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BIRATIONAL DARBOUX COORDINATES ON NILPOTENT COADJOINT ORBITS CLASSICAL COMPLEX LIE GROUPS, JORDAN BLOCKS 2×2

Abstract. A problem of the constructing of the birational Darboux coordinates on the nilpotent coadjoint orbits of the complex Lie groups $SO(N, \mathbb{C})$ and $Sp(N, \mathbb{C})$ is considered. The nilpotent case is the most difficult case of the orbits. The difficulties arise if the Jordan blocks of the different parities of the sizes present in the Jordan form of the matrices from the orbit. The desired coordinates has been found on the orbits consisting of the matrices with the Jordan blocks of the sizes one and two. The explicit formulae for the coordinates are presented.

§1. Introduction. Notations

Any coadjoint orbit of a Lie group equipped with the canonical Lie-Poison–Kirillov–Kostant two-form is the symplectic manifold. Classical Darboux theorem states that there are such coordinates that the form has the canonical form $\sum\limits_k dp_k \wedge dq_k$, but it is non-trivial problem to find such coordinates.

The rational Darboux coordinates on the orbits of the classical Lie groups were constructed in [2]. The only restriction was made for the Jordan type of the orbits for the orthogonal and the symplectic groups: there should not be Jordan blocks in the zero root-space. We consider the complicated case of the nilpotent orbits now, and present the canonical coordinates for the case of the presence of 2×2 Jordan blocks, in other words we consider the case of matrices from the algebras $\mathrm{so}(N, \mathbb{C})$ and $\mathrm{sp}(N, \mathbb{C})$ satisfying $A^2 = 0$.

From the beginning we consider the orthogonal and the symplectic cases simultaneously. The groups $\mathrm{SO}(N)$ and $\mathrm{Sp}(N)$ preserve the scalar product $\langle \cdots, \cdots \rangle$ on $\mathbb{C}^N$ that is either symmetric or skew-symmetric.

Key words and phrases: coadjoint orbit, classical Lie groups, Lie–Poisson–Kirillov–Kostant form, symplectic fibration, projection-flag coordinates.

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Let us numerate basic vectors $\mathbf{e_i}$ symmetrically, index i belongs to a set $\pm[N/2], \pm([N/2]-1), \dots, \pm 1$ and the zero value for odd N.

Definition 1. *A basis is called **standard** if its Gram matrix*[1] *g is*

$$\begin{pmatrix} 0 & 0 & \tau \\ 0 & 1 & 0 \\ \tau & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \tau \\ -\tau & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & \tau \\ \tau & 0 \end{pmatrix},$$

where τ is a square anti-diagonal matrix, consisting of units. It is the matrix of the inversion:

$$\tau = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

For the orthogonal groups $g^2 = \mathrm{I}$, for the symplectic groups $g^2 = -\mathrm{I}$, in any cases $g^T = g^{-1}$.

Our ground field is $\mathbb{C}$, consequently there are non-zero isotropic vectors $\langle \xi, \xi \rangle = 0$ even in the orthogonal case.

Definition 2. *The subspace L is called **isotropic**, if it consists of the isotropic vectors: $\xi \in L \Rightarrow \langle \xi, \xi \rangle = 0$.*

If a standard basis has given, an example of isotropic space is a coordinate subspace enveloping several coordinate vectors with the indexes of the same sign.

An *orthogonal complement* $L^\perp$ to the space L is called a set of all vectors orthogonal to all vectors of L:

$$\eta \in L^\perp \Leftrightarrow \langle \eta, \xi \rangle = 0 \ \ \forall \xi \in L.$$

An orthogonal complement to a subspace is a subspace too. For the non-zero isotropic L, $L \subset L^\perp \neq V$, consequently $L + L^\perp = L^\perp \neq V$. Nevertheless some usual identities take place

- $(L^\perp)^\perp = L$,
- $\dim L + \dim L^\perp = \dim V$.

[1] A Gram matrix of the set of vectors $f_1, f_2, \dots$ is a matrix of their pairwise products $g_{ij} := \langle f_i, f_j \rangle$.

There is a simple orthogonal transformation that transforms a given *isotropic* subspace to the coordinate subspace of the same dimension. The transformation has the triangular form and can be easily factorized on the product of two projections parallel to the coordinate subspaces of the standard basis:

$$\begin{pmatrix} \mathrm{I} & 0 & 0 \\ \star & \mathrm{I} & 0 \\ \star & \star & \mathrm{I} \end{pmatrix} = \begin{pmatrix} \mathrm{I} & 0 & 0 \\ \star & \mathrm{I} & 0 \\ \star & 0 & \mathrm{I} \end{pmatrix} \begin{pmatrix} \mathrm{I} & 0 & 0 \\ 0 & \mathrm{I} & 0 \\ 0 & \star & \mathrm{I} \end{pmatrix}.$$

An eigenspace (the kernel) is not isotropic in the case of the orthogonal or symplectic nilpotent matrix. *It is the sours of all the difficulties in the case.* Nevertheless there are isotropic subspaces of the kernel, namely the intersections of the kernel and the images of the powers of the matrix. Always the image and the kernel are the orthogonal complements of each other. In our case $A^2 = 0$, the kernel contains the image, so the isotropic subspace in question is just the image. It simplifies the formulae and makes possible to tolwe the problem.

Let us split the standard basis on three sets of vectors in accordance with the dimension of the image of A:

$$\{\mathbf{e}\} = \{\mathbf{e}_-, \mathbf{e}_\mathbf{0}, \mathbf{e}_+\},$$

the number of vectors in $\mathbf{e}_\pm$ is equal to the dimension of the image of A, it is the number of the Jordan blocks 2×2. The number of vectors in $\mathbf{e}_\mathbf{0}$ is the number of eigenvectors without the generalized eigenvectors.

If the first set of the basic vectors forms the image of A (the image is contained in the kernel), and the second set of the basic vectors compleats the kernel, the matrix from the orbit takes the form $\begin{pmatrix} 0 & 0 & \rho \\ 0 & 0 & \widetilde{\rho} \\ 0 & 0 & 0 \end{pmatrix}$, where ρ and $\widetilde{\rho}$ are some matrix blocks.

Let us denote a matrix of the projection of the the first set $\mathbf{e}_-$ of the basic vectors on the image of A parallel to $\mathbf{e}_\mathbf{0} \cup \mathbf{e}_-$ by $\begin{pmatrix} \mathrm{I} & 0 & 0 \\ q & \mathrm{I} & 0 \\ q_\Box & 0 & I \end{pmatrix}$, and denote the matrix of the projection of the set $\mathbf{e}_\mathbf{0}$ on the kernel of A parallel to $\mathbf{e}_+$ by $\begin{pmatrix} \mathrm{I} & 0 & 0 \\ 0 & \mathrm{I} & 0 \\ 0 & \widetilde{q} & I \end{pmatrix}$.

We get a representation

$$A = \begin{pmatrix} \mathrm{I} & 0 & 0 \\ q & \mathrm{I} & 0 \\ q_\Box & \widetilde{q} & I \end{pmatrix} \begin{pmatrix} 0 & 0 & \rho \\ 0 & 0 & \widetilde{\rho} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{I} & 0 & 0 \\ q & \mathrm{I} & 0 \\ q_\Box & \widetilde{q} & I \end{pmatrix}^{-1}$$

of A from the algebraically open subset of the orbit.

Let us denote the transposing with respect to the antidiagonal by the superscript "$\vdash$": $A^\vdash = \tau A^T \tau$, $(A^\vdash)_{ij} = (A^\vdash)_{-j-i}$.

Conditions that f belongs to the group and A belongs to the algebra are $f^T g f = g$ and $A^T g + gA = 0$. In the standard basis $g = \widetilde{I}\tau$, where $\widetilde{\mathrm{I}} = \mathrm{I}$ for the orthogonal group, and $\widetilde{\mathrm{I}} = \mathrm{diag}(\mathrm{I}, -\mathrm{I})$ for the symplectic group. The conditions can be rewritten as

$$\widetilde{\mathrm{I}} f^\vdash \widetilde{\mathrm{I}} f = I, \qquad A^\vdash = -\widetilde{\mathrm{I}} A \widetilde{\mathrm{I}},$$

in the standard basis.

From this symmetry follows that $\widetilde{\rho} = 0$, consequently

$$A = \begin{pmatrix} \mathrm{I} & 0 & 0 \\ q & \mathrm{I} & 0 \\ q_\Box & \widetilde{q} & I \end{pmatrix} \begin{pmatrix} 0 & 0 & \rho \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{I} & 0 & 0 \\ q & \mathrm{I} & 0 \\ q_\Box & \widetilde{q} & I \end{pmatrix}^{-1} .$$

Let us treat $\widetilde{\mathrm{I}}$ as symbol of variable size, like unit matrix I. So in our formulae *square matrices* $\widetilde{\mathrm{I}}, \mathrm{I}$ *have the sizes that are necessary for the present situation.*

The factor $\begin{pmatrix} \mathrm{I} & 0 & 0 \\ q & \mathrm{I} & 0 \\ q_\Box & \widetilde{q} & I \end{pmatrix}$ belongs to the group, it implies $\widetilde{q} = -q^\vdash \widetilde{\mathrm{I}}^\vdash$, and $q_\Box \pm q_\Box^\vdash \pm q^\vdash \widetilde{\mathrm{I}} q = 0$. The simple verifications shows that these are the only restrictions on the matrix-value parameters $q, q_\Box, \rho$:

$$\rho^\vdash \pm \rho = 0, \qquad q_\Box \pm q_\Box^\vdash = \mp q^\vdash \widetilde{\mathrm{I}} q,$$

where the upper sign corresponds to the orthogonal groups and the lower sign corresponds to the symplectic groups. In other words q is arbitrary, ρ is arbitrary skew-symmetric (or symmetric), $q_\Box$ has arbitrary skew-symmetric (or symmetric) part and its symmetric (skew-symmetric) part is uniquely defined by q.

§2. Calculation of the symplectic form.

The Lie–Poisson–Kirillov–Kostant forms ω_{so} and ω_{sp} on the orbits in $\mathrm{so}(N,\mathbb{C})$ and $\mathrm{sp}(N,\mathbb{C})$ are the contractions of the form ω_{gl} living on the corresponding orbit in $\mathrm{gl}(N,\mathbb{C})$, the ω_{so} and ω_{sp} -orbits are the submanifolds of the orbit of the general linear group, consequently we can use the formula for $\mathrm{gl}(N,\mathbb{C})$ from [1] for the calculation in $\mathrm{so}(N,\mathbb{C})$ and $\mathrm{sp}(N,\mathbb{C})$.

The calculation gives:

$$A = \begin{pmatrix} \mathrm{I} & 0 & 0 \\ q & \mathrm{I} & 0 \\ q_\square & 0 & \mathrm{I} \end{pmatrix} \begin{pmatrix} 0 & -\rho\widetilde{q} & \rho \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{I} & 0 & 0 \\ q & \mathrm{I} & 0 \\ q_\square & 0 & \mathrm{I} \end{pmatrix}^{-1}, \quad \widetilde{q} = -q^{\vdash}\widetilde{\mathrm{I}}^{\vdash},$$

consequently for both types of groups

$$\omega = \mathrm{tr}\, d(\rho q^{\vdash}) \wedge \widetilde{\mathrm{I}}^{\vdash} dq + \mathrm{tr}\, d\rho \wedge dq_\square .$$

It is not the finite answer because elements of the matrices in the formula are not independent. The further consideration depends on the parity of the number of rows of q, it is the dimension of $\mathbf{e_0}$. In the case of the symplectic group $\mathrm{Sp}(N,\mathbb{C})$ this number is always even, it is a simple case that we consider in the afterword. The same method can be applied to the orbit of the orthogonal group too, but in that partial case when the number of the eigenvectors without the generalized eigenvectors is even. Otherwise we should apply the general formulae from the next section.

§3. Orbits of the orthogonal group.

From now let all indexes takes the natural values and numerates rows and columns of our blocks in a usual way. Let ρ be nondegenerated antisymmetrical $2n \times 2n$ matrix: $\rho^{\vdash} = -\rho$ and $\widetilde{\mathrm{I}} = \mathrm{diag}(\mathrm{I}, -\mathrm{I})$ be constant antisymmetrical matrix[2]. We need a following version of the Lagrangian method of the diagonalization of the quadratic form by the triangular transformation.

Lemma 1. *Matrix ρ can be rationally factorized to the product $\rho = \sigma\widetilde{\mathrm{I}}\sigma^{\vdash}$ on the algebraically open subset of the set of antisymmetrical matrices.*

[2]We consider the symmetry with respect to the antidiagonal.

Matrix σ has the form

$$\sigma = \begin{pmatrix} \begin{matrix} c_1 & 0 \\ b_{2,1} & c_2 \end{matrix} & \cdots & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \cdots & \begin{matrix} 0 & 0 \\ 0 & a_{2,2n} \end{matrix} \\ \vdots & \ddots & \cdots & \ddots & \vdots \\ b_{ij} & \cdots & \begin{matrix} c_n & 0 \\ 0 & 1 \end{matrix} & \cdots & a_{k,l} \\ \vdots & \ddots & \cdots & \ddots & \vdots \\ \begin{matrix} b_{2n-1,1} & 0 \\ 0 & 0 \end{matrix} & \cdots & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \cdots & \begin{matrix} 1 & a_{2n-1,2n} \\ 0 & 1 \end{matrix} \end{pmatrix}$$

The indexes i, j satisfy $i > j, i + j < 2n + 1$, and indexes k, l satisfy $i < j, i + j > 2n + 1$. Functions $a_{i,j}(\rho), b_{k,l}(\rho), c_m(\rho)$ are $2n(2n-1)/2$ free rational parameters of an arbitrary antisymmetrical $2n \times 2n$ matrix ρ.

Proof. Let us denote the values "on the boundary" of ρ in the following way:

$$\rho = \begin{pmatrix} c & ca & 0 \\ b & \widetilde{\rho} & -a^{\vdash}c \\ 0 & -b^{\vdash} & -c \end{pmatrix}, \quad c \in \mathbb{C}, \quad b \in \mathbb{C}^{2n-2}, \quad a \in (\mathbb{C}^{2n-2})^{\vdash},$$

it can be done on the open subset $\rho_{1,1} =: c \neq 0$. The multiplication gives

$$\begin{pmatrix} c & ca & 0 \\ b & \widetilde{\rho} & -a^{\vdash}c \\ 0 & -b^{\vdash} & -c \end{pmatrix} = \begin{pmatrix} c & 0 & 0 \\ b & \mathrm{I} & a^{\vdash} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ b & \rho_1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & a & 0 \\ 0 & \mathrm{I} & 0 \\ 0 & b^{\vdash} & c \end{pmatrix},$$

where $\rho_1 = \widetilde{\rho} + a^{\vdash}b^{\vdash} - ba$ is a new antisymmetrical matrix of the smaller size. The process can be iterated. □

Lemma 2. *For any matrices A, B, C with the proper sizes*

$$\mathrm{tr}\, d(AB) \wedge dC = \mathrm{tr}\, dA \wedge d(CB) + \mathrm{tr}\, dB \wedge d(CA).$$

Proof. Let us rewrite the summand from the left-hand side in coordinates:

$$\begin{aligned} d(A_{ij}B_{jk}) \wedge dC_{ki} &= dA_{ij} \wedge B_{jk}dC_{ki} + dB_{jk} \wedge (dC_{ki})A_{ij} \\ &= dA_{ij} \wedge d(B_{jk}C_{ki}) + dB_{jk} \wedge d(C_{ki}A_{ij}) \\ &- (dA_{ij} \wedge (dB_{jk})C_{ki} + dB_{jk} \wedge C_{ki}dA_{ij}). \end{aligned}$$

The last bracket is zero because $df \wedge gdh = g(df \wedge dh) = -dh \wedge gdf$. □

For $\mathrm{SO}(N,\mathbb{C})$ the symplectic form is equal $\operatorname{tr} d(\rho q^{\vdash}) \wedge dq + \operatorname{tr} d\rho \wedge dq_{\square}$. The substitution of $\rho = \sigma\widetilde{\mathrm{I}}\sigma^{\vdash}$ and transformation using the derived formula gives

$$\omega = \operatorname{tr} d(\sigma\widetilde{\mathrm{I}}\sigma^{\vdash} q^{\vdash}) \wedge dq + \operatorname{tr} d\sigma\widetilde{\mathrm{I}}\sigma^{\vdash} \wedge dq_{\square}$$
$$= \operatorname{tr} \widetilde{\mathrm{I}} dq_1^{\vdash} \wedge dq_1 + \operatorname{tr} d\sigma \wedge \widetilde{\mathrm{I}}\sigma^{\vdash} \left(q^{\vdash} q + q_{\boxtimes}\right),$$

where $q_1 := q\sigma$, $q_{\boxtimes} := q_{\square} - q_{\square}^{\vdash}$.

Let us consider the summand $\operatorname{tr} \widetilde{\mathrm{I}} dq_1^{\vdash} \wedge dq_1$. Matrix q (and q_1 too) has an even number of columns $2n$ because orthogonal matrix has even number of the Jordan blocks of the even sizes, let us split q_1 on two halves $(q_-, q_+) := q_1 = q\sigma$. We can transform the summand:

$$\operatorname{tr} \widetilde{\mathrm{I}} dq_1^{\vdash} \wedge dq_1 = \operatorname{tr} d \begin{pmatrix} \mathrm{I} & 0 \\ 0 & -\mathrm{I} \end{pmatrix} \begin{pmatrix} q_+^{\vdash} \\ q_-^{\vdash} \end{pmatrix} \wedge d\,(q_-, q_+) = 2 \operatorname{tr} dq_+^{\vdash} \wedge dq_-$$

We can see that *the canonical coordinates on the orbit are the corresponding couples of the matrix elements* $(q_+^{\vdash})_{ij}$ *and* $(q_-)_{ji}$ *and the couples of the non-trivial matrix elements of* σ *and the corresponding matrix elements of* $\widetilde{\mathrm{I}}\sigma^{\vdash}\left(q^{\vdash} q + q_{\square} - q_{\square}^{\vdash}\right)$.

§4. Orbits of the symplectic group.

In the symplectic case the formula for ω can be transformed much easier. Matrix q has an even number of rows, we split it on two halves: $q =: \begin{pmatrix} q_- \\ q_+ \end{pmatrix}$, it gives: $\operatorname{tr} d(\rho q^{\vdash}) \wedge \widetilde{\mathrm{I}}^{\vdash} dq = \operatorname{tr} d(2\rho q_+^{\vdash}) \wedge dq_- - \operatorname{tr} d\rho \wedge d(q_-^{\vdash} q_+)$. The formula for ω becomes $\operatorname{tr} d(2\rho q_+^{\vdash}) \wedge dq_+ - \operatorname{tr} d\rho \wedge d(q_{\square} - q_-^{\vdash} q_+)$, and finitely

$$\omega = \operatorname{tr} d(2\rho q_+^{\vdash}) \wedge dq_+ + \operatorname{tr} d\rho \wedge d(q_{\square} + q_{\square}^{\vdash} - q_-^{\vdash} q_+ - q_+^{\vdash} q_-)/2.$$

We symmetrized matrix $q_{\square} - q_-^{\vdash} q_+$ because ρ is symmetric.

Finitely *the canonical coordinates for the orbit of* $\mathrm{Sp}(N,\mathbb{C})$ *are the couples* $(2\rho q_+^{\vdash})_{ij}$, $(q_+)_{ji}$, *and the couples* $(\rho)_{ij}$, $(q_{\square} + q_{\square}^{\vdash} - q_-^{\vdash} q_+ - q_+^{\vdash} q_-)_{ji}$ *for* $i+j < 2n+1$ *and the anti-diagonal entries* $(\rho)_{i,2n+1-i}$, $(q_{\square} - q_-^{\vdash} q_+)_{i,2n+1-i}$ *for* $i = 1, 2n$.

References

1. M. V. Babich, *Birational Darboux Coordinates on (Co)Adjoint Orbits of*$GL(N,\mathbb{C})$. — Functional Analysis and Its Applications, **50**, No. 1 (2016), 17–30.
2. M. V. Babich, *On birational Darboux coordinates on coadjoint orbits of classical complex Lie groups* — Zap. Nauchn. Semin. POMI, **432** (2015), 36–57. ftp://ftp.pdmi.ras.ru/pub/publicat/znsl/v432/p068.pdf
3. M. V. Babich *Young tableaux and stratification of space of complex square matrices.* — Zap. Nauchn. Semin. POMI, **433** (2015), 41–64. ftp://ftp.pdmi.ras.ru/pub/publicat/znsl/v433/p041.pdf
4. S. E. Derkachov, A. N. Manashov, *R-matrix and Baxter Q-operators for the noncompact SL(N;C) invariant spin chain SIGMA* 2. 084 arXiv:nlin.SI/0612003, (2006).
5. M. V. Babich, S. E. Derkachov, *On rational symplectic parametrization of coajoint orbit of* $GL(N)$. *Diagonalizable case.* — Algebra and Analiz, **22**, No. 3 (2010), 16–30.
6. M. V. Babich *About Coordinates on the Phase Spaces of the Schlesinger System and the Garnier-Painlevé 6 System.* — Doklady Mathematics, **75**, No. 1 (2007), 71.
7. A. G. Reiman, M. A. Semenov-Tyan-Shanskii, *Integrable systems. Group-theoretic approach.* — Sovremennaya Matematika, Inst. for Computer Research, Moscow–Izhevsk, (2003).
8. I. M. Gelfand, M. I. Najmark , *Unitary representation of classical groups.* — Zap. Nauchn. Semin. POMI, **36** (1950).

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