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ON THE DUAL COMPLEXITY AND SPECTRA OF SOME COMBINATORIAL FUNCTIONS

Abstract. In a recent paper, A. M. Vershik and the author started the study of representation-theoretic aspects of well-known combinatorial functions on the symmetric groups $\mathfrak{S}_n$. The note presents a series of further results in this direction.

§1. Introduction

In [9], A. M. Vershik and the author started the project concerned with the study of representation-theoretic aspects of combinatorial functions on the symmetric groups $\mathfrak{S}_n$. The idea is, given a combinatorial statistic a (i.e., a function on $\mathfrak{S}_n$, or an element of the group algebra $\mathbb{C}[\mathfrak{S}_n]$), to study the representation of $\mathfrak{S}_n$ which is the restriction of the left regular representation to the left ideal $\mathbb{C}[G]a$ generated by a. In particular, we considered the notion of the *dual complexity* of a, originally suggested by A. M. Vershik, which is the dimension of this representation. The statistics considered in [9] are the major index, the descent number, and the inversion number of a permutation. It turned out that each of them generates the same ideal, and the corresponding representation of $\mathfrak{S}_n$ is isomorphic to its representation in the space of $n\times n$ skew-symmetric matrices, which allowed us to obtain formulas for the functions under consideration in terms of matrices of an exceptionally simple form, which, in turn, were applied to find their spectra in the regular representation, as well as to deduce a series of identities relating them to one another and to the number of fixed points. In this note, we consider a series of further examples. First, these are the so-called m-pattern and consecutive m-pattern functionals (see Definitions 2 and 3), which essentially count the number of occurrences of subsequences of length m with certain order structures in permutations (regarded as words in the alphabet $1,2,\dots,n$). The study of such functionals (from the combinatoric point of view) goes back to Knuth [5]; for a comprehensive survey on classical and generalized patterns, see, e.g., [3].

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The number of inversions and the number of descents studied in [9] are a 2-pattern functional and a consecutive 2-pattern functional, respectively. In Sec. 3, we state a general result on the dual complexity of a pattern functional (Theorem 2, Corollaries 1 and 2), and then in Sec. 4 consider some examples, namely, the number of peaks, valleys, double ascents, and double descents, which are 3-pattern functionals. The techniques used in dealing with this series of examples are related to the so-called Solomon descent algebra, so that in Sec. 2 we present the necessary background. Another type of statistics, with a quite different behavior as regards the representation-theoretic aspects, is considered in Sec. 5; these are the excedance number and the number of fixed points.

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§2. The Solomon descent algebra and Lie characters of the symmetric group

For $n \in \mathbb{N}$, denote by $\mathrm{Comp}(n)$ the set of compositions of n and by $\mathrm{Part}(n)$ the set of partitions of n.

Given a permutation $\sigma \in \mathfrak{S}_n$, we denote by $\mathrm{Des}(\sigma)$ its descent set: $\mathrm{Des}(\sigma) = \{i \in \{1, \dots, n-1\} : \sigma(i) > \sigma(i+1)\}$. For $p = (\alpha_1, \dots, \alpha_k) \in \mathrm{Comp}(n)$, put

$$B_p = \sum_{\sigma:\mathrm{Des}(\sigma) \subset \{\alpha_1, \alpha_1+\alpha_2, \dots, \alpha_1+\dots+\alpha_{k-1}\}} \sigma \in \mathbb{C}[\mathfrak{S}_n].$$

In particular, $B_{(1^n)} = \sum_{\sigma \in \mathfrak{S}_n} \sigma$, and for $p_k^{(2)} = (1, \dots, 1, 2, 1, \dots, 1)$ (where 2 is in the kth position) $B_{p_k^{(2)}} = \sum_{\sigma: k \notin \mathrm{Des}(\sigma)} \sigma$.

The elements $\{B_p\}_{p \in \mathrm{Comp}(n)}$ form a basis of a subalgebra Σ_n of the group algebra $\mathbb{C}[\mathfrak{S}_n]$ called the Solomon algebra. In [2], another important basis $\{I_p\}_{p \in \mathrm{Comp}(n)}$ of Σ_n was introduced. Denote by $k(p)$ the number of parts of a composition p. Given two compositions $p, q \in \mathrm{Comp}(n)$ with $p \leqslant q$ (i.e., p being a refinement of q), let k_i be the number of parts of p that subdivide the ith part of q, and denote $\mathbf{k}(p,q) = k_1 k_2 \dots k_s$ and $\mathbf{k}!(p,q) = k_1! k_2! \dots k_s!$. Then the elements of the two bases are related as follows:

$$I_q = \sum_{p \leqslant q} \frac{(-1)^{k(p)} - (-1)^{k(q)}}{\mathbf{k}(p,q)} B_p, \qquad B_r = \sum_{q \leqslant r} \frac{1}{\mathbf{k}!(q,r)} I_q.$$

In particular,

$$B_{(1^n)} = I_{(1^n)}, \qquad B_{p_k^{(2)}} = I_{p_k^{(2)}} + \frac{1}{2} I_{(1^n)}. \tag{1}$$

Consider the representation

$$\tau_\lambda = \mathrm{Ind}_{\mathfrak{S}_{m_1}[\mathfrak{S}_1]\times\ldots\times\mathfrak{S}_{m_k}[\mathfrak{S}_k]\times\ldots}^{\mathfrak{S}_n}(\mathrm{Id}_{m_1}[\mathrm{Lie}_1]\otimes\ldots\otimes\mathrm{Id}_{m_k}[\mathrm{Lie}_k]\otimes\ldots), \tag{2}$$

where $\mathfrak{S}_{m_k}[\mathfrak{S}_k]$ is the wreath product $(\mathfrak{S}_k)^{m_k} \wr \mathfrak{S}_{m_k}$ and $\mathrm{Id}_{m_k}[\mathrm{Lie}_k]$ is the representation of $\mathfrak{S}_{m_k}[\mathfrak{S}_k]$ constructed from the identity representation Id_{m_k} of $\mathfrak{S}_{m_k}$ and the so-called Lie representation Lie_k (see, e.g., [2]) of $\mathfrak{S}_k$. The structure of the representation Lie_k is described in [4, 6] (see also [8, Ex. 7.88]).

The following result is essentially proved in [2, Theorem 4.4] (see also [1, Theorem 2.2 and Corollary 2.3]).

Theorem 1 ([1,2]). *Let $a = \sum_q a_q I_q$ be an element of Σ_n, and let M_a be the matrix of the right multiplication by a in $\mathbb{C}[\mathfrak{S}_n]$. Then the eigenvalues s_λ of M_a are indexed by the partitions $\lambda \in \mathrm{Part}(n)$ and*

$$s_\lambda = b_\lambda \sum_{p:\lambda(p)=\lambda} a_p, \tag{3}$$

where $b_\lambda = \prod m_k!$ for $\lambda = (k^{m_k})$. The restriction of the left regular representation of $\mathfrak{S}_n$ to the corresponding eigenspace V_λ is isomorphic to τ_λ. In particular, $\dim V_\lambda$, i.e., the multiplicity of the eigenvalue s_λ, is $\frac{n!}{z_\lambda}$, where $z_\lambda = \prod k^{m_k} m_k!$, that is, the cardinality of the conjugacy class of $\mathfrak{S}_n$ corresponding to λ.

In particular, we have the following decomposition of the left regular representation Reg_l of $\mathfrak{S}_n$:

$$\mathbb{C}[\mathfrak{S}_n] = \sum_{\lambda\in\mathrm{Part}(n)} V_\lambda, \quad \text{i.e.,} \quad \mathrm{Reg}_l = \sum_{\lambda\in\mathrm{Part}(n)} \tau_\lambda. \tag{4}$$

§3. The dual complexity of pattern functionals

Recall the definition of dual complexity from [9].

Definition 1. The *dual complexity* $\mathrm{dc}(\phi)$ of an element ϕ of the group algebra $\mathbb{C}[G]$ is the dimension of the cyclic subspace (ideal) $\mathrm{Ide}(\phi) = \mathbb{C}[G]\phi$ generated by all left translations of this element.

The purpose of this section is to study the dual complexity of so-called pattern functionals on the symmetric groups.

Let $m \leqslant n$. For every element $a \in \mathbb{C}[\mathfrak{S}_m]$ (regarded as a function on $\mathfrak{S}_m$), consider the following functional on $\mathfrak{S}_n$:

$$\phi_a(g) = \sum_{1 \leqslant k_1 < k_2 < \ldots < k_m \leqslant n} a(g|_{\{k_1,\ldots,k_m\}}),$$

where the notation $g|_{\{k_1,\ldots,k_m\}} = \tau$ for $\tau \in \mathfrak{S}_m$ means that $g(k_i) < g(k_j) \iff \tau(i) < \tau(j)$ for any $i, j = 1, \ldots, m$.

Definition 2. *The functionals of the form ϕ_a, $a \in \mathbb{C}[\mathfrak{S}_m]$, are called m-pattern functionals.*

Given $a \in \mathbb{C}[\mathfrak{S}_m]$, denote by ρ_a the restriction of the left regular representation Reg_l of $\mathfrak{S}_n$ to the ideal $\mathrm{Ide}(\phi_a)$.

Theorem 2. *Consider the decomposition* (4), *and let $a \in V_\lambda$, $\lambda \in \mathrm{Part}(m)$. Assume that λ has $m-k$ rows of length* 1, *and denote by μ the diagram of size k obtained from λ by removing all these rows. Then*

$$\rho_a = \mathrm{Ind}_{\mathfrak{S}_k \times \mathfrak{S}_{n-k}}^{\mathfrak{S}_n}(\tau_\mu \times \mathrm{Id}_{n-k})$$

and, in particular, $\mathrm{dc}(\phi_a) = \dim \tau_\mu \cdot \binom{n}{k}$.

Proof. Given $b \in V_\lambda \subset \mathbb{C}[\mathfrak{S}_m]$ and $1 \leqslant i_1 < \ldots < i_m \leqslant n$, consider the functional $F^b_{\{i_1,\ldots,i_m\}}$ on $\mathbb{C}[\mathfrak{S}_n]$, where

$$F^b_{\{i_1,\ldots,i_m\}}(g) := b(g|_{\{i_1,\ldots,i_m\}}), \qquad g \in \mathbb{C}[\mathfrak{S}_n].$$

It is not difficult to check that the subset W_λ of $\mathbb{C}[\mathfrak{S}_n]$ spanned by $F^b_{\{i_1,\ldots,i_m\}}$ for all $b \in V_\lambda$ and all sets of m indices $\{i_1, \ldots, i_m\}$ is a left ideal and the restriction of Reg_l to W_λ is isomorphic to $\mathrm{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_{n-m}}^{\mathfrak{S}_n}(\tau_\lambda \times \mathrm{Id}_{n-m})$. Moreover, it is clear that this ideal contains $\mathrm{Ide}(\phi_a)$. If we realize the induced representation in the space of functions on $\mathfrak{S}_n/(\mathfrak{S}_m \times \mathfrak{S}_{n-m})$ with values in V_λ, then ϕ_a corresponds to the constant function identically equal to $a \in V_\lambda$.

On the other hand, since $\tau_\lambda = \mathrm{Ind}_{\mathfrak{S}_k \times \mathfrak{S}_{m-k}}^{\mathfrak{S}_m}(\tau_\mu \times \mathrm{Id}_{m-k})$ by (2), we can realize V_λ as the space of functions on $\mathfrak{S}_m/(\mathfrak{S}_k \times \mathfrak{S}_{m-k})$ with values in V_μ, and then it is not difficult to see, using the properties of induced representations, that $\mathrm{Ide}(\phi_a)$ is in fact the space of $\mathrm{Ind}_{\mathfrak{S}_k \times \mathfrak{S}_{n-k}}^{\mathfrak{S}_n}(\tau_\mu \times \mathrm{Id}_{n-k})$. $\square$

Definition 3. *An m-pattern functional on $\mathfrak{S}_n$ of the form ϕ_a where $a \in V_\lambda$ and λ has k rows of length greater than 1 will be called a k-ary pattern functional.*

Corollary 1. *For a k-ary functional ϕ_a, the representation ρ_a is isomorphic to a representation of $\mathfrak{S}_n$ in a space of tensors of rank k.*

An element $a \in \mathfrak{S}_m$ determines, along with ϕ_a, another natural functional on $\mathfrak{S}_n$:

$$\psi_a(g) = \sum_{k=1}^{n-m+1} a(g|_{\{k,k+1,\dots,k+m-1\}}).$$

Definition 4. *A functional of the form ψ_a is called a consecutive m-pattern functional on $\mathfrak{S}_n$.*

Corollary 2. *For every $a \in \mathbb{C}[\mathfrak{S}_m]$ we have* $\mathrm{Ide}(\psi_a) = \mathrm{Ide}(\phi_a)$*, and the conclusions of the theorem hold for ψ_a, too.*

Proof. Follows from the proof of Theorem 2. □

§4. Examples; the number of peaks, valleys, double ascents, and double descents

Given $\lambda \in \mathrm{Part}(n)$, by π_λ we denote the irreducible representation of $\mathfrak{S}_n$ corresponding to λ.

For $k \leqslant 3$, from (2) we have

$$\begin{aligned}
&\tau_{(1)} \simeq \pi_{(1)};\\
&\tau_{(1^2)} \simeq \pi_{(2)}, \quad \tau_{(2)} \simeq \pi_{(1^2)};\\
&\tau_{(1^3)} \simeq \pi_{(3)}, \quad \tau_{(2,1)} \simeq \pi_{(21)} \oplus \pi_{(1^3)}, \quad \tau_{(3)} \simeq \pi_{(21)}.
\end{aligned}$$

Thus, a 1-pattern functional on $\mathfrak{S}_n$ is just a *constant* functional, which is 0-ary. There are no 1-ary functionals.

The set of 2-pattern functionals is spanned by the 0-ary constant functionals and the 2-ary functional ϕ_a with $a = \delta_e - \delta_{(1,2)} \in \mathbb{C}[\mathfrak{S}_2]$. Clearly, this is the centered[1] *number of inversions* $\widetilde{\mathrm{inv}}(g) = \mathrm{inv}(g) - \frac{n(n-1)}{4}$. Its dual complexity is equal to $\frac{n(n-1)}{2}$, and the corresponding representation is isomorphic to the representation of $\mathfrak{S}_n$ in the space of $n \times n$ skew-symmetric matrices. The corresponding consecutive functional ψ_a is the

[1] By a centered functional we mean a functional orthogonal to the subspace of constants, i.e., such that the sum of its values over all elements of the group vanishes.

centered *number of descents* $\widetilde{\operatorname{des}}(g) = \operatorname{des}(g) - \frac{n-1}{2}$. For more details, see [9].

For $k = 3$, we have also the set of 3-ary functionals, for which the corresponding representation ρ_a is isomorphic to a representation of $\mathfrak{S}_n$ in the space of tensors of rank 3 and decomposes as

$$\operatorname{Ind}_{\mathfrak{S}_3\times\mathfrak{S}_{n-3}}^{\mathfrak{S}_n}(\pi_{(2,1)}\times\pi_{(n-3)}) = \pi_{(n-1,1)} + \pi_{(n-2,2)} + \pi_{(n-2,1^2)} + \pi_{(n-3,2,1)}.$$

The dual complexity of such a functional is equal to $\frac{n(n-1)(n-2)}{3}$.

Consider, for example, the following well-known consecutive 3-pattern functionals ψ_a, $a \in \mathfrak{S}_3$: the number of *peaks, valleys, double ascents*, and *double descents*

$$\begin{aligned}
\operatorname{peaks}(\sigma) &= \{i \in \{2,\dots,n-1\} : \sigma(i-1) < \sigma(i) > \sigma(i+1)\},\\
\operatorname{valleys}(\sigma) &= \{i \in \{2,\dots,n-1\} : \sigma(i-1) > \sigma(i) < \sigma(i+1)\},\\
\operatorname{dasc}(\sigma) &= \{i \in \{2,\dots,n-1\} : \sigma(i-1) < \sigma(i) < \sigma(i+1)\},\\
\operatorname{ddes}(\sigma) &= \{i \in \{2,\dots,n-1\} : \sigma(i-1) > \sigma(i) > \sigma(i+1)\}.
\end{aligned}$$

It is easy to see that these functionals are of the form ψ_a for the functions $a_{\text{peaks}}(g) = \delta_{[132]} + \delta_{[231]}$, $a_{\text{valleys}}(g) = \delta_{[213]} + \delta_{[312]}$, $a_{\text{dasc}}(g) = \delta_{[123]}$, and $a_{\text{ddes}}(g) = \delta_{[321]}$, respectively.

Observe that all these functionals belong to the Solomon algebra, so that Theorem 1 gives all the necessary information on the spectra. Namely, denote by $p_{2,j}$ the composition of n of the form $(1,\dots,1,2,1,\dots,1)$ with 2 at position j, and by $p_{3,j}$ the composition of n of the form $(1,\dots,1,3,1,\dots,1)$ with 3 at position j. Then it is easy to see that

$$\operatorname{peaks} = \sum_{j=1}^{n-2}(B_{p_{2,j}} - B_{p_{3,j}}) = \frac{n-2}{3}I_{(1^n)} - \sum_{j=1}^{n-2} I_{p_{3,j}} + \frac{1}{2}(I_{p_{2,1}} - I_{p_{2,n-1}}),$$

so that, in the notation of Theorem 1, we have only two nonzero eigenvalues:

$$s_{(1^n)} = n!\cdot\frac{n-2}{3}, \qquad s_{(31^{n-3})} = -(n-3)!\cdot(n-2),$$

the first one (of multiplicity 1) corresponding to the identity representation, and the second one (of multiplicity $\frac{n(n-1)(n-2)}{3}$) corresponding to the 3-ary functional obtained from the number of peaks by centering.

Obviously, exactly the same results hold for the number of valleys.

In a similar way, we have

$$\mathrm{dasc} = \sum_{j=1}^{n-2} B_{p_{3,j}} = \frac{n-2}{6} I_{(1^n)} + \sum_{j=1}^{n-2} I_{p_{3,j}} + \frac{1}{2}\sum_{j=2}^{n-1}(I_{p_{2,j-1}} - I_{p_{2,j}}),$$

so that in this case there are three nonzero eigenvalues:

$$s_{(1^n)} = n!\cdot\frac{n-2}{6}, \quad s_{(21^{n-2})} = (n-2)!\cdot(n-2), \quad s_{(31^{n-3})} = (n-3)!\cdot(n-2),$$

of multiplicities 1, $\frac{n(n-1)}{2}$, and $\frac{n(n-1)(n-2)}{3}$, respectively, corresponding to the decomposition of dasc into a sum of a 0-ary, 2-ary, and 3-ary functionals.

It is not difficult to see that the case of the number of double descents differs from this one only in that the second and third eigenvalues have the opposite sign.

§5. The excedance number and the number of fixed points

In this section, we consider statistics of another type, which are not pattern functionals and demonstrate a quite different behavior of spectra.

The *excedance number* of a permutation $\sigma \in [\mathfrak{S}_n]$ is defined as follows:

$$\mathrm{exc}(\sigma) = \#\{i = 1, \dots, n-1 : \sigma(i) > i\}.$$

This statistic was first studied by MacMahon [7], and it is an Eulerian statistics, that is, its generating function is given by the Euler polynomials:

$$\sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{exc}(\sigma)} = A_n(q), \quad \text{where} \quad \sum_{n\geqslant 0} A_n(q)\frac{z^n}{n!} = \frac{(1-q)e^z}{e^{qz} - qe^z}.$$

It is not difficult to deduce that

$$C_n = \sum_{\sigma\in\mathfrak{S}_n} \mathrm{exc}(\sigma) = n!\cdot\frac{n-1}{2}. \tag{5}$$

Denote by u_{exc} the corresponding element of $\mathbb{C}[\mathfrak{S}_n]$:

$$u_{\mathrm{exc}} = \sum_{g\in\mathfrak{S}_n} \mathrm{exc}(g) g \in \mathbb{C}[\mathfrak{S}_n].$$

Theorem 3. *The dual complexity of the function* exc *is equal to* $(n-1)^2+1$, *and* $\mathrm{Ide}(u_{\mathrm{exc}})$ *coincides with the space of the primary component of the representation* $\pi_{(n-1,1)}$ *plus the subspace of constants.*

Proof. Consider the following elements of $\mathbb{C}[\mathfrak{S}_n]$:

$$e_{ij} = \sum_{\sigma\in\mathfrak{S}_n} \varepsilon_{ij}(\sigma)\sigma, \quad \text{where} \quad \varepsilon_{ij}(\sigma) = \begin{cases} 1, & \sigma(i) > j, \\ 0 & \text{otherwise}, \end{cases} \tag{6}$$

for $i = 1, \ldots, n$, $j = 1, \ldots, n-1$. Then it is not difficult to see that for every $g \in \mathfrak{S}_n$,

$$ge_{ij} = e_{g^{-1}(i),j}. \tag{7}$$

It easily follows that for every $j = 1, \ldots, n-1$, the subspace L_j spanned by e_{ij} for $i = 1, \ldots, n$ is invariant for Reg_l and the corresponding subrepresentation is isomorphic to $\pi_{(n-1,1)} \oplus \pi_{(n)}$. Then the whole space $L = \bigoplus_{j=1}^{n-1} L_j$ spanned by all e_{ij} is the primary component of the representation $\pi_{(n-1,1)}$ plus the subspace of constants.

Now we obviously have $u_{\mathrm{exc}} = \sum\limits_{i=1}^{n} e_{ii}$, and the theorem follows. □

Theorem 4. *Let $M_{\mathrm{exc}} = \mathrm{Reg}_l(u_{\mathrm{exc}})$ be the operator of the left multiplication by u_{exc} in $\mathbb{C}[\mathfrak{S}_n]$. Then the eigenvalues of M_{exc} are*

$$\begin{aligned} s_0 &= n!\cdot\frac{n-1}{2} \quad (\textit{with multiplicity } 1) \quad \textit{and} \\ s_k &= -\frac{n(n-2)!}{1-\omega_k}, \quad \textit{where} \quad \omega_k = e^{\frac{2\pi i k}{n}}, \qquad k = 1, \ldots, n-1, \end{aligned}$$

each having the multiplicity $n-1$. The eigenspace of s_0 is the subspace of constants, and the eigenspace corresponding to s_k with $k = 1, \ldots, n-1$ is

$$\mathrm{span}\left\{ \sum_{m=1}^{n} \omega_k^{m-1} e_{mj},\ j = 1, \ldots, n-1 \right\}.$$

Remark. Another form of s_k is

$$s_k = -n(n-2)!\cdot\frac{1 + i\cot\frac{\pi k}{n}}{2}, \qquad k = 1, \ldots, n-1.$$

Lemma 1. *In each of the spaces L_j, the matrix of the operator M_{exc} in the basis (6) is the following circulant matrix:*

$$(n-2)!\cdot\begin{pmatrix} a_n & a_n+1 & a_n+2 & \ldots & a_n+n-1 \\ a_n+n-1 & a_n & a_n+1 & \ldots & a_n+n-2 \\ & & \ldots & & \\ a_n+1 & a_n+2 & a_n+3 & \ldots & a_n \end{pmatrix},$$

where $a_n = \frac{(n-1)(n-2)}{2}$.

Proof. It follows from (7) that the entries of the matrix in question are

$$m_{ik}=\sum_{\sigma\in\mathfrak{S}_n:\,\sigma^{-1}(i)=k}\operatorname{exc}(\sigma).$$

It is clear that for every i we have $m_{ii}=C_{n-1}=\frac{(n-1)!(n-2)}{2}=(n-2)!a_n$ by (5). Now, it is not difficult to show that for $k\geqslant i$,

$$\sum_{\sigma\in\mathfrak{S}_n:\,\sigma^{-1}(i)=k+1}\operatorname{exc}(\sigma)=\sum_{\sigma\in\mathfrak{S}_n:\,\sigma^{-1}(i)=k}\operatorname{exc}(\sigma)+(n-2)!,$$

by considering the bijection

$$\begin{pmatrix}\dots & k & k+1 & \dots\\ \dots & x & i & \dots\end{pmatrix}\leftrightarrow\begin{pmatrix}\dots & k & k+1 & \dots\\ \dots & i & x & \dots\end{pmatrix},$$

which does not change the number of excedances if $x\neq k+1$ and changes it by 1 if $x=k+1$. In a similar way one can check that $m_{i-1,i}=m_{ii}+(n-1)!$ and that $m_{k-1,i}=m_{ki}-(n-2)!$ for every $k<i$. The lemma now follows by induction. □

Proof of Theorem 4. The theorem follows from Lemma 1 and the well-known description of the spectrum of a circulant matrix. Namely, for the circulant matrix with the first row $(c_0,c_{n-1},\dots,c_2,c_1)$, the eigenvalues are given by

$s_k=c_0+c_{n-1}\omega_k+c_{n-2}\omega_k^2+\ldots+c_1\omega_k^{n-1}$, where $\omega_k=e^{\frac{2\pi ik}{n}}$, $k=1,\dots,n$, and the corresponding eigenvectors are $(1,\omega_k,\omega_k^2,\dots,\omega_k^{n-1})^T$. □

Now consider the *number of fixed points*

$$\operatorname{fix}(\sigma)=\#\{i=1,\dots,n:\sigma(i)=i\}.$$

Theorem 5. *The dual complexity of the function* fix *is equal to* $(n-1)^2+1$, *and* $\operatorname{Ide}(u_{\rm exc})$ *coincides with the space of the primary component of the representation* $\pi_{(n-1,1)}$ *plus the subspace of constants.*

Proof. The proof is similar to that of Theorem 3; it suffices to observe that

$$u_{\rm fix}=\sum_{i=1}^{n}(e_{i,i-1}-e_{ii}).$$ □

Theorem 6. *Let* $M_{\rm fix}=\operatorname{Reg}_l(u_{\rm fix})$ *be the operator of the left multiplication by* $u_{\rm fix}$ *in* $\mathbb{C}[\mathfrak{S}_n]$. *Then* $M_{\rm fix}$ *has two nonzero eigenvalues:* $s_0=n!$ *with multiplicity* 1 *and* $s_1=n\cdot(n-2)!$ *with multiplicity* $(n-1)^2$, *the eigenspace* *of s_0 being the subspace of constants and that of s_1 being the primary component of the representation $\pi_{(n-1,1)}$.*

Proof. Like Theorem 4, follows from the lemma below, which can be proved similarly to Lemma 1. $\square$

Lemma 2. *In each of the spaces L_j, the matrix of the operator M_{fix} in the basis* (6) *is the matrix whose all diagonal entries are equal to* $2(n-1)!$ *and all off-diagonal entries are equal to* $(n-2)!(n-2)$.

Denote by $\widetilde{\mathrm{exc}}$ and $\widetilde{\mathrm{fix}}$ the centered versions of the corresponding statistics:

$$\widetilde{\mathrm{exc}}(\sigma) = \mathrm{exc}(\sigma) - \frac{n-1}{2}, \qquad \widetilde{\mathrm{fix}}(\sigma) = \mathrm{fix}(\sigma) - 1.$$

Corollary 3.

$$u_{\widetilde{\mathrm{exc}}} * u_{\widetilde{\mathrm{fix}}} = u_{\widetilde{\mathrm{fix}}} * u_{\widetilde{\mathrm{exc}}} = n(n-2)! \cdot u_{\widetilde{\mathrm{exc}}}.$$

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