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SOME EXPLICIT RESULTS FOR THE GENERALIZED EMPTINESS FORMATION PROBABILITY OF THE SIX-VERTEX MODEL

Abstract. We study a multi-point correlation function of the six-vertex model on the square lattice with the domain wall boundary conditions which is called the generalized emptiness formation probability. This function describes probability of observing the ferroelectric order around all the vertices of any Ferrer diagram λ at the top-left corner of the lattice. For the free-fermion model we derive and compare explicit formulas for this correlation function for two cases of diagram λ: the square and triangle. We found a connection of our formulas with the τ-function of the sixth Painlevé equation.

§1. Introduction

One of the most interesting properties of the six-vertex model with domain wall boundary conditions [1–3] is existence of the limiting shape, or phase separation phenomena [4, 5]; for recent advances see [6] and references therein. These phenomena arise in the thermodynamic limit and their detailed description is closely related with the problem of calculation of the corresponding correlation functions. The general approach to this problem for the six-vertex model is based on the quantum integrability, whilst in the study and description of the quantum correlation functions an important role play mathematical structures related with classical integrable systems [7], random matrices and combinatorics [8].

A notable progress in calculation of correlation functions of the six-vertex model with the domain wall boundary conditions is achieved for correlations near the boundaries [9, 10]. An example of correlation function which can be computed away from the boundary, i.e., for the bulk, is provided by the so-called emptiness formation probability (EFP), which can represented as a multiple contour integral [11]. By using this integral

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representation an explicit analytic expression for the spatial curve separating ferroelectric order from disorder — the so-called arctic curve is obtained in [12].

At the free-fermion point the EFP can be evaluated in various forms: as a Hankel or a Fredholm determinant, as a discrete random matrix model, and as solution of Toda lattice hierarchy [13]. These representations lead to an interpretation of the arctic curve as a curve of the third-order phase transition [14] and determine thermodynamics the six-vertex model on an L-shaped domain [15]. In [16], we showed that the EFP at the free-fermion point is nothing but the τ-function of the sixth Painlevé equation (refered below as Painlevé VI), and also that this fact can be used to construct full asymptotic expansions in the thermodynamic limit for various regimes of parameters.

In this paper, we address similar problems in relation to the generalized emptiness formation probability (GEFP), introduced and evaluated in the form of a multiple contour integral in [17]. The GEFP is the probability of the ferroelectric order around all vertices which belongs to a Ferrer diagram λ located at the top-left corner of the lattice; the case of EFP corresponds to λ of a rectangular form. Here, we focus our main attention to the GEFP in the case of λ having the shape of triangle. In [18], this case is mentioned as the six-vertex model on a pentagonal domain.

Specifically, here we present and compare various explicit formulas for GEFP, which correspond to λ with the shape of square and triangle. The former case is just a special case of the EFP already studied in [13, 16], while the results for the triangle are new. In particular, for this case we discover a random matrix like representation for GEFP. Additionally, for special values of the discrete parameters we find relations GEFP with Painlevé VI and point out intriguing combinatorial interpretations for the related σ-function. We believe that these results would prove to be useful in further study of the limit shape phenomena in the six-vertex model.

§2. The generalized emptiness formation probability

2.1. Definition and integral representation. Consider the six-vertex model (see, e.g., [19]) on the $N \times N$ square lattice with domain wall boundary conditions. This means that the lattice is obtained by intersection of N horizontal and N vertical lines, and the arrows on the external edges are fixed as follows: on the vertical lines (top and bottom) they are incoming (pointing downward and upward, respectively), while on the horizontal ones (left and right) they are outgoing (pointing left and right, respectively) [1–3].

Let λ be a Young diagram consisting out of s lines, $\lambda = (\lambda_1, \dots, \lambda_s)$, with the rows satisfying the conditions $\lambda_j \leqslant N - j$. Let

$$r_j = N - \lambda_j, \qquad j = 1, \dots, s. \tag{1}$$

According to [17], the GEFP $G_{N,s}^{(r_1,\dots,r_s)}$ is defined as the probability of obtaining a configuration of the model in which arrows of the jth (counted from the top) horizontal line lying to the left after the r_jth vertical line (counted from the right), are all pointing left. Equivalently, treating λ as a Ferrer diagram, one may define GEFP as the probability that the vertices of the top-left corner of the lattice forming the shape λ are all having the same configuration of arrows around them: horizontal arrow are pointing left and vertical ones are pointing downward. (The equivalence of two definitions is due to both the ice rule of the six-vertex model and the domain wall boundary conditions.)

In [17] the following integral representation for GEFP was derived:

$$G_{N,s}^{(r_1,\dots,r_s)} = (-1)^s \oint\limits_{C_0} \cdots \oint\limits_{C_0} \prod_{j=1}^{s} \frac{[(t^2 - 2\Delta t) z_j + 1]^{s-j}}{z_j^{r_j} (z_j - 1)^{s-j+1}} \\ \times \prod_{1 \leqslant j < k \leqslant s} \frac{z_j - z_k}{t^2 z_j z_k - 2\Delta t z_j + 1}\, h_{N,s}(z_1, \dots, z_s)\, \frac{\mathrm{d}z_1 \cdots \mathrm{d}z_s}{(2\pi \mathrm{i})^s}. \tag{2}$$

Here, C_0 is a simple counterclockwise oriented contour surrounding the point $z = 0$ and no other singularity of the integrand. The Boltzmann weights a, b, c of the six-vertex model are encoded in the parameters

$$t \equiv \frac{b}{a}, \qquad \Delta = \frac{a^2 + b^2 - c^2}{2ab}.$$

and they also enter implicitly into the function

$$h_{N,s}(z_1, \dots, z_s) = \frac{\det \left[z_j^{k-1} (z_j - 1)^{s-k} h_{N-k+1}(z_j) \right]_{j,k=1,\dots,s}}{\prod\limits_{1 \leqslant j < k \leqslant s} (z_j - z_k)}. \tag{3}$$

Here, the functions $h_{N-k+1}(z)$ are polynomials of degree $N - k$ in z, with the coefficients depending on the weights of the model (i.e., on the parameters t and Δ). These polynomials are generating functions of certain boundary correlation function of the six-vertex model with domain-wall boundary conditions on the $(N-k+1)\times(N-k+1)$ lattices, $k=1,\dots,s$. The explicit form of these functions is very complicated, being simple only in the case of the weights obeying the free-fermion condition.

2.2. The free-fermion model. The free-fermion condition for the six-vertex model means that the weights satisfy the following equation

$$a^2+b^2=c^2, \quad \text{that is} \quad \Delta=0.$$

For this model, it is convenient to use the parameter

$$\alpha=\frac{t^2}{1+t^2},$$

so that the weights are parametrized such that

$$a/c=\sqrt{1-\alpha}, \qquad b/c=\sqrt{\alpha}.$$

In this case [10, 11]:

$$h_N(z)=\left(\frac{1+t^2z}{1+t^2}\right)^{N-1}=(1-\alpha+\alpha z)^{N-1}.$$

Hence,

$$h_{N,s}(z_1,\dots,z_s)=\prod_{j=1}^{s}(1-\alpha+\alpha z_j)^{N-s}\prod_{1\leqslant j<k\leqslant s}(1-\alpha+\alpha z_j z_k).$$

Therefore, from (2) it follows that at $\Delta=0$ the GEFP reads

$$G_{N,s}^{(r_1,\dots,r_s)}=(-1)^s\oint\limits_{C_0}\cdots\oint\limits_{C_0}\prod_{1\leqslant j<k\leqslant s}(z_j-z_k)\\ \times\prod_{j=1}^{s}\frac{(1-\alpha+\alpha z_j)^{N-j}}{z_j^{r_j}(z_j-1)^{s-j+1}}\,\frac{\mathrm{d}z_1\cdots\mathrm{d}z_s}{(2\pi\mathrm{i})^s}. \tag{4}$$

The presence of the Vandemonde determinant times a single product in the integrand implies that GEFP can be rewritten as a determinant.

2.3. Determinant representation. Before bringing (4) in the form of a determinant, it is useful, to reduce to minimum subsequent calculations, to make a change of the integration variables $z_j \mapsto x_j = (1-\alpha+\alpha z_j)/z_j$. This yields

$$G_{N,s}^{(r_1,\ldots,r_s)} = (-1)^{\frac{s(s-1)}{2}} \prod_{j=1}^{s} (1-\alpha)^{N-r_j} \oint\limits_{C_\infty} \cdots \oint\limits_{C_\infty} \prod_{1\leqslant j<k\leqslant s} (x_k - x_j)$$
$$\times \prod_{j=1}^{s} \frac{x_j^{N-j}}{(x_j-\alpha)^{N-r_j}(x_j-1)^{s-j+1}} \frac{\mathrm{d}x_1\cdots\mathrm{d}x_s}{(2\pi\mathrm{i})^s}, \quad (5)$$

where C_∞ denotes a closed contour of large radius, counterclockwise oriented around the origin. Using

$$\prod_{1\leqslant j<k\leqslant s} (x_k - x_j) = (-1)^{\frac{s(s-1)}{2}} \det \left[x_j^{s-k}\right]_{j,k=1,\ldots,s},$$

one can readily rewrite (5) as a determinant

$$G_{N,s}^{(r_1,\ldots,r_s)} = \prod_{j=1}^{s} (1-\alpha)^{N-r_j} \times \det_{1\leqslant j,k\leqslant s} \left[P_{N-s+j+k-2}^{(N-r_{s-k+1},k)}\right], \quad (6)$$

where we have used the following notation:

$$P_l^{(j,k)} = \oint\limits_{C_\infty} \frac{x^l}{(x-\alpha)^j(x-1)^k} \frac{\mathrm{d}x}{2\pi\mathrm{i}}. \quad (7)$$

The integral in (7) can be evaluated, using

$$P_l^{(j,k)} = \frac{1}{(j-1)!(k-1)!} \partial_\alpha^{j-1} \partial_\beta^{k-1} \oint\limits_{C_\infty} \frac{x^l}{(x-\alpha)(x-\beta)} \frac{\mathrm{d}x}{2\pi\mathrm{i}} \bigg|_{\beta=1}$$
$$= \frac{1}{(j-1)!(k-1)!} \partial_\alpha^{j-1} \partial_\beta^{k-1} \frac{\alpha^l - \beta^l}{\alpha-\beta} \bigg|_{\beta=1}. \quad (8)$$

It turns out that (8) appears rather convenient for calculations discussed below, though one can also obtain more explicit expressions like

$$P_l^{(j,k)} = \sum_{n=0}^{l-j-k+1} \binom{j-1+n}{j-1} \binom{l-j-n}{k-1} \alpha^n,$$

or

$$P_l^{(j,k)} = \binom{l-j}{k-1} {}_2F_1 \left(\begin{matrix} -l+j+k-1,\, j \\ -l+j \end{matrix} \bigg| \alpha \right).$$

These expressions show that $P_l^{(j,k)}$ is in fact a polynomial of degree $l-j-k+1$ in α.

§3. The case of square domain

3.1. Explicit formulas. The special case where λ has a rectangular shape corresponds to the usual emptiness formation probability (EFP) [11]. In turn, its important particular case is where λ has the form of a square. This case is very interesting since EFP has rather simple explicit form, while still represent a nontrivial correlation function of the model.

It is convenient to set $N = r + s$, so that $\lambda = (s, \dots, s)$, that is $r_j = r$, $j = 1, \dots, s$. For the EFP we introduce a short notation

$$f_s := G_{r+s,s}^{(r,\dots,r)}.$$

Equation (6) implies

$$f_s = (1-\alpha)^{s^2} \det_{1\leqslant j,k\leqslant s} \left[P_{r+j+k-2}^{(s,k)} \right],$$

so that, taking into account (8), we get

$$\begin{aligned} f_s = {} & \frac{(1-\alpha)^{s^2}}{[(s-1)!]^s \prod_{k=0}^{s-1} k!} \\ & \times \det_{1\leqslant j,k\leqslant s} \left[\partial_\alpha^{s-1} \partial_\beta^{k-1} \frac{\alpha^{r+j+k-2} - \beta^{r+j+k-2}}{\alpha - \beta} \bigg|_{\beta=1} \right]. \end{aligned} \tag{9}$$

With the help of equation (9) one proves that

$$f_s \equiv f_s(\alpha) \equiv 0 \quad \text{for} \quad r < s.$$

and find explicit expressions for EFP for some small values of s, namely,

$$\begin{aligned} f_0 &= 1, \\ f_1 &= 1 - \alpha^r, \\ f_2 &= 1 - r^2\alpha^{r-1} + 2(r^2-1)\alpha^r - r^2\alpha^{r+1} + \alpha^{2r}, \\ f_3 &= 1 - \frac{r^2(r-1)^2}{4}(\alpha^{r-2} - \alpha^{2r+2}) + r^2(r-2)(r+1)(\alpha^{r-1} - \alpha^{2r+1}) \\ &\quad - \frac{3(r^2-1)(r^2-2)}{2}(\alpha^r - \alpha^{2r}) + r^2(r+2)(r-1)(\alpha^{r+1} - \alpha^{2r-1}) \\ &\quad - \frac{r^2(r+1)^2}{4}\left(\alpha^{r+2} - \alpha^{2r-2}\right) - \alpha^{3r}. \end{aligned}$$

As a function of α, the EFP $f_s = f_s(\alpha)$ has the following structure

$$f_s = (1-\alpha)^{s^2}\widetilde{f}_s(\alpha).$$

where $\widetilde{f}_s(\alpha)$ is a self-reciprocal (palindromic) polynomial of the degree $(r-s)s$, whose coefficients are positive integers, and

$$\widetilde{f}_s(0) = 1, \qquad r \geqslant s.$$

In particular,

$$\widetilde{f}_1 = \sum_{m=0}^{r-1} \alpha^m,$$

$$\widetilde{f}_2 = \sum_{m=0}^{2r-2} \binom{r+1-|r-m-2|}{3}\alpha^m,$$

$$\widetilde{f}_3 = \sum_{m=0}^{r-1} \binom{m+8}{8}(\alpha^{3r-3-m}+\alpha^m) + \sum_{m=r-2}^{\left[\frac{3}{2}(r-3)\right]} p_m\left(\alpha^{3(r-3)-m}+\alpha^m\right),$$

$$p_{r-2} = \binom{r+6}{8} - \frac{r^2(r-1)^2}{4}, \quad p_{r-1} = \binom{r+7}{8} - \frac{r^2(5r^2-14r+17)}{4}, \ldots$$

$$p_{r+k} = \binom{r+k+8}{8} - \frac{r^2}{4}\left(\left(r^2 - \frac{2}{5}(2k+9)r\right)\binom{k+6}{4} + C_k\right) - 3\binom{k+8}{8},$$

$$C_k = \binom{k+6}{4}\frac{2k^2+18k+33}{5},$$

$$m = r+k, \quad k = -2, -1, 0, 1, \ldots, \left[\frac{3}{2}(r-3)\right] - r.$$

Here $[\,\cdot\,]$ denotes the integer part of the corresponding number, and

$$\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0. \end{cases}$$

The polynomial $\widetilde{f}_3$ has the following intriguing property: for odd r it is irreducible over $\mathbb{Z}$. For odd r apart with the "trivial" divisor $(1+\alpha)$ it is divisible by exactly two self-reciprocal polynomials with coefficients in $\mathbb{Z}_+$

of the degrees

$$6\left[\frac{r+2}{4}\right]-6, \qquad 6\left[\frac{r}{4}\right]-4.$$

For example,

$$\begin{aligned}
\widetilde{f}_3\big|_{r=4} &= (\alpha+1)(\alpha^2+8\alpha+1),\\
\widetilde{f}_3\big|_{r=5} &= \alpha^6+9\alpha^5+45\alpha^4+65\alpha^3+45\alpha^2+9\alpha+1,\\
\widetilde{f}_3\big|_{r=6} &= (\alpha+1)(\alpha^2+5\alpha+1)(\alpha^6+3\alpha^5+21\alpha^4+20\alpha^3+21\alpha^2+3\alpha+1),\\
\widetilde{f}_3\big|_{r=7} &= \alpha^{12}+1+9(\alpha^{11}+\alpha)+45(\alpha^{10}+\alpha^2)+165(\alpha^9+\alpha^3)\\
&\quad +495(\alpha^8+\alpha^4)+846(\alpha^7+\alpha^5)+994\alpha^6,\\
\widetilde{f}_3\big|_{r=8} &= (\alpha+1)(\alpha^6+6\alpha^5+21\alpha^4+28\alpha^3+21\alpha^2+6\alpha+1)\\
&\quad \times(\alpha^8+2\alpha^7+4\alpha^6+34\alpha^5+2\alpha^4+34\alpha^3+4\alpha^2+2\alpha+1).
\end{aligned}$$

We finish this subsection by noting, that for $r = s$ the polynomial $\widetilde{f}_s(\alpha)$ is equal to 1, that implies the following relation for the EFP:

$$f_s\big|_{r=s} = (1-\alpha)^{s^2}.$$

If $r = s + 1$, then one can find that

$$\widetilde{f}_s\big|_{r=s+1} = \sum_{m=0}^{s}\binom{s}{m}^2\alpha^m.$$

This last relation is, in fact, a particular case of a more general equivalent representation for the EFP in terms of random matrix model, considered below.

3.2. Equivalent representations. The function f_s has an interesting property: its second logarithmic derivative with respect to the variable $\log\alpha$ is expressed in terms of square of some polynomial in α, namely,

$$f_s^2\partial_{\log\alpha}^2\log f_s = f_s(\alpha\partial_\alpha)^2 f_s - (\alpha\partial_\alpha f_s)^2 = -r^2\alpha^{r-s}w_s^2,$$

where w_s is a self-reciprocal polynomial of the degree $(s-1)(r+1)$. The first few polynomials read:

$$\begin{aligned}
w_1 &= 1,\\
w_2 &= (1+\alpha)(1-\alpha^r) - r(1-\alpha)(1+\alpha^r),\\
w_3 &= \binom{r-1}{2}(1+\alpha^{2r+2}) - (r^2-1)(\alpha+\alpha^{2r+1}) + \binom{r+1}{2}(\alpha^2+\alpha^{2r})
\end{aligned}$$

$$+\frac{1}{4}r^2(r^2-1)(1+\alpha^4)\alpha^{r-1}-(r^4-3r^2+2)(1+\alpha^2)\alpha^r$$
$$+\frac{1}{2}\left(3r^4-11r^2-4\right)\alpha^{r+1}.$$

These polynomials satisfy the initial condition $w_s(0)=\binom{r-1}{s-1}$. It is fairly easy to see that these polynomials possess the property

$$f_s^2-\frac{r^2(1-\alpha)^2\alpha^{r-s-1}}{s^2}w_s^2=f_{s+1}f_{s-1},$$

which implies that the EFP satisfies the equation

$$f_s^2(\alpha\partial_\alpha)^2\log f_s=\frac{s^2\alpha}{(1-\alpha)^2}(f_{s+1}f_{s-1}-f_s^2). \tag{10}$$

One more relation of the similar kind is

$$f_s^2-(1-\alpha)^2\alpha^{r-s-1}w_s^2=f_s[r+1]f_s[r-1],\qquad f_s[r\pm 1]\equiv f_s\big|_{r\mapsto r\pm1}.$$

This implies the following equation

$$f_s^2(\alpha\partial_\alpha)^2\log f_s=\frac{r^2\alpha}{(1-\alpha)^2}(f_s[r+1]f_s[r-1]-f_s^2). \tag{11}$$

The equations (10) and (11) are, in fact, equations of the Toda hierarchy. For example, the equation (10) is closely related with the fact that the EFP can be represented as the following Hankel determinant:

$$f_s=\frac{1}{\prod_{j=0}^{s-1}(j!)^2}\frac{(1-\alpha)^{s^2}}{\alpha^{\frac{s(s-1)}{2}}}\det_{1\leqslant j,k\leqslant s}\left[(\alpha\partial_\alpha)^{j+k-2}P_r^{(1,1)}\right]. \tag{12}$$

This representation can be obtained by making use various linear relations satisfied by the polynomials $P_l^{(j,k)}$; for full details see [13, Sect. 2].

Finally, using $P_r^{(1,1)}=(1-\alpha^r)/(1-\alpha)=\sum_{m=0}^{r-1}\alpha^m$ one can notice that (12) can be written as a multiple sum:

$$f_s=\frac{1}{s!\prod_{j=0}^{s-1}(j!)^2}\frac{(1-\alpha)^{s^2}}{\alpha^{\frac{s(s-1)}{2}}}\sum_{m_1,\dots,m_s=0}^{r-1}\prod_{1\leqslant j<k\leqslant s}(m_j-m_k)^2\alpha^{m_1+\dots+m_s}. \tag{13}$$

This formula represents EFP as a discrete random matrix model or a as model of discrete Coulomb gas. Its peculiar is the presence of two hard walls in addition to the constraint of discreteness, which both are responsible for appearance of the third-order phase transition [14].

3.3. Relation with the Painlevé VI. We consider here the Painlevé VI in its σ-form [22–24]. This is the following nonlinear ordinary differential equation:

$$(1-\alpha)^2\alpha^2(\sigma'')^2 - 4\Big[(1-\alpha)\sigma' + \sigma\Big]\Big[\alpha\sigma' - \sigma\Big]\sigma' \\ + 2s_0\Big[(1-2\alpha)\sigma' + 2\sigma\Big] - s_1(\sigma')^2 - s_2\sigma' - s_3 = 0 \quad (14)$$

here and below the prime denotes derivative with respect to α, and $s_0, \dots, s_3$ are constants, which are expressed in terms of the usual Painlevé VI monodromy parameters $\nu_1, \dots, \nu_4$ as follows:

$$\begin{aligned} s_0 &= \nu_1\nu_2\nu_3\nu_4, \\ s_1 &= \nu_1^2 + \nu_2^2 + \nu_3^2 + \nu_4^2, \\ s_2 &= \nu_1^2\nu_2^2 + \nu_1^2\nu_3^2 + \nu_1^2\nu_4^2 + \nu_2^2\nu_3^2 + \nu_2^2\nu_4^2 + \nu_3^2\nu_4^2, \\ s_3 &= \nu_1^2\nu_2^2\nu_3^2 + \nu_1^2\nu_2^2\nu_4^2 + \nu_1^2\nu_3^2\nu_4^2 + \nu_2^2\nu_3^2\nu_4^2. \end{aligned}$$

It is known [22] that function σ is related with the Hamiltonian function (H_{VI}) of the Painlevé VI, $\sigma = \alpha(\alpha-1)H_{\mathrm{VI}}$, and the corresponding τ-function,

$$\sigma = \alpha(\alpha-1)\frac{\tau'}{\tau} + B\alpha + C, \quad (15)$$

where B and C are some normalizing constants which can be explicitly expressed in terms of the parameters $\nu_1, \dots, \nu_4$.

The EFP is related to the Painlevé VI upon setting

$$\tau = f_s, \qquad B = -\frac{(r+s)^2}{4}, \qquad C = \frac{rs}{2}, \quad (16)$$

and

$$\nu_1 = \nu_3 = -\frac{r+s}{2}, \qquad \nu_2 = -\nu_4 = -\frac{r-s}{2}.$$

This statement is particular case of a more general result proven in [16] (see Th. 1.1 therein).

Here, as a complement to the derivation provided in [16], it seems useful to give a hint how the connection of the EFP with the Painlevé VI can be deduced from (12). Indeed, the essential factor here, which can be identified with the τ-function, is the Hankel determinant. To see that this is indeed the case, let us introduce new variable $t = \frac{\alpha}{\alpha-1}$. Then

$$\alpha\partial_\alpha = -t(t-1)\partial_t,$$

and

$$P_r^{(1,1)} = (1-t)^{r-1} {}_2F_1(-r+1, -r, -r+1; t),$$

where ${}_2F_1$ is the Gauss hypergeometric function. These formulas show that the determinant in (12) is, in fact, a τ-function of Painlevé VI describing its classical solutions [23, 24]. Since the change of the variables $\alpha \mapsto t$ is a symmetry of Painlevé VI, the same holds true with respect the variable α for the determinant in (12), or, equivalently, for the whole expression in (12).

§4. The case of trianglar domain

4.1. Explicit formulas. Another interesting case is where λ has a triangular shape. Again, we set $N = r + s$, and then we choose $\lambda = (s, s-1, \dots, 2, 1)$, that is $r_j = r + j - 1$, $j = 1, \dots, s$. For this triangular domain EFP (TDEFP) we introduce a short notation

$$g_s := G_{r+s,s}^{(r,r+1,\dots,r+s-1)}.$$

Equation (6) implies

$$g_s = (1-\alpha)^{\frac{s(s+1)}{2}} \det_{1 \leqslant j,k \leqslant s} \left[P_{r+j+k-2}^{(k,k)} \right],$$

and from (8) we get

$$g_s = \frac{(1-\alpha)^{\frac{s(s+1)}{2}}}{\prod_{k=0}^{s-1} (k!)^2} \det_{1 \leqslant j,k \leqslant s} \left[\partial_\alpha^{k-1} \partial_\beta^{k-1} \frac{\alpha^{r+j+k-2} - \beta^{r+j+k-2}}{\alpha - \beta} \bigg|_{\beta=1} \right]. \quad (17)$$

By using (17) one can find explicit expressions for TDEFP for some small values of s, namely,

$$\begin{aligned}
g_1 &= f_1 = 1 - \alpha^r, \\
g_2 &= 1 - (2r+1)(1-\alpha)\alpha^r - \alpha^{2r+1}, \\
g_3 &= 1 - (r+1)(2r+1)(\alpha^r + \alpha^{2r+3}) + (2r+1)(2r+3)(\alpha^{r+1} + \alpha^{2r+2}) \\
&\quad - (r+1)(2r+3)\left(\alpha^{r+2} + \alpha^{2r+1}\right) + \alpha^{3r+3}, \\
g_4 &= 1 - \binom{2r+3}{3} \left(\alpha^r + \alpha^{3r+6}\right) \\
&\quad + (r+2)(2r+1)(2r+3)\left(\alpha^{r+1} + \alpha^{3r+5}\right) \\
&\quad - (r+1)(2r+3)(2r+5)\left(\alpha^{r+2} + \alpha^{3r+4}\right)
\end{aligned}$$

$$+\binom{2r+5}{3}\left(\alpha^{r+3}+\alpha^{3r+3}\right)-\frac{2r+3}{2}\binom{2r+4}{3}\left(1+\alpha^4\right)\alpha^{2r+1}$$
$$+\frac{(2r+1)(2r+3)^2(2r+5)}{3}\left(1+\alpha^2\right)\alpha^{2r+2}+\alpha^{4r+6}.$$

As a function of α, the TDEFP $g_s=g_s(\alpha)$ has the following structure

$$g_s=(1-\alpha)^{\frac{s(s+1)}{2}}\widetilde{g}_s(\alpha), \qquad \widetilde{g}_s(0)=1, \tag{18}$$

where $\widetilde{g}_s(\alpha)$ is a self-reciprocal polynomial of the degree $s(r-1)$, with positive integer coefficients. In particular,

$$\widetilde{g}_1=\widetilde{f}_1=\sum_{m=0}^{r-1}\alpha^m,$$
$$\widetilde{g}_2=\sum_{m=0}^{2r-2}\binom{r+1-|r-m-1|}{2}\alpha^m,$$
$$\widetilde{g}_3=\sum_{m=0}^{r-1}\binom{m+5}{5}(\alpha^{3r-3-m}+\alpha^m)+\sum_{m=r}^{2\left[\frac{3}{2}(r-1)\right]+2}\frac{1+\operatorname{sgn}\left(\frac{3}{2}(r-1)-m\right)}{2}$$
$$\times\left\{\binom{m+5}{5}-\frac{(2r+1)(5r+4-m)}{4}\binom{m-r+3}{3}-\binom{m-r+3}{5}\right\}$$
$$\times\left(\alpha^{3r-3-m}+\alpha^m\right).$$

Note that for r even, the polynomial $\widetilde{g}_3$ is divisible by $1+\alpha$.

For $r=1$ the polynomial $\widetilde{g}_s$ is equal to 1, that implies the following relation for the TDEFP:

$$g_s\big|_{r=1}=(1-\alpha)^{\frac{s(s+1)}{2}}.$$

In the case of $r=2$, by studying various values of s we observe that

$$\widetilde{g}_s\big|_{r=2}=\sum_{m=0}^{s}\frac{s+1}{(s-m+1)(m+1)}\binom{s}{m}^2\alpha^m. \tag{19}$$

This formula can be rewritten in terms of the Gegenbauer polynomials,

$$\widetilde{g}_s\big|_{r=2}=\frac{2(1-\alpha)^{\frac{1}{2}s(s+3)}}{(s+1)(s+2)}C_s^{3/2}\left(\frac{1+\alpha}{1-\alpha}\right),$$

where $C_s^{3/2}(\cdot)$ is the Gegenbauer polynomial of degree s [21].

4.2. Equivalent representations. Even though the explicit expressions for functions g_s look even simpler than those for f_s, it seems that functions g_s do not possess any simple analogues of the representations listed in Sect. 3.2 for functions f_s.

There is, however, one exception which worth to be mentioned here, since it is in fact provides a representation for the TDEFP which reminds very much a discrete random matrix model (though, strictly speaking, it is not). Namely, by investigating explicit expressions for functions $\widetilde{g}_s$, related to g_s modulo factor $(1-\alpha)^{s(s+1)/2}$, see (18), we discover, for $s = 2, 3, 4, 5$, that

$$\widetilde{g}_s = \sum_{0\leqslant m_1\leqslant\ldots\leqslant m_s\leqslant r-1} \left(\prod_{1\leqslant j<k\leqslant s} \frac{2m_k - 2m_j + k - j}{k-j} \right) \alpha^{m_1+\ldots+m_s}. \quad (20)$$

We also mention here that (20) is agreement with (19) in the case of $r = 2$, which have been verified for values of $s = 3, 4, 5$, against direct counting of configurations of the model contributing to the TDEFP.

We note that in (20) the summation is running over weakly increasing sequences. We also note that the sum cannot be put in a fully symmetric form, namely, with the summand invariant under permutations of the summation variables, like in (13). For this reason, formula (20) is not a usual random matrix model representation, but it is fact an analogue of such for the EFP (13).

Definitely, the structure of the product in (20) strongly suggests that this representation could be derived from (17) by making use known relations for the Schur polynomials [20]. However, because of peculiar structure of the matrix in (17), the proof seems to be not straightforward, and we leave this problem for a separate publication.

4.3. Relation with the Painlevé VI. We find that the TDEFP can be related to the Painlevé VI is several special cases of the discrete parameters. Besides the trivial case of $s = 1$, in which $g_1 = f_1$, and therefore (16) applies, we find that the relation can also be provided for $s = 2$, r is arbitrary, and $r = 2$, s is arbitrary; the case of $s = r = 2$ admits even more special treatment.

Case $s = 2$, r is arbitrary. In this case the TDEFP is given by

$$g_2 = (1-\alpha)^3 \sum_{m=0}^{2r-2} \binom{r+1-|r-m-1|}{2} \alpha^m.$$

As it can be directly verified, for arbitrary values of $r \geqslant 3$ (with the exception of the case of $r = 2$ considered separately below), the σ-form of Painlevé VI (14) cannot be fulfilled with any choice of parameters B, C and $s_0, \dots, s_3$. Nevertheless, let us consider the first-order derivative of this function,

$$g_2' = -(2r+1)(1-\alpha)^2\alpha^{r-1}\sum_{m=0}^{r-1}(r-m)\alpha^m.$$

Since the sum here is in fact a truncating Gauss hypergeometric series, one might expect that it can serve as the τ-function for Painlevé VI.

Indeed, by direct inspection for various values of $r = 3, 4, 5, 6, \dots$, we have found that (14) is satisfied with

$$\tau = g_2', \qquad B = -\frac{(r+1)^2+4}{4}, \qquad C = \frac{r+3}{4},$$

and

$$\{\nu_1, \dots, \nu_4\} = \left\{-\frac{r-1}{2}, \frac{r-1}{2}, \frac{r+1}{2}, \frac{r+3}{2}\right\},$$

where the values of the parameters $\nu_1, \dots, \nu_4$ can be rearranged arbitrarily (below we use the increasing order).

Case $r = 2$, s is arbitrary. In this case, as it follows from (18) and (19),

$$g_s\big|_{r=2} = (1-\alpha)^{\frac{s(s+1)}{2}}\sum_{m=0}^{s}\frac{s+1}{(s-m+1)(m+1)}\binom{s}{m}^2\alpha^m. \tag{21}$$

The σ-form of Painlevé VI (14) is fulfilled with the following choice in (15):

$$\tau = g\big|_{r=2}, \qquad B = -\binom{s+2}{2}, \qquad C = \binom{s+2}{2} - \frac{1}{2},$$

and we have also set

$$\nu_1 = 0, \quad \nu_2 = 1, \quad \nu_3 = s+1, \quad \nu_4 = s+2.$$

The function (21) can be also presented in various ways in terms of finite hypergeometric series. For example,

$$g_s\big|_{r=2} = (1-\alpha)^{\frac{1}{2}s(s+1)}\left(1 + \frac{(s+1)s}{2}\,\alpha\;{}_3F_2(1, -s, 1-s; 2, 3; \alpha)\right).$$

Using such representations one can find the following asymptotics:

$$\sigma \underset{\alpha\to\infty}{=} -\alpha + \frac{1}{2} + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{a_k}{\alpha^k},$$

$$\sigma \underset{\alpha\to 0}{=} C + B\alpha + \alpha\sum_{k=1}^{\infty}(-1)^{k+1}a_k\alpha^k,$$

where a_k $k = 1, 2, \ldots$ is the following sequence of positive integers:

$$\begin{aligned}
a_1 &= \frac{1}{12}s(s+3)(s+2)(s+1),\\
a_2 &= \frac{1}{48}s^2(s+1)(s+2)(s+3)^2,\\
a_3 &= \frac{a_2}{15}(4s^2+12s-1),\\
a_4 &= \frac{a_2}{180}(13s^4+78s^3+109s^2-24s+4),\\
a_5 &= \frac{a_2}{1680}(3s^2+9s-2)(11s^4+66s^3+95s^2-12s+8).
\end{aligned}$$

Case $s = r = 2$. In the case of $s = r = 2$, the TDEFP has the following form

$$g_2\big|_{r=2} = (1-\alpha)^3(\alpha^2+3\alpha+1).$$

It can be easily seen that σ-form of Painlevé VI (14) is fulfilled with the following choice in (15)

$$\tau = g_2\big|_{r=2}, \qquad B = -6, \qquad C = \frac{11}{2},$$

and

$$\nu_1 = 0, \quad \nu_2 = 1, \quad \nu_3 = 3, \quad \nu_4 = 4.$$

Explicitly, the σ-function in this case has the form

$$\sigma = \frac{11}{2} - \alpha - \alpha\frac{5(1+\alpha)}{1+3\alpha+\alpha^2}.$$

Here, the rational function given by the ratio in the last term, which is obviously invariant under the change $\alpha \mapsto 1/\alpha$, has the following $\alpha \to 0$ expansion

$$\frac{5(1+\alpha)}{1+3\alpha+\alpha^2} = 5 - 10\alpha + 25\alpha^2 - 65\alpha^3 + 170\alpha^4 - 445\alpha^5 + 1165\alpha^6 + O(\alpha^7).$$

The integer sequence

$$5,\ 10,\ 25,\ 65,\ 170,\ 445,\ 1165,\ 3050,\ 7985, \ldots,$$

can be found in OEIS [25]. These numbers are the sums of squares of the Lucas and Fibonacci numbers,

$$\begin{aligned} L(n)^2 + L(n+1)^2 &= 5(F(n)^2 + F(n+1)^2) \\ &= F(n-2)^2 + (n+3)^2, \qquad n \in \mathbb{Z}_+. \end{aligned}$$

We recall that the Lucas numbers are defined as

$$L(0) = 2, \quad L(1) = 1, \quad L(n+1) = L(n) + L(n-1),$$

and they form the sequence $\{2, 1, 3, 4, 7, 11, \ldots\}$; the Fibonacci numbers are defined as

$$F(-1) = 1, \quad F(0) = 0, \quad F(n+1) = F(n) + F(n-1),$$

and they form the sequence $\{1, 0, 1, 1, 2, 3, 5, \ldots\}$.

As a comment to this intriguing combinatorial interpretation in the $s = r = 2$ case, we mention that the similar sequences arising in $\alpha \to 0$ (or $\alpha \to \infty$) expansions for the σ-functions discussed above, even if they are all given purely in terms of integers, seem to have no obvious combinatorial treatment.

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