Записки научных
семинаров ПОМИ
Том 467, 2018 г.

I. Vasilyev

LOCAL BOUNDARY SMOOTHNESS OF AN ANALYTIC FUNCTION AND ITS MODULUS IN SEVERAL DIMENSIONS: AN ANNOUNCEMENT

Abstract. The drop of the smoothness of an analytic function compared to the smoothness of its modulus is discussed for the unit ball of $\mathbb{C}^n$. The paper is devoted to local aspects of the problem.

This paper is an announcement; all the proofs of the new results discussed here will be published later elsewhere.

The following theorem was first published in the paper [7].

Theorem A. (*Carleson–Jacobs–Havin–Shamoyan*) *Let α be a real number such that $\alpha \in (0,1)$. Suppose that $f : \mathbb{D} \to \mathbb{C}$ is an analytic function without zeros inside $\mathbb{D}$ that has an α-Hölder modulus on the boundary circle $\mathbb{T}$ of the open unit disc $\mathbb{D}$ and that is also continuous on the closed disk. Then f itself is $\frac{\alpha}{2}$-Hölder on $\mathbb{T}$.*

In fact, Theorem A is true for all indexes $\alpha \in \mathbb{R}_+$. It seems that this result was first proved by L. Carleson. Nevertheless, the only published proof of this theorem is that contained in the book [9]. We refer the reader to the paper [6] for a more detailed history of the subject.

The following local version of Theorem A was proved in the paper [6].

Theorem B. (*Kislyakov–Vasin–Medvedev*) *Let $\alpha \in (0,2)$. Suppose that $f : \mathbb{D} \to \mathbb{C}$ is an analytic function without zeros inside $\mathbb{D}$ which is α-Hölder at some point ξ that belongs to the boundary circle $\mathbb{T}$. Suppose also that $\int_{\mathbb{T}} |\log|f||^p < \infty$ for some $p \geqslant 1$. Then for all intervals I containing the point ξ, the mean oscillation $\nu(f,I)$ enjoys the following property:*

$$\nu(f,I)\left(:= \inf_{a\in\mathbb{C}} \frac{1}{|I|}\int_I |f(z)-a|\,d\sigma(z)\right) \leqslant Cl(I)^{\frac{\alpha}{2-\frac{1}{q}}},$$

where q is the Hölder conjugate of p : $\frac{1}{p}+\frac{1}{q}=1$.

Key words and phrases: Hölder classes, functions analytic in the ball.

Supported by RFBR, grant no. 18-31-00250.

We refer the reader to the papers [11] and [6] for the properties of the function ν and its relationship with the local and global Hölder and Lipschitz classes.

Several remarks are in order. First, we would like to remark that Theorem A was used by J. Brennan in his paper [2], where with the help of this result he characterized sufficiently regular planar domains on which any analytic function admits polynomial approximation in the L^p metric. Another application of the Carleson–Jacobs–Havin–Shamoyan theorem was found in the paper [1], where the authors used it in order to establish new sufficient conditions for a function in the harmonic Dirichlet space to be cyclic. We also mention the paper [3] by Mashreghi and Shabankhah, where the Carleson–Jacobs–Havin–Shamoyan theorem was used to construct new examples of zero sets for the Dirichlet space. One more remark is that Theorem A was cited in the papers [5] and [4]. Alas, it is yet unclear whether the local theorem has any of the mentioned applications.

This article is an honest attempt to treat the multidimensional case in the local problem. The following theorems are the main results of the paper.

Theorem 1. *Let $\alpha \in (0,1)$. Suppose that $f : \mathbb{B}^n \to \mathbb{C}$ is an outer function such that for all $t \in \mathbb{S}^n$ one has $|\phi(t) - \phi(\mathbb{1})| \leqslant C_0 d(t, \mathbb{1})^\alpha$, where $\phi := |f|, \mathbb{1} := (1, 0, \ldots, 0), d(u,v) := |1 - \langle u, v\rangle|^{\frac{1}{2}}$ is the nonisotropic metric on the n-dimensional unit sphere $\mathbb{S}^n$. Suppose also that $B_p := \int\limits_{\mathbb{S}^n} |\log \phi|^p < \infty$ for some $p > 1$. Then for all nonisotropic balls Q containing the point $\mathbb{1}$ the quantity $\nu(f,Q)$ measuring smoothness enjoys the following property:*

$$\nu(f,Q) \left(:= \inf_{a \in \mathbb{C}} \frac{1}{|Q|} \int\limits_Q |f(z) - a| \, d\sigma(z) \right) \leqslant Cl(Q)^{\frac{\alpha}{n+1-\frac{n}{q}}},$$

where $l(Q)$ is the radius of the ball Q and q is the Hölder conjugate of p.

Theorem 2. *Let $\alpha \in (0,1)$. Suppose that $f : \mathbb{B}^n \to \mathbb{C}$ is an analytic function without zeros inside $\mathbb{B}^n$, continuous up to the boundary n-dimensional unit sphere $\mathbb{S}^n$ such that for all $t \in \mathbb{S}^n$ one has $|\phi(t) - \phi(\mathbb{1})| \leqslant C_0 d(t, \mathbb{1})^\alpha$. Then for all nonisotropic balls Q containing the point $\mathbb{1}$ the function $\nu(f,Q)$ enjoys the following property:*

$$\nu(f,Q) \leqslant Cl(Q)^{\frac{\alpha}{2}}.$$

We explain the difference between these results. In the first theorem, the method yields nothing for $p = 1$ unless $n = 1$. However, if we formally put $p = 1$ there for comparison with the assumptions of Theorem 2, we arrive at a restriction on the surface integral. Namely, we impose the condition on the surface integral of the modulus of the logarithm of our function ϕ whereas in the second theorem we demand the boundness of the integrals for the all slice functions, which is an (*a priori*) stronger condition (consult [10] for more details). We also remark that in the second case the “slice” condition holds automatically once we know that our function f has no zeros in $\mathbb{B}^n$ and is both analytic in $\mathbb{B}^n$ and continuous up to the unit sphere $\mathbb{S}^n$. Another observation is that if $n = 1$ then there is no technical difference between these two situations and both results coincide basically with Theorem C of the paper [6]. We also refer the reader to the paper [8], where the global problem in the case of the unit ball was resolved.

As it seems to the author, it is plausible that there are versions of these theorems that hold true in a more general setting, namely in the context of the holomorphic functions defined on more general domains in $\mathbb{C}^n$. The exponent $\frac{p}{p+n}$ in Theorem 1 is conjectured to be sharp. The author plans to prove both these generalizations in the nearest future.

References

1. E. Abakumov, O. El-Fallah, K. Kellay, T. Ransford, *Cyclicity in the harmonic Dirichlet space.* — Harmonic Analysis, Function Theory, Operator Theory, and Their Applications, 1–10, Theta Ser. Adv. Math., Theta, Bucharest, 2017.
2. J. E. Brennan, *Approximation in the mean by polynomials on non-Carathéodory domains.* — Ark. Mat., **15**, No. 1–2 (1977), 117–168.
3. J. Mashreghi, M. Shabankhah, *Zero sets and uniqueness sets with one cluster point for the Dirichlet space.* — J. Math. Analys. Applic., **357**, No. 2 (2009), 498–503.
4. B. A. Taylor, D. L. Williams, *Zeros of Lipschitz functions analytic in the unit disc.* — Michigan Math. J., **18**, No. 2 (1971), 129–139.
5. J. Bruna, J. Ortega, *Closed finitely generated ideals in algebras of holomorphic functions and smooth to the boundary in strictly pseudoconvex domains.* — Math. Annalen, **268** (1984), 137–158.
6. A. V. Vasin, S. V. Kislyakov, A. N. Medvedev, *Local smoothness of an analytic function compared to the smoothness of its modules.* — Algebra Analiz, **25**, No. 3 (2013), 52–85.
7. V. P. Havin, F. A. Shamoyan, *Analytic functions with the boundary values having Lipschitz module.* — Zap. Nauchn. Semin. LOMI **19** (1970), 237–239.
8. N. A. Shirokov, *Smoothness of a holomorphic function in a ball and smoothness of its modulus on the sphere.* — Zap. Nauchn. Semin. POMI **447** (2016), 123–128.

9. N. A. Shirokov, *Analytic functions smooth up to the boundary.* Springer Verlag (1988).
10. W. Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$. Springer-Verlag, New York (1980).
11. R. A. DeVore, R. C. Sharpley, *Maximal functions measuring smoothness.* — Mem. Amer. Math. Soc. **47**, No. 293 (1984).

С.-Петербургское отделение
Математического института
им. В. А. Стеклова РАН,
наб. р. Фонтанки, д. 27,
191023 С.-Петербург, Россия
E-mail: `milavas@pdmi.ras.ru`

Поступило 13 июня 2018 г.