nontangential paths in $\mathbb{D}$ to the point $\zeta_0 := e^{it_0}\in\partial\mathbb{D}$.

Proof. Indeed, by Proposition 1 with $g(t) := \Phi(t)$ and $f(t) := P_r(\vartheta - t)$, $t \in \mathbb{R}$, for every fixed $z = re^{i\vartheta}$, $r < 1$, $\vartheta \in \mathbb{R}$, we obtain

$$\int_{-\pi}^{\pi} P_r(\vartheta - t)\, d\Phi(t) \;=\; \int_{-\pi}^{\pi} \Phi(t) \cdot \frac{\partial}{\partial \vartheta} P_r(\vartheta - t)\, dt, \quad r \in (0,1), \quad \vartheta \in \mathbb{R}, \tag{3.5}$$

because of the 2π-periodicity of the given functions g and f the right hand side in (2.2) is equal to zero, $f \in C^1$ and

$$\frac{\partial}{\partial \vartheta} P_r(\vartheta - t) \;=\; -\, \frac{\partial}{\partial t} P_r(\vartheta - t), \quad r \in (0,1), \quad \vartheta, t \in \mathbb{R}.$$

Now, considering the Poisson integral

$$u(re^{i\vartheta}) \;:=\; \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\vartheta - t)\ \Phi(t)\ dt\ ,$$

we see by the Fatou result, see, e.g., 3.441 in [32], p. 53, or Theorem IX.1.2 in [8], that $\frac{\partial}{\partial \vartheta}\, u(z) \to \Phi'(t_0)$ as $z \to \zeta_0$ along any nontangential path in $\mathbb{D}$ ending at ζ_0. Thus, the conclusion follows because $\mathbb{U}_\Phi(z) = \frac{\partial}{\partial \vartheta}\, u(z)$ by (3.5). $\Box$

Corollary 1. *If $\Phi : \mathbb{R} \to \mathbb{R}$ is a 2π-periodic continuous function that is differentiable a.e., then $\mathbb{U}_\Phi(z) \to \Phi'(\arg \zeta)$ as $z \to \zeta$ for a.e. $\zeta \in \partial\mathbb{D}$ along all nontangential paths in $\mathbb{D}$ to the point ζ.*

Here we denote by $\arg \zeta$ the *principal branch of the argument* of $\zeta \in \mathbb{C}$ with $|\zeta| = 1$, i.e., a unique number $\tau \in (-\pi, \pi]$ such that $\zeta = e^{i\tau}$.

Remark 2. Note that, generally speaking, the function of interval

$$\Phi_*([a,b]) := \Phi(b) - \Phi(a)$$

generates no finite signed measure (charge) if Φ is not of bounded variation. Hence we cannot apply the known Fatou result on the angular boundary limits directly to the Poisson–Stieltjes integrals, see, e.g., Theorem I.D.3 in [12].

Corollary 2. *If $\Phi : \mathbb{R} \to \mathbb{R}$ is a 2π-periodic bounded function that is differentiable a.e., then $\mathbb{U}_\Phi(z) \to \Phi'(\arg \zeta)$ as $z \to \zeta$ for a.e. $\zeta \in \partial\mathbb{D}$ along all nontangential paths in $\mathbb{D}$ to the point ζ.*

A function $\Phi : \mathbb{R} \to \mathbb{C}$ is said to be of *countably bounded variation* (in symbols: $\Phi \in \mathcal{CBV}(\mathbb{R})$) if there is a countable collection of mutually disjoint intervals (a_n, b_n), $n = 1, 2, \ldots$, such that the restriction of Φ to each of them is of bounded variation and the set $\mathbb{R}\setminus\bigcup\limits_{1}^{\infty}(a_n, b_n)$ is countable.

Corollary 3. *If $\Phi : \mathbb{R} \to \mathbb{R}$ is a 2π-periodic bounded function of class $\mathcal{CBV}(\mathbb{R})$, then $\mathbb{U}_\Phi(z) \to \Phi'(\arg\zeta)$ as $z \to \zeta$ for a.e. $\zeta \in \partial\mathbb{D}$ along all nontangential paths in $\mathbb{D}$ to the point ζ.*

§4. On the conjugate Poisson–Stieltjes integrals

Recall that the *conjugate Poisson kernel* is the 2π-periodic function

$$Q_r(\Theta) = \frac{2r\sin\Theta}{1 - 2r\cos\Theta + r^2}, \quad r < 1, \quad \Theta \in \mathbb{R}. \tag{4.1}$$

By Proposition 1 the *conjugate Poisson-Stieltjes integral*

$$\mathbb{V}(z) = \mathbb{V}_\Phi(z) := \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} Q_r(\vartheta - t)\; d\Phi(t),\; z = re^{i\vartheta},\; r < 1,\; \vartheta \in \mathbb{R}, \tag{4.2}$$

is well defined for 2π-periodic bounded functions $\Phi : \mathbb{R} \to \mathbb{R}$ whose set of points of discontinuity is of measure zero because the function $Q_r(\Theta)$ is continuously differentiable and hence it is absolutely continuous. Again, directly by the definition of the Riemann–Stieltjes integral and the Weierstrass type theorem, $\mathbb{V}_\Phi$ is a conjugate harmonic function for $\mathbb{U}_\Phi$ in the unit disk $\mathbb{D}$ because the function $Q_r(\vartheta - t)$ is the imaginary part of the same analytic function (3.3).

By Theorem 1 in [27], see also [28], we have the following important consequences of Theorem 1 and Corollaries 1–3.

Corollary 4. *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic continuous function that is differentiable a.e. Then $\mathbb{V}_\Phi(z)$ has a finite limit $\varphi(\zeta)$ as $z \to \zeta$ along all nontangential paths in $\mathbb{D}$ to a.e. $\zeta \in \partial\mathbb{D}$.*

Corollary 5. *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic bounded function that is differentiable a.e. Then $\mathbb{V}_\Phi(z)$ has a finite limit $\varphi(\zeta)$ as $z \to \zeta$ along all nontangential paths in $\mathbb{D}$ to a.e. $\zeta \in \partial\mathbb{D}$.*

Corollary 6. *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic bounded function of class $\mathcal{CBV}(\mathbb{R})$. Then $\mathbb{V}_\Phi(z)$ has a finite angular limit $\varphi(\zeta)$ as $z \to \zeta$ for a.e. $\zeta \in \partial\mathbb{D}$.*

The function $\varphi(\zeta)$ will be calculated explicitly in terms of $\Phi(\zeta)$ via the so-called Hilbert–Stieltjes integral. To prove this fact we need first establish an auxiliary result in the next section.

§5. On the Hilbert–Stieltjes integral

Lemma 2. *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic bounded function whose set of points of discontinuity has measure zero. Suppose that Φ is differentiable at a point $t_0 \in \mathbb{R}$. Then the difference*

$$\mathbb{V}_\Phi(z) - \frac{1}{\pi} \int\limits_{1-|z|}^{\pi} \frac{d\{\Phi(t_0 - t) - \Phi(t_0 + t)\}}{2\tan\frac{t}{2}} \tag{5.1}$$

converges to zero as $z \to \zeta_0 := e^{it_0} \in \partial\mathbb{D}$ along the radius in $\mathbb{D}$ to the point ζ_0.

Proof. First of all, applying (if necessary) simultaneous rotations of ζ_0 to $\zeta = e^{it} \in \partial\mathbb{D}$ and of $z \in \mathbb{D}$ in (3.3), we may assume that $t_0 = 0$. Moreover, there is no loss of generality in assuming that $\Phi(0) = 0$ and $\Phi'(0) = 0$ because, for the linear function $\Phi_*(t) := \Phi(0) + \Phi'(0) \cdot t$ from $(-\pi, \pi]$ to $\mathbb{R}$ extended 2π-periodically to $\mathbb{R}$, the relation $d\,\Phi_*(t) \equiv \Phi'(0)\,dt$ gives identical zero in the difference (5.1) in view of the oddness of the kernel Q_r and of $\tan\frac{t}{2}$.

Note that by the oddness of Q_r we also have

$$\mathbb{V}_\Phi(r) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} Q_r(-t)\; d\,\Phi(t) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} Q_r(t)\; d\,\Phi(-t), \quad r \in (0,1).$$

Then the difference (5.1) is split into two parts with $\varepsilon = \varepsilon(r) := 1 - r$:

$$\mathrm{I} := \frac{1}{2\pi}\int\limits_{-\varepsilon}^{\varepsilon} Q_r(t)\; d\,\Phi(-t),$$

$$\mathrm{II} := \frac{1}{2\pi}\int\limits_{\varepsilon\leqslant|t|\leqslant\pi} \{Q_r(t) - Q_1(t)\}\; d\,\{\Phi(-t) - \Phi(t)\}.$$

Integrating I by parts, by Proposition 1 and the oddness of $Q_r(t)$ we have

$$\mathrm{I} = \frac{1}{2\pi} Q_r(\varepsilon)\{\Phi(-\varepsilon) - \Phi(\varepsilon)\} + \frac{1}{\pi}\int\limits_{0}^{\varepsilon}\{\Phi(-t) - \Phi(t)\}\; d\,Q_r(t)\;. \tag{5.2}$$

The first summand converges to zero as $\varepsilon \to 0$ because $|\Phi(\pm\varepsilon)| = o(\varepsilon)$ and

$$Q_r(\varepsilon) = \frac{2r\sin\varepsilon}{1-2r\cos\varepsilon + r^2} = \frac{2r\sin\varepsilon}{\varepsilon^2 + 4r\sin^2\frac{\varepsilon}{2}} \leqslant 2\frac{\sin\varepsilon}{\varepsilon^2} \leqslant \frac{2}{\varepsilon}. \tag{5.3}$$

To estimate the second summand in (5.2), we observe that $\sin^2\frac{t}{2} \leqslant \left[\frac{1-r}{2}\right]^2$ and, thus,

$$\begin{aligned}
Q_r'(t) &= \frac{2r\cos t}{1-2r\cos t + r^2} - \frac{4r^2\sin^2 t}{(1-2r\cos t+r^2)^2} \\
&= 2r\frac{(1+r^2)\cos t - 2r}{(1-2r\cos t+r^2)^2} = 2r\frac{(1-r)^2 - 2(1+r^2)\sin^2\frac{t}{2}}{(1-2r\cos t+r^2)^2} \\
&\geqslant 2r\frac{(1-r)^2[1-(1+r^2)/2]}{(1-2r\cos t+r^2)^2} = \frac{r(1+r)(1-r)^3}{(1-2r\cos t+r^2)^2},
\end{aligned}$$

i.e., $Q_r'(t) > 0$ for all $t \in [0,\varepsilon]$. Since $Q_r(t)$ is smooth, it is strictly monotone increasing on $[0,\varepsilon]$. Hence the modulus of the second summand is dominated by

$$\frac{1}{\pi}\cdot Q_r(\varepsilon)\cdot \sup_{t\in[0,\varepsilon]}\{|\Phi(-t)| + |\Phi(t)|\} \leqslant \frac{1}{\pi}\cdot\frac{2}{\varepsilon}\cdot o(\varepsilon) = o(1),$$

where the inequality follows by (5.3). Thus, the second summand in (5.2) also converges to zero as $\varepsilon \to 0$.

Now, by the oddness of the kernels $Q_r(t)$, $r \in (0,1)$, and $Q_1(t)$, we obtain

$$\mathrm{II} := \frac{1}{\pi}\int_{\varepsilon}^{\pi}\{Q_r(t) - Q_1(t)\}\, d\{\Phi(-t) - \Phi(t)\},$$

where

$$\begin{aligned}
Q_1(t) - Q_r(t) &= \frac{2\sin t}{2(1-\cos t)} - \frac{2r\sin t}{\varepsilon^2 + 2r(1-\cos t)} \\
&= \frac{2\sin t}{4\sin^2\frac{t}{2}} - \frac{2r\sin t}{\varepsilon^2 + 4r\sin^2\frac{t}{2}} = \frac{2\varepsilon^2\sin t}{4\left(\varepsilon^2 + 4r\sin^2\frac{t}{2}\right)\sin^2\frac{t}{2}}.
\end{aligned}$$

Integrating by parts, we see that the last integral is equal to

$$\{Q_1(\varepsilon) - Q_r(\varepsilon)\}\cdot\{\Phi(-\varepsilon) - \Phi(\varepsilon)\} + \int_{\varepsilon}^{\pi}\{\Phi(-t) - \Phi(t)\}\, d\{Q_1(t) - Q_r(t)\}.$$

Here the first summand converges to zero because $\Phi(-\varepsilon)-\Phi(\varepsilon)=o(\varepsilon)$ and

$$Q_1(\varepsilon)-Q_r(\varepsilon) \;=\; \frac{2\varepsilon^2\sin\varepsilon}{4(\varepsilon^2+4r\sin^2\frac{\varepsilon}{2})\sin^2\frac{\varepsilon}{2}} \;\sim\; \frac{1}{\varepsilon} \quad \text{as} \quad \varepsilon\to 0\ . \tag{5.4}$$

Thus, it remains to estimate the integral

$$\mathrm{III} := \int\limits_{\varepsilon}^{\pi}\{\Phi(-t)-\Phi(t)\}\ d\{Q_1(t)-Q_r(t)\} = \int\limits_{\varepsilon}^{\pi}\varphi(t)\ d\,\alpha_r(t),$$

where $\varphi(t)=\Phi(-t)-\Phi(t)$ and $\alpha_r(t)=Q_1(t)-Q_r(t)$. To do this, we first observe that $\alpha_r'(t)<0$ for $t\in(\varepsilon,\pi)$ because

$$\begin{aligned}
\alpha_r'(t) &= \frac{2\varepsilon^2\cos t}{4(\varepsilon^2+4r\sin^2\frac{t}{2})\sin^2\frac{t}{2}} - \frac{2\varepsilon^2\sin^2 t(\varepsilon^2+8r\sin^2\frac{t}{2})}{4[(\varepsilon^2+4r\sin^2\frac{t}{2})\sin^2\frac{t}{2}]^2}\\
&= 2\cdot\frac{\varepsilon^2}{\delta^2}\cdot\left[\cos t\cdot\sin^2\frac{t}{2}\cdot\left(\varepsilon^2+4r\sin^2\frac{t}{2}\right)\right.\\
&\qquad\left.-\sin^2 t\cdot\left(\varepsilon^2+8r\sin^2\frac{t}{2}\right)\right]\\
&= 2\cdot\frac{\varepsilon^2}{\delta^2}\cdot\left[\varepsilon^2\cdot\left(\cos t\cdot\sin^2\frac{t}{2}-\sin^2 t\right)\right.\\
&\qquad\left.-4r\sin^2\frac{t}{2}\cdot\left(2\sin^2 t-\cos t\cdot\sin^2\frac{t}{2}\right)\right]\\
&= 2\cdot\frac{\varepsilon^2}{\delta^2}\cdot\left[\varepsilon^2\cdot\sin^2\frac{t}{2}\cdot\left(\cos t-4\cos^2\frac{t}{2}\right)\right.\\
&\qquad\left.-4r\sin^4\frac{t}{2}\cdot\left(8\cos^2\frac{t}{2}-\cos t\right)\right]\\
&= -2\cdot\frac{\varepsilon^2}{\delta^2}\cdot\sin^2\frac{t}{2}\cdot\left[\varepsilon^2\cdot\left(1{+}2\cos^2\frac{t}{2}\right)+4r\sin^2\frac{t}{2}\cdot\left(1{+}6\cos^2\frac{t}{2}\right)\right],
\end{aligned}$$

where we have applied many times the trigonometric identities $\sin t = 2\sin\frac{t}{2}\ \cos\frac{t}{2}$ and $1-\cos t = 2\sin^2\frac{t}{2}$, $1+\cos t = 2\cos^2\frac{t}{2}$, and we have employed the notation

$$\delta \;:=\; 2\sin^2\frac{t}{2}\left(\varepsilon^2+4r\sin^2\frac{t}{2}\right)\ .$$

The above expression for $\alpha_r'(t)$ also implies that $|\alpha_r'(t)| \leqslant c \cdot \frac{\varepsilon^2}{t^4}$. Thus,

$$|\mathrm{III}| \;\leqslant\; c \cdot \varepsilon^2 \int\limits_{\varepsilon}^{\pi} |\varphi(t)| \,\frac{d\,t}{t^4}\;.$$

We fix an arbitrary $\epsilon > 0$ and choose $\eta > 0$ so small that $|\varphi(t)|/t < \epsilon$ for all $t \in (0, \eta)$. Note that we may assume here that $\varepsilon < \eta^2\sqrt{\epsilon}$ for sufficiently small ε. Consequently, we have the following estimates:

$$|\mathrm{III}| \;\leqslant\; c \cdot \varepsilon^2 \cdot \epsilon \int\limits_{\varepsilon}^{\eta} \frac{d\,t}{t^3} \;+\; c \cdot \varepsilon^2 \int\limits_{\eta}^{\pi} |\varphi(t)| \,\frac{d\,t}{t^4} \;\leqslant\; \frac{c}{2} \cdot \epsilon \;+\; c\pi \cdot \epsilon \cdot M,$$

where $M = \sup\limits_{t \in [0,\pi]} |\varphi(t)|$. Since ε and ϵ are arbitrary, we conclude that the integral III converges to zero as $\varepsilon \to 0$. □

Theorem 2. *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic bounded function. Suppose that Φ is differentiable a.e. Then*

$$\lim_{z \to \xi} \mathbb{V}_{\Phi}(z) = \frac{1}{\pi} \int\limits_{0}^{\pi} \frac{d\,\{\Phi(\tau - t) - \Phi(\tau + t)\}}{2 \tan \frac{t}{2}}, \qquad \xi := e^{i\tau} \in \partial\mathbb{D}, \tag{5.5}$$

for a.e. $\tau \in \mathbb{R}$ along all nontangential paths in $\mathbb{D}$ to the point ξ.

Here the singular integral on the right hand side in (5.5) is understood as a limit of the corresponding proper integrals (*Cauchy principal value*):

$$\begin{aligned} \mathbb{H}_{\Phi}(\tau) : &= \frac{1}{\pi} \lim_{\varepsilon \to +0} \int\limits_{\varepsilon}^{\pi} \frac{d\,\{\Phi(\tau - t) - \Phi(\tau + t)\}}{2 \tan \frac{t}{2}} \\ &= \frac{1}{2\pi} \lim_{\varepsilon \to 0} \int\limits_{|\tau - t| \geqslant \varepsilon} \frac{d\,\Phi(t)}{\tan \frac{\tau - t}{2}}. \end{aligned} \tag{5.6}$$

It will be called the *Hilbert–Stieltjes integral* of the function Φ at the point τ.

Proof. The claim of Theorem 2 follows immediately from Lemma 2 and Corollary 5. □

Corollary 7. *The Hilbert–Stieltjes integral converges a.e. for every 2π-periodic bounded function $\Phi : \mathbb{R} \to \mathbb{R}$ that is differentiable a.e.*

Remark 3. In particular, the claims of Theorem 2 and Corollary 7 hold true for every 2π-periodic bounded function of class $\mathcal{CBV}(\mathbb{R})$.

Of course, Lemmas 1–2, Theorems 1–2, Corollaries 1–7 and the definition of the Hilbert–Stieltjes integral are extended in a natural way to complex-valued functions Φ.

§6. On Schwartz–Stieltjes and Cauchy–Stieltjes integrals

Given a 2π-periodic bounded function $\Phi : \mathbb{R} \to \mathbb{R}$ whose set of points of discontinuity has measure zero, the *Schwartz–Stieltjes integral*

$$\mathbb{S}_\Phi(z) \ := \ \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \, d\,\Phi(t), \quad z \in \mathbb{D}, \tag{6.1}$$

is well defined by the previous sections and the function $\mathbb{S}_\Phi(z)$ is analytic by the definition of the Riemann–Stieltjes integral and the Weierstrass theorem, see, e.g., Theorem I.1.1 in [8]. By Theorem 2 and Corollary 2, we have also the following.

Corollary 8. *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic bounded function that is differentiable a.e. Then $\mathbb{S}_\Phi(z)$ has finite angular limit*

$$\Phi'(\arg \zeta) + i \cdot \mathbb{H}_\Phi(\arg \zeta)$$

as $z \to \zeta$ for a.e. $\zeta \in \partial\mathbb{D}$.

It is clear that the definition (6.1), as well as Corollary 8, are extended in a natural way to the case of the complex valued functions Φ.

Given a 2π-periodic bounded function $\Phi : \mathbb{R} \to \mathbb{C}$ whose set of points of discontinuity has measure zero, we see that the integral

$$\mathbb{C}_\Phi(z) \ := \ \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \frac{e^{it} d\,\Phi(t)}{e^{it} - z}, \quad z \in \mathbb{D}, \tag{6.2}$$

is also well defined and we call it the *Cauchy–Stieltjes integral.* It is easy to see that $\mathbb{C}_\Phi(z) = \frac{1}{2}\,\mathbb{S}_\Phi(z)$.

Corollary 9. *Let $\Phi : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic bounded function that is differentiable a.e. Then*

$$\lim_{z\to\zeta} \mathbb{C}_\Phi(z) \;=\; \frac{1}{2}\,\{\,\Phi'(\arg\,\zeta) \;+\; i\cdot\mathbb{H}_\Phi(\arg\zeta)\} \tag{6.3}$$

for a.e. $\zeta \in \partial\mathbb{D}$ along all nontangential paths in $\mathbb{D}$ to the point ζ.

In this connection, note that the Hilbert–Stieltjes integral can be described in another way for functions Φ of bounded variation.

Namely, let us denote by $C(\zeta_0,\varepsilon)$, $\varepsilon \in (0,1)$, $\zeta_0 \in \partial\mathbb{D}$, the rest of the unit circle $\partial\mathbb{D}$ after removing its arc $A(\zeta_0,\varepsilon) := \{\zeta \in \partial\mathbb{D} : |\zeta-\zeta_0| < \varepsilon\}$ and, setting

$$\mathrm{I}_\Phi(\zeta_0,\varepsilon) = \frac{1}{2\pi}\int\limits_{C(\zeta_0,\varepsilon)} \frac{\zeta d\,\Phi_*(\zeta)}{\zeta-\zeta_0}, \quad \zeta_0\in\partial\mathbb{D}, \quad \text{where} \quad \Phi_*(\zeta) := \Phi(\arg\zeta),$$

define the *singular integral of the Cauchy–Stieltjes type*

$$\mathbb{I}_\Phi(\zeta_0) = \frac{1}{2\pi}\int\limits_{\partial\mathbb{D}} \frac{\zeta\,d\,\Phi}{\zeta-\zeta_0}, \quad \zeta_0\in\partial\mathbb{D},$$

as a limit of the integrals $\mathrm{I}(\zeta_0,\varepsilon)$ as $\varepsilon\to 0$. By the paper [21], we have

$$\lim_{z\to\zeta} \mathbb{C}_\Phi(z) = \frac{1}{2}\cdot\Phi'(\arg\,\zeta) + i\cdot\mathbb{I}_\Phi(\zeta) \tag{6.4}$$

for a.e. $\zeta\in\partial\mathbb{D}$ along all nontangential paths in $\mathbb{D}$ to the point ζ. Comparing relations (6.3) and (6.4), we arrive at the following conclusion.

Corollary 10. *Let $\Phi : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic function with bounded variation on $[-\pi,\pi]$. Then for a.e. $\tau\in[-\pi,\pi]$ we have*

$$\mathbb{H}_\Phi(\tau) = 2\cdot\mathbb{I}_\Phi(e^{i\tau}). \tag{6.5}$$

§7. Representation of the Luzin construction

The following deep (nontrivial) result of Luzin was among the main theorems of his Thesis, see, e.g., his paper [13], his Thesis [14], p. 35, and its reprint [15], p. 78, where one may assume that $\Phi(0)=\Phi(1)=0$, cf. also [29].

Theorem A. *For any measurable function $\varphi : [0,1]\to\mathbb{R}$, there is a continuous function $\Phi : [0,1]\to\mathbb{R}$ such that $\Phi'=\varphi$ a.e.*

It is on the basis of Theorem A that Luzin proved the next important result of his Thesis, see, e.g., [15], p. 80, which was a key to establish the corresponding result on the Hilbert boundary value problem for analytic functions in [25].

Theorem B. *Let $\varphi(\vartheta)$ be real, measurable, finite almost everywhere and have period 2π. Then there exists a harmonic function U in the unit disk $\mathbb{D}$ such that $U(z) \to \varphi(\vartheta)$ for a.e. ϑ as $z \to e^{i\vartheta}$ along any nontangential path.*

Note that Luzin's Thesis was published in Russian as the book [15] with comments of his pupils Bari and Men′shov only after his death. A part of its results was also printed in Italian [16]. However, Theorem A was published with a complete proof in English in the book [30] as Theorem VII(2.3). Later, Frederick Gehring in [7] rediscovered Theorem B and his proof on the basis of Theorem A coincided in fact with the original proof of Luzin. Since the proof is very short and nice and has a common interest, we give it for completeness here.

Proof. By Theorem A, we can find a continuous function $\Phi(\vartheta)$ such that $\Phi'(\vartheta) = \varphi(\vartheta)$ for a.e. ϑ. Considering the Poisson integral

$$u(re^{i\vartheta}) = \frac{1}{2\pi}\int\limits_0^{2\pi} \frac{1-r^2}{1-2r\cos(\vartheta-t)+r^2}\Phi(t)\ dt$$

for $0 < r < 1$, $u(0) := 0$, we see by the Fatou result, see, e.g., 3.441 in [32], p. 53, or Theorem IX.1.2 in [8], that $\frac{\partial}{\partial\vartheta}\ u(z) \to \Phi'(\vartheta)$ as $z \to e^{i\vartheta}$ along any nontangential path whenever $\Phi'(\vartheta)$ exists. Thus, the conclusion follows for the function $U(z) = \frac{\partial}{\partial\vartheta}\ u(z)$. $\Box$

Remark 4. Note that the given function U is harmonic in the punctured unit disk $\mathbb{D}\setminus\{0\}$ because the function u is harmonic in $\mathbb{D}$ and the differential operator $\frac{\partial}{\partial\vartheta}$ commutes with the Laplace operator Δ. Setting $U(0) = 0$, we see that

$$U(re^{i\vartheta}) = -\frac{r}{\pi}\int\limits_0^{2\pi} \frac{(1-r^2)\sin(\vartheta-t)}{(1-2r\cos(\vartheta-t)+r^2)^2}\Phi(t)\ dt \ \to\ 0 \quad \text{as} \quad r \to 0,$$

i.e., $U(z) \to U(0)$ as $z \to 0$ and, moreover, the integral of U over each circle $|z| = r$, $0 < r < 1$, is equal to zero. Thus, by the criterion for a function to be harmonic in terms of the averages over circles, we see that

U is harmonic in $\mathbb{D}$. An alternative argument for the last statement is the removability of isolated singularities for harmonic functions, see, e.g., [18].

In connection with Theorem B, it should also be mentioned that the paper [17] contains the Men′shov theorem on the existence of a trigonometric series that is convergent a.e. to a prescribed measurable function $\varphi : (0, 2\pi) \to \mathbb{R}$.

Corollary 5.1 in [25] refines Theorem B in the following way, see also [26].

Theorem C. *For each (Lebesgue) measurable function $\varphi : \partial\mathbb{D} \to \mathbb{R}$, the space of all harmonic functions $u : \mathbb{D} \to \mathbb{R}$ with the angular limits $\varphi(\zeta)$ for a.e. $\zeta \in \partial\mathbb{D}$ has infinite dimension.*

Remark 5. One can find in [29] finer results, which are counterparts of Theorems A, B, and C in terms of logarithmic capacity. This makes it possible to extend the theory of boundary value problems to the so-called A-harmonic functions corresponding to generalizations of the Laplace equation in inhomogeneous and anisotropic media, see also [33].

By the well-known Lindelöf maximum principle, see, e.g., Lemma 1.1 in [6], we obtain the uniqueness theorem for the Dirichlet problem in the class of bounded harmonic functions u on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. In general, there is no uniqueness theorem for the Dirichlet problem for the Laplace equation, even with zero boundary data.

Many such nontrivial solutions for the Laplace equation can be given merely by the Poisson–Stieltjes integral

$$\mathbb{U}_\Phi(z) := \frac{1}{2\pi} \int\limits_0^{2\pi} P_r(\vartheta - t)\ d\Phi(t), \quad z = re^{i\vartheta}, \quad r < 1, \tag{7.1}$$

with an arbitrary *singular function* $\Phi : [0, 2\pi] \to \mathbb{R}$, i.e., where Φ is of bounded variation and $\Phi' = 0$ a.e. Indeed, by the Fatou theorem, see, e.g., Theorem I.D.3.1 in [12], $\mathbb{U}_\Phi(z) \to \Phi'(\Theta)$ as $z \to e^{i\Theta}$ along any nontangential path whenever $\Phi'(\Theta)$ exists, i.e., $\mathbb{U}_\Phi(z) \to 0$ as $z \to e^{i\Theta}$ for a.e. $\Theta \in [0, 2\pi]$ along any nontangential paths for every singular function Φ.

Example 2. The first natural example is given by formula (7.1) with $\Phi(t) = \varphi(t/2\pi)$ where $\varphi : [0, 1] \to [0, 1]$ is the well-known *Cantor function*, see, e.g., [3] and further references therein.

Example 3. However, the simplest example of this kind is merely

$$u(z) := P_r(\vartheta - \vartheta_0) = \frac{1 - r^2}{1 - 2r\cos(\vartheta - \vartheta_0) + r^2}, \quad z = re^{i\vartheta}, \quad r < 1.$$

We see that $u(z) \to 0$ as $z \to e^{i\Theta}$ for all $\Theta \in (0, 2\pi)$ except $\Theta = \vartheta_0$.

The construction of Luzin can be described as a Poisson–Stieltjes integral.

Theorem 3. *The harmonic function U in Theorem B has the representation*

$$\mathbb{U}_\Phi(z) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} P_r(\vartheta - t)\ d\Phi(t), \ \ z = re^{i\vartheta}, \ \ r \in (0,1), \ \ \vartheta \in [-\pi, \pi], \tag{7.2}$$

where $\Phi : [-\pi, \pi] \to \mathbb{R}$ is the continuous Luzin function with $\Phi' = \varphi$ a.e.

Corollary 11. *The conjugate harmonic function $\mathbb{V}_\Phi$ has finite angular limits*

$$\lim_{z \to \zeta} \mathbb{V}_\Phi(z) = \mathbb{H}_\Phi(\arg \zeta) \quad \textit{for a.e.} \quad \zeta \in \partial\mathbb{D}.$$

Proof. Indeed, choosing in Proposition 1 $g(t) = \Phi(t)$ and $f(t) = P_r(\vartheta - t)$, $t \in [-\pi, \pi]$, for every fixed $z = re^{i\vartheta}$, $r < 1$, $\vartheta \in [-\pi, \pi]$, we obtain

$$\int\limits_{-\pi}^{\pi} \Phi(t) \cdot \frac{\partial}{\partial\vartheta} P_r(\vartheta - t)\ dt$$

$$= \int\limits_{-\pi}^{\pi} P_r(\vartheta - t)\ d\Phi(t), \quad r \in (0,1), \quad \vartheta \in [-\pi, \pi] \tag{7.3}$$

because by the 2π-periodicity of the given functions g and f, the right hand side in (2.2) is equal to zero, $f \in C^1$, and

$$\frac{\partial}{\partial\vartheta} P_r(\vartheta - t) = -\frac{\partial}{\partial t} P_r(\vartheta - t), \quad r \in (0,1), \quad \vartheta \in [-\pi, \pi], \quad t \in [-\pi, \pi].$$

Relation (7.2) follows from (7.3) because it was in the proof of Theorem B:

$$U(z) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \Phi(t) \cdot \frac{\partial}{\partial\vartheta} P_r(\vartheta - t)\ dt,$$

$$z = re^{i\vartheta}, \ \ r \in (0,1), \ \ \vartheta \in [-\pi, \pi]. \qquad \square$$

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