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A NOTE ON A CONJECTURE BY KHABIBULLIN

Abstract. We show that for $n = 2$ and $\alpha > 1/2$ Khabibullin's conjecture is not true.

§1. Introduction

Conjecture 1.1 (Khabibullin's Conjecture). *Let $\alpha > 1/2$. For any non-negative increasing function $h(t)$ on the interval $[0, \infty)$ and for any $n \geqslant 2$ if*

$$\int_0^t \frac{h(x)}{x}(t-x)^{n-1}\,dx \leqslant t^{\alpha+n-1}, \qquad 0 \leqslant t < \infty, \tag{1.1}$$

then

$$\int_0^\infty \frac{h(t)}{t}\frac{dt}{1+t^{2\alpha}} \leqslant \frac{\pi}{2}\prod_{k=1}^{n-1}\Big(1+\frac{\alpha}{k}\Big). \tag{1.2}$$

This conjecture, though in a different form, was initially proposed by Khabibullin [3] in a paper related to the Paley problem about plurisubharmonic functions of finite lower order. It is also possible to express Khabibullin's conjecture in three equivalent forms (Khabibullin [4], Baladai, Khabibullin [1]) but in this paper we work only with the version stated above. This conjecture is related to extremal problems in the theory of entire functions of several variables, but primarily it is important because the upper bound in (1.2) is an estimate for the rate of growth of a plurisubharmonic function on the unit sphere. For more details about this conjecture's connection with the class of plurisubharmonic functions one could refer to an extensive survey by Khabibullin [5] and the references therein. It has been proved that these inequalities are true whenever $0 \leqslant \alpha \leqslant 1/2$, a result known as Khabibullin's theorem. Several proofs of this theorem exist in the literature, for example one could refer to Khabibullin [3], Sharipov [6], or more recently Bërdëllima [2]. Elementary

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observations [2, Prop. 3.2] show that there exists a unique nonnegative, increasing function $h^*(t)$ that satisfies both inequalities with identity where $h^*(t)$ is given by

$$h^*(t) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(n)} t^\alpha, \quad \alpha > 0, \ \ n \geqslant 2, \ \ n \in \mathbb{N}. \tag{1.3}$$

Here $\Gamma(\,\cdot\,)$ denotes Euler's gamma function. The existence of h^* shows that the upper estimates in the integral inequalities are sharp. However Khabibullin's conjecture is not true in general. Sharipov [7] explicitly constructed a counterexample when $\alpha = 2$ and $n = 2$ by using a method of spline functions. However it is still an open question whether there exists some $n \geqslant 2$ and $\alpha > 1/2$ for which this conjecture is true. In this short note we show that in general one can construct a counterexample whenever $n = 2$ and $\alpha > 1/2$.

§2. Preliminary discussion

The integral inequality (1.1) can be regarded as an inequality of two functions $f(t) \leqslant g(t)$ for all $t \in [0, \infty)$ where $f(t)$ denotes the integral and $g(t) := t^{\alpha+n-1}$. On the other hand inequality (1.2) is an inequality of quantities that are fixed for any given parameters α and n. Therefore it is reasonable to think of getting from (1.1) to (1.2) through some integration over $[0, \infty)$ with some appropriate real valued function $v(t)$. If so then there must exist some relation of $v(t)$ with the function $\phi(t) := (1+t^{2\alpha})^{-1}$ which appears in (1.2). To find an appropriately related function v first we must study ϕ and in particular its derivatives. In general for ϕ we have the following representation for its derivatives

$$\frac{d^n}{dt^n}\phi(t) = \frac{P(t^\alpha)}{t^n \cdot [Q(t^\alpha)]^{n+1}}, \qquad n \in \mathbb{N}, \tag{2.1}$$

where $Q(t^\alpha) = 1+t^{2\alpha}$ and $P(t^\alpha)$ is a polynomial function in t^α of degree $2n$. One can show (2.1) by using an argument of mathematical induction on n. An important implication of (2.1) is the growth order of $\phi^{(n)}(t)$ like $\mathcal{O}(t^{-n-2\alpha})$ as $t \to \infty$[1]. In particular this fact together with inequality

[1]With slight abuse of notation $\frac{d^n}{dt^n}\phi$ and $\phi^{(n)}(t)$ will be used interchangeably to mean the nth order derivative of a function ϕ with respect to a real variable t.

(1.1) imply the following vanishing limits

$$\lim_{t\to\infty} |\phi^{(k)}(t)| \int_0^t \frac{h(x)}{x}(t-x)^k\,dx = 0, \qquad \forall k \in \mathbb{N}. \tag{2.2}$$

If additionally the function $h(t)$ satisfies a restriction of the form $h(t) = o(t^{2\alpha})$ as $t \to \infty$ then (2.2) holds true as well for $k = 0$. Another key characteristic of $\phi(t)$ is its sign variation over the interval $[0, \infty)$. It is possible, though technical, to show that when $0 < \alpha \leqslant 1/2$ the derivatives $\phi^{(n)}(t)$ are of constant sign for every $n \in \mathbb{N}$. In fact they satisfy the following equation[2]

$$\operatorname{sign} \phi^{(n)}(t) = (-1)^n, \qquad \forall t > 0, \;\; \forall n \in \mathbb{N}, \tag{2.3}$$

where $\operatorname{sign}(\,\cdot\,)$ is the usual sign function which takes the value 1 on the positive numbers, -1 on the negative numbers and 0 at the origin. Since successive integrations bring $\phi^{(n)}(t)$ to $\phi(t)$ it is justifiable to assume that there exists some relation of our sought function $v(t)$ with the derivatives of $\phi(t)$. But integrating both sides of (1.1) with $v(t)$ over $[0, \infty)$ is valid for our problem only if $v(t) \geqslant 0$ for all $t \geqslant 0$ because otherwise the order of inequality in (1.1) would reverse. At least when $0 < \alpha \leqslant 1/2$ we can set $v(t) := (-1)^n \phi^{(n)}(t)$. By (2.3) it is immediate that $v(t) \geqslant 0$. Then multiplying both sides of (1.1) by v and integrating over $[0, \infty)$ yields

$$\int_0^\infty v(t) f(t)\,dt \leqslant \int_0^\infty v(t) g(t)\,dt. \tag{2.4}$$

Using the vanishing limit conditions in (2.2) and the assumption that $h(t) = o(t^{2\alpha})$ as $t \to \infty$ we get from integrating n times by parts in (2.4) the following inequality

$$(-1)^n \Gamma(n) \int_0^\infty \frac{h(t)}{t}(-1)^n \phi(t)\,dt \leqslant (-1)^n \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1}(-1)^n \phi(t)\,dt,$$

where we use the definitions of f and g and the fact that ϕ can be obtained from v by integrating successively n times. After proper simplifications in

[2] For rigorous derivations of (2.1), (2.2), and (2.3) please refer to the appendix.

the last inequality we arrive at

$$\int_0^\infty \frac{h(t)}{t}\phi(t)\,dt \leqslant \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(n)}\int_0^\infty t^{\alpha-1}\phi(t)\,dt = \frac{\pi}{2\alpha}\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(n)} = \frac{\pi}{2}\prod_{k=1}^{n-1}\Big(1+\frac{\alpha}{k}\Big) \tag{2.5}$$

which is exactly inequality (1.2). This is but a special case of Khabibullin's theorem, however one might want to apply this method also when $\alpha > 1/2$. But in this case the derivatives of our function $\phi(t)$ are not guaranteed to be of constant sign over $[0,\infty)$ and we cannot follow a similar argument. In particular for $n = 2$ it is easy to see that $\phi''(t)$ changes sign on $[0,\infty)$. Explicit calculations show that

$$\phi''(t) = \frac{2\alpha t^{2\alpha-2}((2\alpha+1)t^{2\alpha} - (2\alpha-1))}{(1+t^{2\alpha})^3} \tag{2.6}$$

and evidently $\phi''(t) < 0$ for $t \in [0,\tau)$ and $\phi''(t) > 0$ for $t > \tau$ where τ is the nonzero solution of $\phi''(t) = 0$ given by the formula

$$\tau(\alpha) = \Big(\frac{2\alpha-1}{2\alpha+1}\Big)^{\frac{1}{2\alpha}}. \tag{2.7}$$

While on one hand the failure of $\phi''(t)$ to be of constant sign over $[0,\infty)$ invalidates the integration by parts technique, on the other hand it provides us with an opportunity to build a counterexample at least for the case where $n = 2$. We use this sign variation of $\phi''(t)$ to successfully set up sufficient conditions for a counterexample to work and then we explicitly show the existence of such a counterexample.

§3. Sufficient conditions

Let $n = 2$ and fix $\alpha > 1/2$. We want to construct a pair of functions $\{f,h\}$ satisfying the following relation

$$f(t) = \int_0^t \frac{h(x)}{x}(t-x)\,dx \text{ or equivalently } h(t) = t\frac{d^2}{dt^2}f(t) \quad \text{for all } \; t \in \mathbb{R}^+. \tag{3.1}$$

In the construction of our functions $\{f,h\}$ we follow an approach similar to Sharipov's method of spline functions (piecewise continuous functions).

Define a function f as follows

$$f(t) = \begin{cases} t^{\alpha+1}(1-\varepsilon\eta(t)), & t \in [0,\tau], \\ t^{\alpha+1}, & t \in (\tau,\infty), \end{cases}$$

where τ is given by (2.7) and $\eta(t)$ is a sufficiently smooth (derivatives of higher orders exist) such that $0 \leqslant \eta(t) \leqslant 1$ for all $t \in [0,\tau]$ and $\eta(t) = 0$ for all $t \notin [0,\tau]$. Clearly we have $0 \leqslant f(t) \leqslant t^{\alpha+1}$ whenever $0 \leqslant \varepsilon \leqslant 1$ and so the first integral inequality is satisfied. From the relation in (3.1) it follows that f must be at least three times continuously differentiable on $\mathbb{R}^+$ whenever $h(t)$ is differentiable on $\mathbb{R}^+$. These requirements lead to the following conditions:

$$\lim_{t\to\tau^-} \frac{d^i}{dt^i}[t^{\alpha+1}\eta(t)] = 0 \quad \text{for} \quad i = 1,2,3. \tag{3.2}$$

The limits in (3.2) illustrate the fact that the right and left limits of the derivatives of f must coincide at $t = \tau$. From (3.1) we can also get a piecewise representation of h:

$$h(t) = \begin{cases} (\alpha+1)\alpha t^\alpha - \varepsilon t \dfrac{d^2}{dt^2}\eta(t), & t \in [0,\tau], \\ (\alpha+1)\alpha t^\alpha, & t \in (\tau,\infty). \end{cases}$$

Consider now the integral expression

$$J_\varepsilon(\alpha) := \int_0^\infty \frac{h(t)}{t}\phi(t)\,dt.$$

By the construction of f it follows that $h = o(t^{2\alpha})$ as $t \to \infty$. Integrating by parts twice on $J_\varepsilon(\alpha)$ and using the vanishing limits (2.2) yields

$$J_\varepsilon(\alpha) = \int_0^\infty \Big[\int_0^t \frac{h(u)}{u}(t-u)\,du\Big]\phi''(t)\,dt.$$

Substituting $f(t)$ in the integrand we obtain

$$\begin{aligned} J_\varepsilon(\alpha) = \int_0^\infty f(t)\phi''(t)\,dt &= \int_0^\tau t^{\alpha+1}(1-\varepsilon\eta(t))\phi''(t)\,dt + \int_\tau^\infty t^{\alpha+1}\phi''(t)\,dt \\ &= -\varepsilon\int_0^\tau t^{\alpha+1}\eta(t)\phi''(t)\,dt + \int_0^\infty t^{\alpha+1}\phi''(t)\,dt. \end{aligned}$$

By assumption $\eta(t) \geqslant 0$. On the other hand $\phi''(t) < 0$ for all $t \in [0, \tau]$ whenever $\alpha > 1/2$, therefore

$$J_\varepsilon(\alpha) = -\varepsilon \int_0^\tau t^{\alpha+1}\eta(t)\phi''(t)\,dt + \int_0^\infty t^{\alpha+1}\phi''(t)\,dt > \int_0^\infty t^{\alpha+1}\phi''(t)\,dt = \frac{\pi}{2}(\alpha+1).$$

This leads to a contradiction for the second integral inequality. However for $\eta(t)$ to be a valid counterexample it needs to satisfy some additional conditions which are derived from the assumptions that $h(t)$ is nonnegative and is an increasing function. These imply

$$\varepsilon \frac{d^2}{dt^2}[t^{\alpha+1}\eta(t)] \leqslant (\alpha+1)\alpha t^{\alpha-1} \qquad \forall t \in (0, \tau), \tag{3.3}$$

$$\varepsilon \frac{d^2}{dt^2}[t^{\alpha+1}\eta(t)] + \varepsilon t \frac{d^3}{dt^3}[t^{\alpha+1}\eta(t)] \leqslant (\alpha+1)\alpha^2 t^{\alpha-1} \qquad \forall t \in (0, \tau) \tag{3.4}$$

for all $t \in (0, \tau)$ and for any $\varepsilon \in [0, 1]$. Condition (3.3) follows from the nonnegativity of h through the equivalences

$$\begin{aligned} h(t) \geqslant 0 \Leftrightarrow t\frac{d^2}{dt^2}f(t) \geqslant 0 \Leftrightarrow \frac{d^2}{dt^2}[t^{\alpha+1}(1 - \varepsilon\eta(t))] \geqslant 0 \\ \Leftrightarrow (\alpha+1)\alpha t^{\alpha-1} \geqslant \varepsilon \frac{d^2}{dt^2}[t^{\alpha+1}\eta(t)]. \end{aligned}$$

Similarly condition (3.4) is derived from the increasing property of h through the equivalences

$$\begin{aligned} h'(t) \geqslant 0 &\Leftrightarrow \frac{d}{dt}\Big[t\frac{d^2}{dt^2}f(t)\Big] \geqslant 0 \\ &\Leftrightarrow \varepsilon\frac{d^2}{dt^2}[t^{\alpha+1}\eta(t)] + \varepsilon t\frac{d^3}{dt^3}[t^{\alpha+1}\eta(t)] \leqslant (\alpha+1)\alpha^2 t^{\alpha-1}. \end{aligned}$$

We collect these observations in the next statement.

Proposition 3.1. *Suppose $\eta(t)$ is a sufficiently smooth nonnegative real valued function such that $0 \leqslant \eta(t) \leqslant 1$ for $t \in [0, \tau]$ and $\eta(t) = 0$ otherwise. Suppose further that $\eta(t)$ satisfies conditions (3.2)–(3.4) on $(0, \tau)$. If*

$$f(t) = \begin{cases} t^{\alpha+1}(1 - \varepsilon\eta(t)), & t \in [0, \tau], \\ t^{\alpha+1}, & t \in (\tau, \infty) \end{cases}$$

and $0 \leqslant \varepsilon \leqslant 1$ then $f(t)$ satisfies the first integral inequality and

$$h(t) = t\frac{d^2}{dt^2}f(t)$$

is a nonnegative increasing function which contradicts the conjecture for any $\alpha > 1/2$.

§4. An explicit counterexample

In this section we explicitly show the existence of a function η which satisfies the prerequisites set in Proposition 3.1. Consider the following function

$$\eta(t) = \begin{cases} \cos^4\left(\dfrac{\pi t}{2\tau}\right), & t \in [0,\tau], \\ 0, & t \in \mathbb{R}^+ \setminus [0,\tau]. \end{cases}$$

Notice that $\eta(t)$ is differentiable on $\mathbb{R}$, nonnegative, and satisfies $0 \leqslant \eta(t) \leqslant 1$ for all $t \in [0,\tau]$. By definition above we also have $\eta(t) = 0$ for $t \notin [0,\tau]$. From Leibniz's formula one obtains

$$\begin{aligned} &\lim_{t\to\tau^-} \frac{d^n}{dt^n}\left[t^{\alpha+1}\cos^4\left(\frac{\pi t}{2\tau}\right)\right] \\ &= \sum_{k=0}^{n}\binom{n}{k}\lim_{t\to\tau^-}\frac{d^k}{dt^k}t^{\alpha+1}\lim_{t\to\tau^-}\frac{d^{n-k}}{dt^{n-k}}\cos^4\left(\frac{\pi t}{2\tau}\right) = 0 \end{aligned}$$

for all $n = 0,1,2,3$ since each summand contains a factor $\cos\left(\dfrac{\pi t}{2\tau}\right)$. Therefore condition (3.2) is satisfied. It remains to verify conditions (3.3) and (3.4) for our choice function $\eta(t)$. By applying Leibniz formula to $\dfrac{d^i}{dt^i}\left[t^{\alpha+1}\eta(t)\right]$ for $i = 2,3$ we rewrite inequalities in (3.3) and (3.4) respectively as follows

$$\begin{aligned} &\varepsilon[(\alpha+1)\alpha t^{\alpha-1}\eta(t) + 2(\alpha+1)t^{\alpha}\eta'(t) + t^{\alpha+1}\eta''(t)] \\ &\qquad \leqslant (\alpha+1)\alpha t^{\alpha-1} \quad \forall t \in (0,\tau), \\ &\varepsilon[(\alpha+1)\alpha^2 t^{\alpha-1}\eta(t) + (3\alpha+2)(\alpha+1)t^{\alpha}\eta'(t) \\ &\qquad\qquad + (3\alpha+4)t^{\alpha+1}\eta''(t)] + t^{\alpha+2}\eta'''(t)] \\ &\qquad \leqslant (\alpha+1)\alpha^2 t^{\alpha-1} \quad \forall t \in (0,\tau). \end{aligned}$$

Since $\alpha > 0$ after proper simplification we get their equivalent form

$$\varepsilon\Big[\eta(t) + \frac{2}{\alpha}t\eta'(t) + \frac{1}{\alpha(\alpha+1)}t^2\eta''(t)\Big] \leqslant 1 \quad \forall t \in (0,\tau), \tag{4.1}$$

$$\varepsilon\Big[\eta(t)+\frac{3\alpha+2}{\alpha^2}t\eta'(t)+\frac{3\alpha+4}{(\alpha+1)\alpha^2}t^2\eta''(t)+\frac{1}{(\alpha+1)\alpha^2}t^3\eta'''(t)\Big]\leqslant 1\ \forall t\in(0,\tau). \tag{4.2}$$

Since η is smooth on $(0,\tau)$, the suprema of its derivatives over the bounded interval $(0,\tau)$ exist and are finite. Define

$$\eta_i^* := \sup_{t\in(0,\tau)}\Big\{\Big|\frac{d^i\eta}{dt^i}\Big|\Big\} \quad \text{for} \quad i=0,1,2,3$$

to be the supremum of the ith derivative of η over the interval $(0,\tau)$. Then we get the following estimates valid for all $t\in(0,\tau)$:

$$\begin{aligned}
&\Big|\eta(t)+\frac{2}{\alpha}t\eta'(t)+\frac{1}{\alpha(\alpha+1)}t^2\eta''(t)\Big| \\
&\quad\leqslant |\eta(t)|+\frac{2}{\alpha}t|\eta'(t)|+\frac{1}{\alpha(\alpha+1)}t^2|\eta''(t)| \\
&\quad\leqslant \eta_0^*+\frac{2}{\alpha}\tau\eta_1^*+\frac{1}{\alpha(\alpha+1)}\tau^2\eta_2^*, \\
&\Big|\eta(t)+\frac{3\alpha+2}{\alpha^2}t\eta'(t)+\frac{3\alpha+4}{(\alpha+1)\alpha^2}t^2\eta''(t)+\frac{1}{(\alpha+1)\alpha^2}t^3\eta'''(t)\Big| \\
&\quad\leqslant |\eta(t)|+\frac{3\alpha+2}{\alpha^2}t|\eta'(t)|+\frac{3\alpha+4}{(\alpha+1)\alpha^2}t^2|\eta''(t)|+\frac{1}{(\alpha+1)\alpha^2}t^3|\eta'''(t)| \\
&\quad\leqslant \eta_0^*+\frac{3\alpha+2}{\alpha^2}\tau\eta_1^*+\frac{3\alpha+4}{(\alpha+1)\alpha^2}\tau^2\eta_2^*+\frac{1}{(\alpha+1)\alpha^2}\tau^3\eta_3^*.
\end{aligned}$$

To prove (4.1) and (4.2) it is sufficient to show the following system of inequalities

$$\varepsilon A\leqslant 1 \ \text{ where } \ A := \eta_0^*+\frac{2}{\alpha}\tau\eta_1^*+\frac{1}{\alpha(\alpha+1)}\tau^2\eta_2^*, \tag{4.3}$$

$$\varepsilon B\leqslant 1 \ \text{ where } \ B= \eta_0^*+\frac{3\alpha+2}{\alpha^2}\tau\eta_1^*+\frac{3\alpha+4}{(\alpha+1)\alpha^2}\tau^2\eta_2^*+\frac{1}{(\alpha+1)\alpha^2}\tau^3\eta_3^*. \tag{4.4}$$

To make (4.3) and (4.4) be valid simultaneously we choose any $\varepsilon\in[0,1]$ if $\max\{A,B\}\leqslant 1$ otherwise we set $\varepsilon=(\max\{A,B\})^{-1}$ if $\max\{A,B\}>1$. In any case we are guaranteed to have such an $0\leqslant\varepsilon\leqslant 1$. We collect these observations in the following statement.

Proposition 4.1. *Let $n = 2$ and let $\eta(t) = \cos^4\left(\frac{\pi t}{2\tau}\right)$ for $t \in [0, \tau]$ and zero otherwise. Then h defined by*

$$h(t) = \begin{cases} (\alpha+1)\alpha t^{\alpha} - \varepsilon t \frac{d^2}{dt^2}\eta(t), & t \in [0, \tau], \\ (\alpha+1)\alpha t^{\alpha}, & t \in (\tau, \infty) \end{cases}$$

is a nonnegative increasing function that violates the conjecture for any $\alpha > 1/2$.

Proof. This follows from the analysis in Section 3 and Section 4. □

§5. Concluding remarks

We have established a new result concerning a nontrivial question raised by Khabibullin. We showed that his conjecture is not true whenever $n = 2$ and $\alpha > 1/2$. While the case of $n > 2$ is still an open problem the method we employed in this paper looks promising in obtaining fruitful results even for the more general case. However it must be pointed out that finding sharp estimates when $n \geqslant 2$ and $\alpha > 1/2$ is still an open problem. Sharipov [6] conjectured that for each $n \geqslant 2$ and $\alpha > 0$ there exists a positive constant $C_{Kh}(n, \alpha)$ that is a sharp upper bound for the integral inequality (1.2). Note that when $0 \leqslant \alpha \leqslant 1/2$ we have

$$C_{Kh}(n, \alpha) = \frac{\pi}{2} \prod_{k=1}^{n-1} \left(1 + \frac{\alpha}{k}\right).$$

It is possible to numerically bound Khabibullin's constants $C_{Kh}(n, \alpha)$ but explicitly finding a formula for them when $\alpha > 1/2$ is yet out of reach. It would certainly be desirable to get a formula for the $C_{Kh}(n, \alpha)$ because they provide sharp estimates for the growth rate on the unit sphere of a plurisubharmonic function of a given finite lower order.

Appendix §A. Technical details

Lemma A.1. *Let $\phi(t) = \frac{1}{1 + t^{2\alpha}}$ then for all $m \geqslant 1, m \in \mathbb{Z}$ we have*

$$\frac{d^m \phi}{dt^m} = \frac{P(t^{\alpha})}{t^m \cdot [Q(t^{\alpha})]^{m+1}},$$

where $Q(t^{\alpha}) = 1 + t^{2\alpha}$ and P is a polynomial function in t^{α} of degree $2m$.

Proof. We prove this lemma using mathematical induction. For $m = 1$ we have

$$\phi'(t) = \frac{-2\alpha t^{2\alpha}}{t(1+t^{2\alpha})^2} = \frac{P(t^\alpha)}{t \cdot [Q(t^\alpha)]^2}$$

where $P(t^\alpha) = -2\alpha t^{2\alpha}$ as a polynomial function in t^α is of degree 2. Let the statement be true for $m = n$, that is

$$\phi^{(n)}(t) = \frac{P(t^\alpha)}{t^n \cdot [Q(t^\alpha)]^{n+1}}$$

where P is a polynomial function in t^α of degree $2n$. Differentiating both sides of the last equation with respect to t yields

$$\begin{aligned} \frac{d}{dt}\phi^{(n)}(t) &= \frac{d}{dt}\Big[\frac{P(t^\alpha)}{t^n \cdot [Q(t^\alpha)]^{n+1}}\Big] \\ \Leftrightarrow \phi^{(n+1)}(t) &= \frac{\alpha t^\alpha P'Q - nPQ - (n+1)t^\alpha PQ'}{t^{n+1}[Q(t^\alpha)]^{n+2}}. \end{aligned}$$

Denote by $\deg P$ the degree of the polynomial P. By the inductive hypothesis since $\deg P = 2n$, we have $\deg P' = 2n - 1$. Notice that $\deg t^\alpha = 1$ and by the definition of Q we have $\deg Q = 2$. Setting

$$P_o = \alpha t^\alpha P'Q - nPQ - (n+1)t^\alpha PQ'$$

and using $\deg PQ = \deg P + \deg Q$ for any two polynomials P and Q, we see that the degree of P_o is

$$\begin{aligned} \deg(P_o) &= \max\{\deg(t^\alpha P'Q), \deg(PQ), \deg(t^\alpha PQ')\} \\ &= \max\{1 + 2n - 1 + 2, 2n + 2, 1 + 2n + 1\} \\ &= 2n + 2 = 2(n+1). \end{aligned}$$

□

Lemma A.2. *Let $h(t) = o(t^{2\alpha})$ as $t \to \infty$ and suppose $h(t)$ satisfies inequality* (1.1). *Then*

$$\lim_{T\to\infty} |\phi^{(k)}(T)| \int_0^T \frac{h(x)}{x}(T-x)^k\,dx = 0 \quad \forall k \in \mathbb{Z}^+ \cup \{0\}.$$

Proof. First we prove the claim for $k = 0$. Notice that both $\phi(T)$ and $\int_0^T \frac{h(x)}{x}\,dx$ are differentiable functions. It is sufficient to prove that the limit vanishes when the improper integral $\lim_{T\to\infty} \int_0^T \frac{h(x)}{x}\,dx$ diverges. We get an indeterminate form ∞/∞. On the other hand since $h(t) = o(t^{2\alpha})$ as $t \to \infty$, we obtain

$$\lim_{T\to\infty} \frac{\frac{d}{dT}\Big[\int_0^T \frac{h(x)}{x}\,dx\Big]}{\frac{d}{dT}(1+T^{2\alpha})} = \frac{1}{2\alpha}\lim_{T\to\infty}\frac{h(T)}{T^{2\alpha}} = 0.$$

Then by L'Hospital's Rule [8] we obtain

$$\lim_{T\to\infty} \phi(T)\int_0^T \frac{h(x)}{x}\,dx = \lim_{T\to\infty}\frac{\int_0^T \frac{h(x)}{x}\,dx}{1+T^{2\alpha}}$$

$$= \lim_{T\to\infty} \frac{\frac{d}{dT}\Big[\int_0^T \frac{h(x)}{x}\,dx\Big]}{\frac{d}{dT}(1+T^{2\alpha})} = \frac{1}{2\alpha}\lim_{T\to\infty}\frac{h(T)}{T^{2\alpha}} = 0.$$

Let $k \geqslant 1$ then from the first integral inequality we obtain

$$0 \leqslant \int_0^T \frac{h(x)}{x}(T-x)^k\,dx \leqslant T^{\alpha+k}, \qquad 0 \leqslant T < \infty.$$

On the other hand from Lemma A.1 we deduce that $\phi^{(k)}(T) = \mathcal{O}(T^{-k-2\alpha})$ as $t \to \infty$. Therefore

$$|\phi^{(k)}(T)|\int_0^T \frac{h(x)}{x}(T-x)^k\,dx \leqslant \mathcal{O}(T^{-k-2\alpha})T^{\alpha+k} = \mathcal{O}(T^{-\alpha}) \to 0 \text{ as } T \to \infty.$$

This completes the proof. $\square$

Lemma A.3. *Then for all* $0 \leqslant \alpha \leqslant 1/2$ *and* $t > 0$ *the following holds true*

$$\operatorname{sign}(\phi^{(n)}(t)) = \begin{cases} (-1)^n, & 0 < \alpha \leqslant 1/2, n \geqslant 0,\\ 0, & \alpha = 0, n \geqslant 1,\\ 1, & \alpha = 0, n = 0. \end{cases}$$

Proof. First notice that
$$\frac{d^m}{dt^m}(1+t^{2\alpha}) = 2\alpha(2\alpha-1)(2\alpha-2)\dots(2\alpha-m+1)t^{2\alpha-m}.$$
This implies
$$\operatorname{sign}\left[\frac{d^m}{dt^m}(1+t^{2\alpha})\right] = \begin{cases} (-1)^{m-1}, & 0<\alpha<1/2, m\geqslant 1,\\ 0, & \alpha\in\{0,1/2\}, m\geqslant 2,\\ 0, & \alpha=0, m=1,\\ 1, & \alpha=1/2, m=1,\\ 1, & \alpha\geqslant 0, m=0\end{cases} \tag{A.1}$$
for all $t>0$ and $m\in\mathbb{Z}$. We will treat the case where $\alpha=0$ and $\alpha=1/2$ separately. So assume that $0<\alpha<1/2$. This reduces to showing that
$$\operatorname{sign}(\phi^{(n)}(t)) = (-1)^n$$
for all $n\geqslant 0$ and $t>0$. We prove this statement by strong mathematical induction. The base case would be for $n=0$. By the definition of $\phi(t)$ we have
$$(1+t^{2\alpha})\phi(t)=1 \Rightarrow \operatorname{sign}((1+t^{2\alpha})\phi(t)) = \operatorname{sign}(1).$$
Since $1+t^{2\alpha}>0$ for all $t>0$, we see that $\operatorname{sign}(1+t^{2\alpha})=1$. Thus we obtain
$$\operatorname{sign}(\phi(t)) = 1 = (-1)^0.$$
Therefore base case is true. Assume that the equation $\operatorname{sign}(\phi^{(n)}(t)) = (-1)^n$ is valid for all $n=1,2,..,k$. Now let $n=k+1$. Differentiating $(1+t^{2\alpha})\phi(t)$ successively $k+1$ times and using Leibniz's formula yields
$$\frac{d^{k+1}}{dt^{k+1}}\Big[(1+t^{2\alpha})\phi(t)\Big] = \sum_{i=0}^{k+1}\binom{k+1}{i}\frac{d^i}{dt^i}(1+t^{2\alpha})\frac{d^{k+1-i}}{dt^{k+1-i}}\phi(t).$$
Since $(1+t^{2\alpha})\phi(t)=1$ for all $t\geqslant 0$, we obtain
$$\sum_{i=0}^{k+1}\binom{k+1}{i}\frac{d^i}{dt^i}(1+t^{2\alpha})\frac{d^{k+1-i}}{dt^{k+1-i}}\phi(t) = 0.$$
From the last equation we arrive at
$$\operatorname{sign}\left[(1+t^{2\alpha})\frac{d^{k+1}}{dt^{k+1}}\phi(t)\right] = (-1)\operatorname{sign}\left[\sum_{i=1}^{k+1}\binom{k+1}{i}\frac{d^i}{dt^i}(1+t^{2\alpha})\frac{d^{k+1-i}}{dt^{k+1-i}}\phi(t)\right]. \tag{A.2}$$

For $1 \leqslant i \leqslant k+1$ the sign of each summand in (A.2) can be computed as follows

$$\operatorname{sign}\Big[\frac{d^i}{dt^i}(1+t^{2\alpha})\frac{d^{k+1-i}}{dt^{k+1-i}}\phi(t)\Big] = \operatorname{sign}\Big[\frac{d^i}{dt^i}(1+t^{2\alpha})\Big]\operatorname{sign}\Big[\frac{d^{k+1-i}}{dt^{k+1-i}}\phi(t)\Big].$$

By (A.1), for all $i = 1, 2, \ldots, k+1$ and $t > 0$ we have

$$\operatorname{sign}\Big[\frac{d^i}{dt^i}(1+t^{2\alpha})\Big] = (-1)^{i-1}.$$

On the other hand applying strong induction for $i = 1, 2, \ldots, k$ to $\operatorname{sign}(\phi^{(k+1-i)}(t))$ gives

$$\operatorname{sign}\Big[\frac{d^{k+1-i}}{dt^{k+1-i}}\phi(t)\Big] = (-1)^{k+1-i}.$$

Therefore we obtain

$$\operatorname{sign}\Big[\frac{d^i}{dt^i}(1+t^{2\alpha})\frac{d^{k+1-i}}{dt^{k+1-i}}\phi(t)\Big] = (-1)^{i-1}(-1)^{k+1-i} = (-1)^k$$

for all $i = 1, 2, \ldots, k$ and $t > 0$. For $i = k+1$ we have

$$\begin{aligned}&\operatorname{sign}\Big[\frac{d^{k+1}}{dt^{k+1}}(1+t^{2\alpha})\frac{d^{k+1-(k+1)}}{dt^{k+1-(k+1)}}\phi(t)\Big]\\ &= \operatorname{sign}\Big[\frac{d^{k+1}}{dt^{k+1}}(1+t^{2\alpha})\cdot\phi(t)\Big] = (-1)^{k+1-1}(-1)^0 = (-1)^k.\end{aligned}$$

This shows that each summand in (A.2) is of the same sign $(-1)^k$ for all $i = 1, 2, \ldots, k+1$ therefore we are led to the conclusion that

$$\operatorname{sign}\Big[(1+t^{2\alpha})\frac{d^{k+1}}{dt^{k+1}}\phi(t)\Big] = (-1)\cdot(-1)^k \Rightarrow \operatorname{sign}\Big[\frac{d^{k+1}}{dt^{k+1}}\phi(t)\Big] = (-1)^{k+1}.$$

This proves the statement for all $0 < \alpha < 1/2$. Now let $\alpha = 0$ then $\phi(t) = 1$ which implies $\operatorname{sign}(\phi(t)) = 1$ and $\operatorname{sign}(\phi^{(n)}(t)) = 0$ for all $n \geqslant 1$ (since $\phi(t)$ in this instance is a constant function, its derivatives of all orders vanish). When $\alpha = 1/2$ we need to prove again that $\operatorname{sign}(\phi^{(n)}(t)) = (-1)^n$ for all $n \geqslant 0$. We need only use mathematical induction. For the base case $n = 0$ we have

$$(1+t)\phi(t) = 1 \Rightarrow \operatorname{sign}(\phi(t)) = 1 = (-1)^0$$

for all $t > 0$. Suppose the statement is true for $n = k$, that is $\operatorname{sign}(\phi^{(k)}(t)) = (-1)^k$. Let $n = k+1$. Differentiating $(1+t)\phi(t)$ successively $k+1$ times yields

$$\frac{d^{k+1}}{dt^{k+1}}\Big[(1+t)\phi(t)\Big] = \sum_{i=0}^{k+1}\binom{k+1}{i}\frac{d^i}{dt^i}(1+t)\frac{d^{k+1-i}}{dt^{k+1-i}}\phi(t)$$
$$= (1+t)\frac{d^{k+1}}{dt^{k+1}}\phi(t) + \binom{k+1}{1}\frac{d^k}{dt^k}\phi(t).$$

Since $(1+t)\phi(t) = 1$ by the definition of $\phi(t)$, it must be the case that

$$(1+t)\frac{d^{k+1}}{dt^{k+1}}\phi(t) = -\binom{k+1}{1}\frac{d^k}{dt^k}\phi(t)$$
$$\Rightarrow \operatorname{sign}\left[\frac{d^{k+1}}{dt^{k+1}}\phi(t)\right] = (-1)\cdot\operatorname{sign}\left[\frac{d^k}{dt^k}\phi(t)\right] = (-1)^{k+1},$$

where at the last step we have used the inductive hypothesis. This completes the proof. □

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