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N. Bogoliubov

CONTINUOUS TIME MULTIDIMENSIONAL WALKS AS AN INTEGRABLE MODEL

Abstract. Continuous time walks in multidimensional symplectical lattices are considered. It is shown that the generating functions of random walks and the transition amplitudes of continuous time quantum walks are expressed through the dynamical correlation functions of the exactly solvable model describing strongly correlated bosons on a chain, the so-called phase model. The number of random lattice paths of fixed number of steps connecting the starting and ending points of the multidimensional lattice is expressed through the solutions of Bethe equations of the phase model. Its asymptotic is obtained in the limit of the large number of steps.

§1. Introduction

Classical random walks on a one-dimensional lattice have been intensively investigated for many years in mathematics and physics [1–6]. Quantum walks are unitary processes that describe the quantum-mechanical analogue of the classical random walk process. Two mathematical models for quantum walks have been developed: the discrete-time quantum walks [7], in which the particle takes discrete steps in a direction given by a dynamic internal degree of freedom (a coin), and the continuous-time quantum walks [8], in which the dynamics are described by Hamiltonian evolution on a lattice in the tight-binding representation. The recent interest to the theory of quantum walks arose in connection with the development of quantum information processing [9–13]. Random and quantum walks on multi-dimensional lattices were studied by many authors [14–19].

In our paper we shall consider the exactly solvable model, known as *phase model*, describing strongly correlated bosons on a chain with D nodes [20,21]. The dynamical variables of this model are the so-called phase operators introduced in quantum optics in connection with the quantum

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phase problem [22]. We shall demonstrate that this model provides a natural description of the continuous time quantum and random single-particle walks on $(D-1)$-dimensional symplectic lattices with vanishing boundary conditions. The asymptotic of the number of lattice random paths in the limit of sufficiently large number of steps is calculated.

The paper is organised as follows. In Section 2 the random walks in a $(D-1)$-dimensional symplectic lattice lattice with vanishing boundary conditions are introduced. In Section 3 the phase model is considered, and the answers for generating functions of "continuous time" random and quantum walks are expressed in terms of the solutions of Bethe equations.

§2. Random walks

We consider a graph embedded in the Euclidean space and the associated random walk. The random walk takes place on the nodes (vertices) of the graph and jumps along the edges with the equal weight.

Consider first a particle (walker) that hops at discrete time-steps between neighboring sites on a one-dimensional chain with unit spacing. Let $G_n(j,l)$ be the number of random paths made by a particle in n steps with the ending nodes j and l. The evolution of this function is described by the master equation

$$G_n(j,l) = G_{n-1}(j-1,l) + G_{n-1}(j+1,l)\,. \tag{1}$$

This states that the number of paths made by a particle from the node l to j in n steps is simply the sum of paths made by a particle from the node l to $j-1$ (contribution of a step from the left) and to $j+1$ (step from the right) in $n-1$ steps see Fig. 1. The initial condition $G_0(j,l) = \delta_{jl}$. The vanishing boundary conditions are $G_n(j,l) = 0$ for $j,l = -1, N+1$ for a random walk on a segment $[0,N]$.

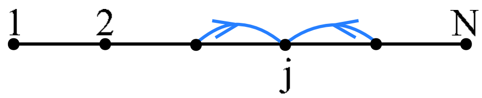

Fig. 1. The hopping processes for the one-dimensional nearest-neighbor random walk.

For the particle that hops between neighboring sites on a triangular lattice (Fig. 2) the difference equation for the function $G_{jl;pq}(n)$ equal to a number of random paths from the node with the coordinates l_1, l_2 to the node j_1, j_2 in n steps takes the form

$$\begin{aligned} G_n(j_1,j_2;l_1,l_2) = G_{n-1}(j_1+1,j_2-1;l_1,l_2) + G_{n-1}(j_1-1,j_2+1;l_1,l_2) \\ + G_{n-1}(j_1+1,j_2;l_1,l_2) + G_{n-1}(j_1-1,j_2;l_1,l_2) \\ + G_{n-1}(j_1,j_2+1;l_1,l_2) + G_{n-1}(j_1,j_2-1;l_1,l_2)\,. \end{aligned} \tag{2}$$

The initial condition $G_0(j_1,j_2;l_1,l_2) = \delta_{j_1 l_1}\delta_{j_2 l_2}$, and the vanishing boundary conditions are

$$\begin{aligned} G_n(-1,j_2;l_1,l_2) = G_n(j_1,-1;l_1,l_2) = G_n(j_1,N+1-j_1;l_1,l_2) \\ = G_n(N+1-j_2,j_2;l_1,l_2) = 0. \end{aligned}$$

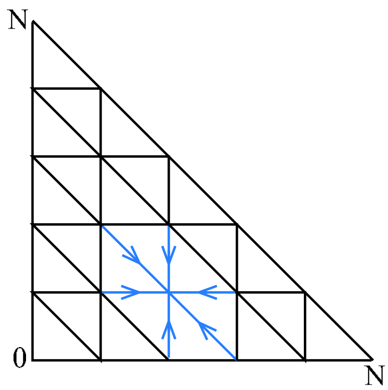

Fig. 2. The hopping processes on the triangular lattice with the vanishing boundary conditions.

The random walks on the introduced lattices with the vanishing boundary conditions are special cases of random walks in the symplectic lattice. Consider a D-dimensional integer lattice $\mathbf{n} \equiv (n_1, n_2, \ldots, n_D) \in \mathbb{Z}^D$, lying in the non-negative orthant $\mathbb{N}_0^D$: $0 \leqslant n_i$ and satisfying the condition $\sum n_i = N$.

Random walks over sites of the *symplectic lattice* $\mathrm{Symp}_{(N)}(\mathbb{Z}^D)$ are defined by a set of admissible steps Ω_D so that at each step an i^{th} coordinate n_i increases by unity, while the nearest neighboring one decreases by unity. Namely, each element of Ω_D is given by sequence $(e_1, e_2, \ldots, e_D)$

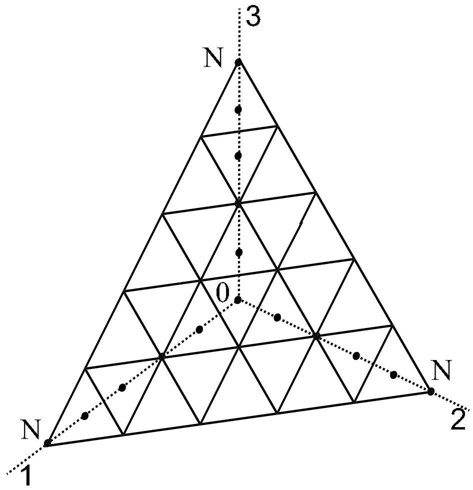

Fig. 3. A two-dimensional triangular simplicial lattice.

so that $e_i = \pm 1$, $e_{i+1} = \mp 1$ for all pairs $(i, i+1)$ with $1 \leqslant i \leqslant D$ and $D+1 = 1(\bmod 2)$, and $e_j = 0$ for all $1 \leqslant j \leqslant D$ and $j \neq i, i+1$. The step set $\Omega_D \equiv \Omega_D(\mathbf{m}_0)$ ensures that trajectory of a random walk determined by the starting point $\mathbf{m}_0$ lies in D-dimensional set $\mathrm{Symp}_{(N)}(\mathbb{Z}^D)$. A two-dimensional triangular simplicial lattice is presented in Fig. 3

The described walks lie on a bounded domain and the behaviour of the walker depends on a boundary conditions. If at least one of the coordinates $n_i = 0$, a particle is said to be on a boundary of a symplectic lattice. The boundary of the simplicial lattice consists of D faces of highest dimensionality $D-2$. Under the free boundary conditions we shall understand the conditions under which the walker happened to be on a border continues to move according the step-sets Ω_D. The random lattice path of a particle on a two-dimensional symplectic lattice is presented in Fig. 4.

The coordinates of the particle $\mathbf{n} \equiv (n_1, n_2, \ldots, n_D)$, $n_1 + n_2 + \ldots + n_D = N$ in the symplectic lattice $\mathrm{Symp}_{(N)}(\mathbb{Z}^D)$ may be expressed as partitions

$$\boldsymbol{\lambda} = (D^{n_D}, (D-1)^{n_{D-1}}, \ldots, 1^{n_1}),$$

where each number S appears n_S times, and hence as Young diagrams. There are N rows in diagram, the row of S squares corresponds to number S and appears n^s times (see Fig. 5).

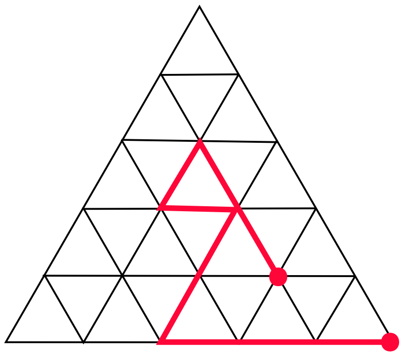

Fig. 4. The path of 9 steps between nodes with coordinates $(0,5,0)$ and $(1,3,1)$.

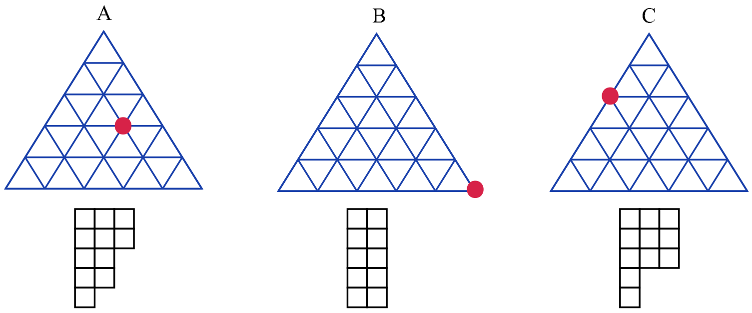

Fig. 5. Walkers with coordinates $(1,2,2) \to \boldsymbol{\lambda} = (3^2, 2^2, 1^1)$ (A), $(0,5,0) \to \boldsymbol{\lambda} = (3^0, 2^5, 1^0)$ (B), and $(2,0,3) \to \boldsymbol{\lambda} = (3^3, 2^0, 1^2)$ (C) on $\mathrm{Symp}_{(5)}(\mathbb{Z}^3)$, and corresponding Young diagrams.

§3. Quantum integrable model

To study random walks in a general $(D-1)$-dimensional symplectic lattice we shall consider the quantum exactly solvable phase model [20, 21]. The dynamical variables of the model are operators of the "exponential phase" $\phi_n, \phi_n^\dagger$ and the number operator N_j satisfying commutation relations:

$$[N_i, \phi_j] = -\phi_i \delta_{ij} \, , \; [N_i, \phi_j^\dagger] = \phi_i^\dagger \delta_{ij} \, , \; [\phi_i, \phi_j^\dagger] = \pi_i \delta_{ij} \, , \qquad (3)$$

where π_j is the local vacuum projector $\phi_j\pi_j = \pi_j\phi_j^\dagger = 0$. On a local orthonormal Fock states $|n_j\rangle_j$ $({}_k\langle n_m|n_i\rangle_j = \delta_{im}\delta_{kj})$:

$$\begin{aligned} &\phi_j|0\rangle_j = 0, \\ &\phi_j|n_j\rangle_j = |n_j-1\rangle_j, \;\; \phi_j^\dagger|n_j\rangle_j = |n_j+1\rangle_j, \\ &N_j|n_j\rangle_j = n_j|n_j\rangle_j, \;\; \pi_j|0\rangle_j = |0\rangle_j\,. \end{aligned} \tag{4}$$

The introduced phase operators are the $q\to\infty$ limit of the q-boson algebra formed by operators $B_j, B_j^\dagger$ and N_j :

$$[N_i, B_j] = -B_i\delta_{ij}\,,\; [N_i, B_j^\dagger] = B_i^\dagger\delta_{ij}\,,\; [B_i, B_j^\dagger] = q^{-2N_j}\delta_{ij}\,. \tag{5}$$

In the local Fock space the phase operators may be represented as

$$\phi_j = \sum_{n_j=0}^{\infty}(|n_j\rangle\langle n_j+1|)_j,\; \phi_j^\dagger = \sum_{n_j=0}^{\infty}(|n_j+1\rangle\langle n_j|)_j,\; N_j = \sum_{n_j=0}^{\infty} n_j(|n_j\rangle\langle n_j|)_j.$$

The integrable phase model on a one-dimensional lattice of $D+1$ nodes is defined by the Hamiltonian [20]:

$$\widehat{H}_D = \sum_{n=1}^{D}(\phi_n^\dagger\phi_{n+1} + \phi_n\phi_{n+1}^\dagger)\,. \tag{6}$$

The periodic boundary conditions are imposed: $D+1 = 0\,(\mathrm{mod}\,2)$. This Hamiltonian commutes with the total number operator $\widehat{N} = \sum\limits_{n=0}^{D} N_n$: $[\widehat{H}_D, \widehat{N}] = 0$. It acts on the Fock space formed from the normalized states

$$|n_0,\ldots,n_D\rangle = \prod_{j=1}^{D}\left(\phi_j^\dagger\right)^{n_j}|0\rangle;\; 0\leqslant n_j\leqslant N,\; \sum_{j=1}^{D} n_j = N,$$

and $|0\rangle$ is given by equation $|0\rangle = |0,\ldots,0\rangle \equiv \prod_{j=1}^{D}|0\rangle_j$.

We can interpret occupation numbers $\mathbf{n} = (n_1, n_2, \ldots, n_D)$ as coordinates of a particle in $\mathrm{Symp}_{(N)}(\mathbb{Z}^D)$ and describe the dynamics of the particle with the help of Fock vectors $|n_1, n_2, \ldots, n_D\rangle$. Operator ϕ_j shifts the value of the jth coordinate downwards $n_j\to n_j-1$, while $\phi_j^\dagger$ upwards $n_j\to n_j+1$. The number operator N_j acts as the coordinate operator $N_j|n_1, n_2, \ldots, n_j, \ldots, n_D\rangle = n_j|n_1, n_2, \ldots, n_j, \ldots, n_D\rangle$. Since the occupation numbers are non-negative integers and their sum is conserved we can regard $\widehat{H}_D$ as a generator of steps of a particle on a $(D-1)$-dimensional symplectic lattice with the reflective boundary conditions.

The number of random paths made by a particle in K steps in $\mathrm{Symp}_{(N)}(\mathbb{Z}^D)$ between the nodes with coordinates $\mathbf{j}$ and $\mathbf{l}$ is given by the expression:

$$\begin{aligned} G_K(j_1, j_2, \dots, j_D; l_1, l_2, \dots, l_D) \\ = \langle j_1, j_2, \dots, j_D | \widehat{H}^k | l_1, l_2, \dots, l_D \rangle ; \\ j_1 + j_2 + \dots + j_D = l_1 + l_2 + \dots + l_D = N . \end{aligned} \tag{7}$$

The random walks of the particle on a segment $[0, N]$ (see Fig. 6) is generated by the Hamiltonian

$$\widehat{H}_1 = \left(\phi_1^\dagger \phi_2 + \phi_1 \phi_2^\dagger \right) .$$

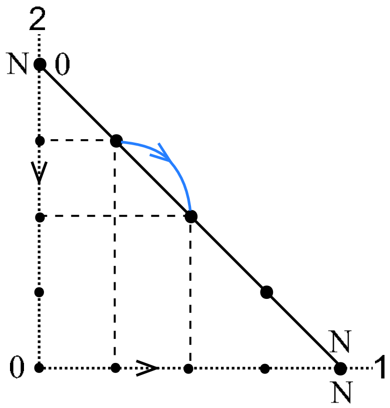

Fig. 6. The hopping processes for the one-dimensional nearest-neighbor random walk on a segment [0,N].

The equation for the number of random paths function (1) is derived in the following way

$$\begin{aligned} G_n(j_1; l_2) = \langle j_1; j_2 | \left(\phi_1^\dagger \phi_2 + \phi_1 \phi_2^\dagger \right) H_1^{n-1} | l_1; l_2 \rangle \\ = \{ \langle j_1 - 1, N - j_1 + 1 | + \langle j_1 + 1, N - j_1 - 1) | \} \, H_1^{n-1} | l_1, N - l_1 \rangle \\ = \{ G_{n-1}(j_1 - 1; l_1) + G_{n-1}(j_1 + 1; l_1) \} . \end{aligned} \tag{8}$$

Random walks of a particle on two-dimensional simplex like lattice (a triangular lattice) (see Fig. 4) is generated by the Hamiltonian

$$\widehat{H}_2 = \left(\phi_1^\dagger\phi_2 + \phi_2^\dagger\phi_3 + \phi_3^\dagger\phi_1 + \phi_1\phi_2^\dagger + \phi_2\phi_3^\dagger + \phi_3\phi_1^\dagger\right) .$$

Substituting this expression into (7) we obtain equation (2).

The model was solved for its eigenstates and eigenvectors in [20,21] by the Quantum Inverse Method [23,24]. The state vectors of the phase model are expressed in the form [25,26]:

$$|\Psi_N(u_1,\dots,u_N)\rangle = \sum_{\lambda\subseteq\{D^N\}} S_{\boldsymbol\lambda}(u_1^2,\dots,u_N^2)(\phi_D^\dagger)^{n_D}\dots(\phi_1^\dagger)^{n_2}(\phi_0^\dagger)^{n_1}|0\rangle\,, \quad (9)$$

where the Schur function

$$S_{\boldsymbol\lambda}(x_1,x_2,...,x_N) = \frac{\det\left(x_j^{N-i+\lambda_i}\right)}{\mathcal{V}(x)} .$$

Here $\boldsymbol\lambda$ denotes the partition $(\lambda_1,...,\lambda_N)$ of non-increasing non-negative integers, $D\geqslant\lambda_1\geqslant\lambda_2\geqslant...\geqslant\lambda_N\geqslant 0$, and the sum in equation is taken over all partitions λ into at most N parts each of which is less or equal to D. The Vandermonde determinant $\mathcal{V}(x) = \prod_{1\leqslant i<j\leqslant N}(x_i - x_j)$. There is a one-to-one correspondence between the configuration of occupation numbers $\mathbf{n} = (n_1,n_2,\dots,n_D)$, $0\leqslant n_1,n_2,\dots,n_D\leqslant N$; $n_1+n_2+\dots+n_D = N$, and the partition $\lambda = (D^{n_D},(D-1)^{n_{D-1}},\dots,1^{n_1})$, where each number S appears n_S times.

The state vectors are the eigenvectors of the Hamiltonian $\widehat{H}_D$, if the parameters $u_1,\dots,u_N$ satisfy the Bethe equations

$$u_n^{2(D+N)} = (-1)^{N-1}\prod_{j=1}^N u_j^2 \equiv (-1)^{N-1}U^2. \quad (10)$$

Using the parametrization $u^2 = \exp(-ip)$ in which p appears as a momentum variable, the Bethe equations take the form

$$e^{ip_n(D+N)} = (-1)^{N-1}e^{iP}, \quad P = \sum_{j=1}^N p_j, \quad (11)$$

where P is the total momentum of the system. These equations are invariant under the transformation $p_n\to p_n+2\pi$, $P\to P+2\pi$, and we can thus restrict the consideration into the domain $-\pi\leqslant p_n<\pi$. The

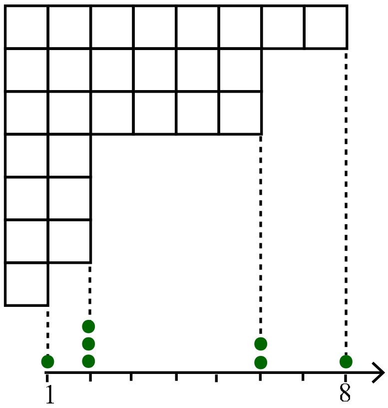

Fig. 7. The configuration of occupation numbers $\mathbf{n} = (1, 3, 0, 0, 0, 2, 0, 1)$ corresponding partition $\boldsymbol{\lambda} = (8^1, 7^0, 6^2, 5^0, 4^0, 3^0, 2^3, 1^1) \equiv (8, 6, 6, 2, 2, 2, 1)$ and its Young diagram.

solutions p_n to the Bethe equations (11) are real numbers and can be parameterized so that $p_n = (2\pi I_n + P)/(D + N)$. Here I_n are integers or half-integers depending on whether N is odd or even, and satisfy the condition $D + N - 1 \geqslant I_1 > I_2 > ... > I_N \geqslant 0$. From (11) it follows in particular that $e^{iPD} = 1$.

The eigenenergies of the Hamiltonian are given by

$$E_N = 2\sum_{k=1}^{N} \cos p_k. \tag{12}$$

Consider now the process in which the discreet number of steps is replaced by a continuous parameter, which will be called "time". We define the "time"-dependent correlation function that describes a continuous "time" walks of a particle in a $\mathrm{Symp}_{(N)}(\mathbb{Z}^D)$ as

$$\begin{aligned} \mathcal{F}_\tau(j_1, j_2, \ldots, j_D; l_1, l_2, \ldots, l_D) = \langle j_1, j_2, \ldots, j_D | e^{\tau \widehat{H}_D} | l_1, l_2, \ldots, l_D \rangle\,; \\ j_1 + j_2 + \ldots + j_D = l_1 + l_2 + \ldots + l_D = N\,. \end{aligned} \tag{13}$$

- If $\tau = t$, then $\mathcal{F}_t$ is the generating function of random lattice walks between the nodes with coordinates $(j_1,j_2,...,j_D)$ and $(l_1,l_2,...,l_D)$.

- If $\tau = it$, then $\mathcal{F}_{it} \equiv F_t$ is a transition amplitude of *continuous time quantum walker* from state $|l_1, l_2, \dots, l_D\rangle$ at time 0 to state $|j_1, j_2, \dots, j_D\rangle$ at time t.

Since eigenvectors of the phase model (9) form a complete orthogonal set [21] we obtain the explicit expression for the correlation function:

$$\begin{aligned}
&\mathcal{F}_\tau(j_1, j_2, \dots, j_D; l_1, l_2, \dots, l_D) \\
&= \sum_{\{p_1,\dots,p_N\}} \frac{e^{\tau E_N}}{\mathcal{N}^2(e^{ip_1}, \dots, e^{ip_N})} S_{\boldsymbol{\lambda}^L}(e^{ip_1}, \dots, e^{ip_N}) S_{\boldsymbol{\lambda}^R}(e^{-ip_1}, \dots, e^{-ip_N}).
\end{aligned} \tag{14}$$

The summation is over all independent sets of the solutions of Bethe equations. The partition $\lambda^L = \left(D^{j_D}, (D-1)^{j_{D-1}}, \dots, 1^{j_1}\right)$ where each number S appears j_S times, and $j_1 + j_2 + \dots + j_D = N$. The partition $\lambda^R = \left(D^{l_D}, (D-1)^{l_{D-1}}, \dots, 1^{l_1}, 0^{l_0}\right)$ and $l_1 + l_2 + \dots + l_D = N$. The norm is equal to [21]:

$$\begin{aligned}
\mathcal{N}^2(e^{ip_1}, \dots, e^{ip_N}) &= \sum_{\lambda \subseteq \{D^N\}} S_{\boldsymbol{\lambda}}(e^{ip_1}, \dots, e^{ip_N}) S_{\boldsymbol{\lambda}}(e^{-ip_1}, \dots, e^{-ip_N}) \\
&= \frac{D(D+N)^{N-1}}{\prod_{1 \leqslant k < j \leqslant N} |e^{ip_k} - e^{ip_j}|^2}\,.
\end{aligned}$$

Note that continuous time is not a simple continuum formed by the discrete variables n, but $\mathcal{F}_\tau$ includes processes with all number of steps. In fact $\mathcal{F}_t$ can be represented in a form

$$\mathcal{F}_\tau(j_1, j_2, \dots, j_D; l_1, l_2, \dots, l_D) = \sum_{m=0}^{\infty} G_m(j_1, j_2, \dots, j_D; l_1, l_2, \dots, l_D) \frac{\tau^m}{m!}\,,$$

where G_m is (7).

Let us consider the case when the walker in $\mathrm{Symp}_{(N)}(\mathbb{Z}^D)$ starts at the node $(N, 0, \dots, 0)$ and finishes the walk at the same node. The corresponding partition of this node is $\boldsymbol{\lambda}_1 = (D^0, \dots, 2^0, 1^N)$ and $S_{\boldsymbol{\lambda}_1}(e^{ip_1}, \dots, e^{ip_N}) = e^{iPD} = 1$. From (14) it follows that the generating function of these walks is:

$$
\begin{aligned}
&\mathcal{F}_\tau(N,0,\dots,0;N,0,\dots,0)\\
&\qquad=\sum_{\{p_1,\dots,p_N\}}\frac{e^{\tau E_N}}{\mathcal{N}^2(e^{ip_1},\dots,e^{ip_N})}\left|S_{\boldsymbol{\lambda}_1}(e^{ip_1},\dots,e^{ip_N})\right|^2\\
&\quad=\frac{1}{D(D+N)^{N-1}}\sum_{\{p_1,\dots,p_N\}}e^{2\tau\sum_{k=1}^N\cos p_k}\prod_{1\leqslant k<j\leqslant N}|e^{ip_k}-e^{ip_j}|^2\,. \qquad (15)
\end{aligned}
$$

The number of random paths made by a particle in K steps in $\mathrm{Symp}_{(N)}(\mathbb{Z}^D)$ (7) which starts at the node $(N,0,\dots,0)$ and finishes the walk at the same node is:

$$
\begin{aligned}
&G_K(N,0,\dots,0;N,0,\dots,0)\\
&=\frac{2^K}{D(D+N)^{N-1}}\sum_{\{p_1,\dots,p_N\}}\left(\sum_{k=1}^N\cos p_k\right)^K\prod_{1\leqslant k<j\leqslant N}|e^{ip_k}-e^{ip_j}|^2\,. \qquad (16)
\end{aligned}
$$

It is of interest to obtain approximate expression of (16) in the limit when K and N are sufficiently large, and $K\gg N$. In this limit, the solutions (11) with sufficiently large N fill a closed interval of length 2π. Hence, a passage from sums to integrals in (16) yields the estimate:

$$
\begin{aligned}
&G_K(N,0,\dots,0;N,0,\dots,0)\\
&\simeq\frac{2^KN}{D}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\dots\int_{-\pi}^{\pi}\left(\sum_{k=1}^N\cos p_k\right)^K\prod_{1\leqslant k<j\leqslant N}|e^{ip_k}-e^{ip_j}|^2\;\frac{dp_1dp_2\dots dp_N}{(2\pi)^N}. \qquad (17)
\end{aligned}
$$

The main contribution to the integrals in this expression comes from near the points $p_j=0$ $(j=1,\dots,\dots N)$ [28], and, therefore, we can replace the first factor of the integrand in (16) by its approximation near the origin of integration variables:

$$
\left(\sum_{k=1}^N\cos p_k\right)^K\propto N^K\exp\left(-\frac{K}{2N}\sum_{k=1}^N p_k^2\right).
$$

The leading term of (16) can be expressed as

$$G_K(N,0,\dots,0;N,0,\dots,0)$$
$$\sim \frac{2^K N^{K+1}}{D}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\cdots\int\limits_{-\infty}^{\infty} e^{-\frac{K}{2N}\sum_{k=1}^{N}p_k^2}\prod_{1\leqslant k<j\leqslant N}(p_k-p_j)^2\,\frac{dp_1dp_2\dots dp_N}{(2\pi)^N}.$$

The integral is the Mehta integral of the gaussian unitary ensemble of random matrices and can be evaluated explicitly:

$$\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\cdots\int\limits_{-\infty}^{\infty} e^{-\frac{K}{2N}\sum_{k=1}^{N}p_k^2}\prod_{1\leqslant k<j\leqslant N}(p_k-p_j)^2\,\frac{dp_1dp_2\dots dp_N}{(2\pi)^N}$$
$$= \frac{N^{N^2/2}\prod_{m=1}^{N}m!}{(2\pi)^{N/2}K^{N^2/2}}\,.$$

Finally, we obtain that the leading term of G_K (16) in the considered limit scales as

$$G_K(N,0,\dots,0;N,0,\dots,0)\sim \mathcal{A}_{N,D}(2N)^K K^{-N^2/2} \tag{18}$$

with

$$\mathcal{A}_{N,D}=\frac{N^{\frac{N^2}{2}+1}\prod_{m=1}^{N}m!}{(2\pi)^{\frac{N}{2}}D}\,. \tag{19}$$

The asymptotical behaviour (18) is similar to the scaling properties of the random-turns version of vicious walkers [4]. The application of the phase model to the description of random walkers was discussed in [30,31].

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St.-Petersburg Department
of Steklov Institute of Mathematics, RAS
Fontanka 27, St.-Petersburg;
ITMO University
Kronverksky 49, St.-Petersburg, Russia
E-mail: bogoliub@yahoo.com

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