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COMBINATORIAL ENCODINGS OF INFINITE SYMMETRIC GROUPS AND DESCRIPTIONS OF SEMIGROUPS OF DOUBLE COSETS

Abstract. Spaces of double cosets of infinite symmetric groups with respect to some special subgroups admit natural structures of semigroups. Elements of such semigroups can be interpreted in combinatorial terms. We present a description of such constructions in a relatively wide degree of generality.

Let G be an infinite-dimensional group and K be a subgroup. Quite often, the double coset space $K \setminus G/K$ admits a natural multiplication. Moreover, for any unitary representation of G, the semigroup $K \setminus G/K$ acts in the space of K-fixed vectors. The first example of such a multiplication was discovered by R. S. Ismagilov in the 1960s, later several series of constructions of this type for classical and symmetric groups were examined by Olshanski (for the case of symmetric groups, see [15]). In [7], it was observed that this phenomenon is quite general; however, it seemed that the spaces $K \setminus G/K$ themselves are unhandable objects. In [9], an explicit geometric description was obtained of the semigroups $K \setminus G/K$ for the case of $G = \mathbb{S}_\infty \times \mathbb{S}_\infty \times \mathbb{S}_\infty$ (where $\mathbb{S}_\infty$ is the infinite symmetric group) and K a diagonal subgroup, in terms of two-dimensional surfaces and their cobordisms. In the preprint [8], the construction was generalized to numerous pairs (G, K) related to infinite symmetric groups. Later, constructions of the preprint [8] were extended in [1, 10, 11]. However, [8] has a "dogmatic part" containig descriptions of the semigroups $K \setminus G/K$ in wide generality (G and K are products of symmetric groups; also, we admit wreath products); this part is the topic of the present paper. Proofs are omitted, because they are one-to-one copies of the proofs given in [10].

It seems that our constructions can be interesting for finite symmetric groups; we give descriptions of various double coset spaces and also produce numerous "parametrizations" of symmetric groups.

Key words and phrases: triangulations, polygonal surfaces, bipartite graphs, unitary representations, representations of categories.

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§1. (G,K)-pairs and their trains. Definitions and a priori theorems

1.1. Notation. – $\mathbb{N}$ is the set of positive integers.

– $\mathbb{M}(\alpha)$ is the initial segment $\{1,2,\dots,\alpha\}$ of $\mathbb{N}$.

– $\mathbb{N}_1$, $\mathbb{N}_2$, ... are disjoint copies of the set $\mathbb{N}$.

– $\mathbb{M}_j(\alpha)$ are the initial segments $\{1,2,\dots,\alpha\}$ of $\mathbb{N}_j$.

– $\mathbb{I}(\zeta)$ are finite sets with ζ elements.

– $\sqcup$, $\coprod$ are symbols for the disjoint union of sets.

– S_n is the finite symmetric group of order n (the group of all permutations of $\{1,2,\dots,n\}$).

1.2. The infinite symmetric group. For a countable set Ω, denote by $\mathbb{S}_\infty(\Omega)$ the group of all finitely supported permutations of Ω. Denote $\mathbb{S}_\infty(\mathbb{N})$ by $\mathbb{S}$.

We represent permutations by infinite 0-1 matrices in the usual way.

By $\mathbb{S}_\infty^\alpha \subset \mathbb{S}_\infty$ we denote the group of permutations having the form

$$\sigma = \begin{pmatrix} 1_\alpha & 0 \\ 0 & * \end{pmatrix},$$

where 1_α is the unit $\alpha\times\alpha$ matrix.

Let $K = \mathbb{S}_\infty(\mathbb{N}_1)\times\dots\times\mathbb{S}_\infty(\mathbb{N}_p)$. For a multi-index $\alpha=(\alpha_1,\dots,\alpha_p)$, we define a subgroup $K^\alpha\subset K$:

$$K^\alpha := \mathbb{S}_\infty^{\alpha_1}(\mathbb{N}_1)\times\dots\times\mathbb{S}_\infty^{\alpha_p}(\mathbb{N}_p).$$

1.3. The topological infinite symmetric group. Denote by $\mathbf{S}_\infty$ the group of all permutations of $\mathbb{N}$. Define the subgroups $\mathbf{S}_\infty^\alpha\subset\mathbf{S}_\infty$ as above. Define a topology on $\mathbf{S}_\infty$ assuming that $\mathbf{S}_\infty^\alpha$ form a fundamental systems of open neighborhoods of the identity. In other words, a sequence σ_j converges to σ if for any $k\in\mathbb{N}$ we have $\sigma_j k=\sigma k$ for sufficiently large j. The group $\mathbf{S}_\infty$ is a totally disconnected topological group[1].

A classification of irreducible unitary representations of $\mathbf{S}_\infty$ was obtained by Lieberman [6], see expositions in [7, 14]. Let $\alpha=0,1,2,\dots$, and let τ be an irreducible representation of S_α. Consider the subgroup $S_\alpha\times\mathbf{S}_\infty^\alpha\subset\mathbf{S}_\infty$. Consider the represetation $\tau\otimes 1$ of $S_\alpha\times\mathbf{S}_\infty^\alpha$, where 1 denotes the trivial (one-dimensional) representation of $\mathbf{S}_\infty^\alpha$, and the corresponding induced representation of $\mathbf{S}_\infty$. Note that the quotient space

[1]This is the unique structure of a separable topological group on $\mathbf{S}_\infty$, see [3].

$\mathbb{S}_\infty/\mathbb{S}_\infty^\alpha = \mathbf{S}_\infty/\mathbf{S}_\infty^\alpha$ is countable and, therefore, the induced representation is well defined (see, e.g., [5, 13.2]).

Theorem 1.1. a) *Every irreducible representation of* $\mathbf{S}_\infty$ *is induced from a representation of the from* $\tau \otimes 1$ *of a subgroup* $S_\alpha \times \mathbf{S}_\infty^\alpha$.

b) *Every unitary representation of* $\mathbf{S}_\infty$ *is a direct sum of irreducible representations.*

In a certain sense, the Lieberman theorem opens and closes the representation theory of the group $\mathbf{S}_\infty$. However, it is an important element of wider theories.

1.4. Reformulations of continuity. Let ρ be a unitary representation of $\mathbb{S}_\infty$ in a Hilbert space H. Denote by $H^\alpha \subset H$ the subspace of all $\mathbb{S}_\infty^\alpha$-fixed vectors. We say that a representation ρ is *admissible* if $\cup H^\alpha$ is dense in H.

Denote by $\mathbf{B}_\infty$ the semigroup of matrices composed of 0 and 1 such that each row and each column contains $\leqslant 1$ ones. We equip $\mathbf{B}_\infty$ with the topology of element-wise convergence; the group $\mathbb{S}_\infty$ is dense in $\mathbf{B}_\infty$.

Theorem 1.2 (see [7, 14])**.** *The following conditions are equivalent:*

– ρ *is continuous in the topology of* $\mathbf{S}_\infty$*;*

– ρ *is admissible;*

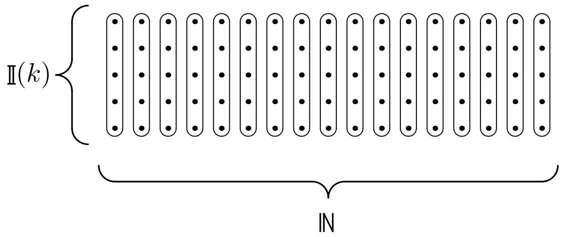
– ρ *admits a continuous extension to the semigroup* $\mathbf{B}_\infty$.

This statement has a straightforward extension to products of symmetric groups $\mathbf{K} = \mathbf{S}_\infty \times \cdots \times \mathbf{S}_\infty$.

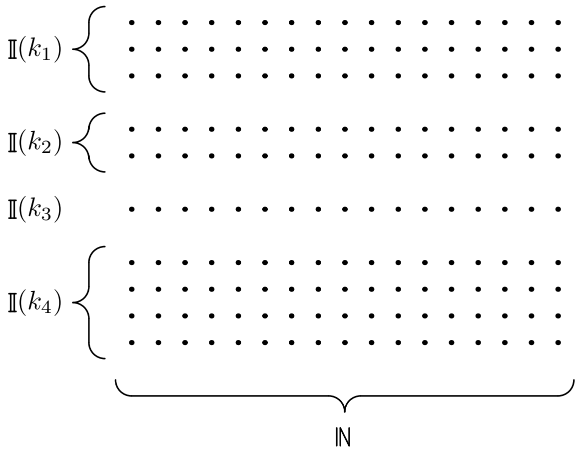
1.5. Wreath products. Let U be a finite group. Consider the countable *direct product* $\mathbf{U}^\infty := U \times U \times U \times \dots$; it is a group whose elements are all infinite sequences $(u_1, u_2, \dots)$. Consider also the *restricted product* U^∞, whose elements are all sequences such that $u_j = 1$ for sufficiently large j. The group $\mathbf{U}^\infty$ is equipped with the direct product topology, the group U^∞ is discrete.

Permutations of sequences $(u_1, u_2, \dots)$ induce automorphisms of U^∞ and $\mathbf{U}^\infty$. Consider the semidirect products $K := \mathbb{S}_\infty \ltimes U^\infty$ and $\mathbf{K} := \mathbf{S}_\infty \ltimes \mathbf{U}^\infty$, see, e.g., [5, 2.4]; they are called the *wreath products* of $\mathbb{S}_\infty$ and U.

Our main example is the wreath product of $\mathbb{S}_\infty$ and a finite symmetric group S_k. We realize it as a group of finite permutations of $\mathbb{N} \times \mathbb{I}(k)$. The



a) The group $\mathbb{S}_\infty \ltimes (S_k)^\infty$. The group $\mathbb{S}_\infty$ acts by permutations of columns. The normal subgroup $(S_k)^\infty$ permutes elements in each column. The semidirect product consists of the permutations preserving the partition of the strip into columns.



b) The group $\mathbb{S}_\infty \ltimes (S_{k_1} \times \cdots \times S_{k_p})^\infty$. The group $\mathbb{S}_\infty$ acts by permutations of columns. The normal subgroup acts by permutations inside each subcolumn.

Fig. 1. Wreath products.

group $\mathbb{S}_\infty$ acts by permutations of $\mathbb{N}$, and the subgroups $S_k \subset (S_k)^\infty$ act by permutations of the sets $\{m\} \times \{1, \dots, k\}$, see Fig. 1a.

1.6. Representations of wreath products. For $K = \mathbb{S}_\infty \ltimes U^\infty$, we define a subgroup K^α as the semidirect product of $\mathbb{S}_\infty^\alpha$ and the subgroup

$$U^{\infty-\alpha} := \underbrace{1 \times \cdots \times 1}_{\alpha \text{ times}} \times U \times U \times \dots .$$

For a unitary representation ρ of K in a Hilbert space H, we denote by H^α the space of K^α-fixed vectors. We say that ρ is *admissible* if $\cup_\alpha H^\alpha$ is dense in H. A unitary representation of $K = \mathbb{S}_\infty \ltimes U^\infty$ is admissible if and only if it is continuous in the topology of the group $\mathbf{K} = \mathbf{S}_\infty \ltimes \mathbf{U}^\infty$.

1.7. (G,K)**-pairs.** See Fig. 2. Fix positive integers q, p and a $q \times p$ matrix

$$\mathcal{Z} := \{\zeta_{ji}\}$$

consisting of nonnegative integers. Assume that it has no zero columns and no zero rows.

Fix a collection $\Lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_q \end{pmatrix}$ of nonnegative integers. Fix sets $L_1, \dots, L_q$ such that L_j has λ_j elements. Consider the collection of sets

$$\mathbb{N}_i \times \mathbb{I}(\zeta_{ji}).$$

Denote by Ω_j the disjoint union

$$\Omega_j := L_j \sqcup \coprod_{i \leqslant p} (\mathbb{N}_i \times \mathbb{I}(\zeta_{ji}))$$

(we assume that all sets Ω_j obtained in this way are mutually disjoint). Set

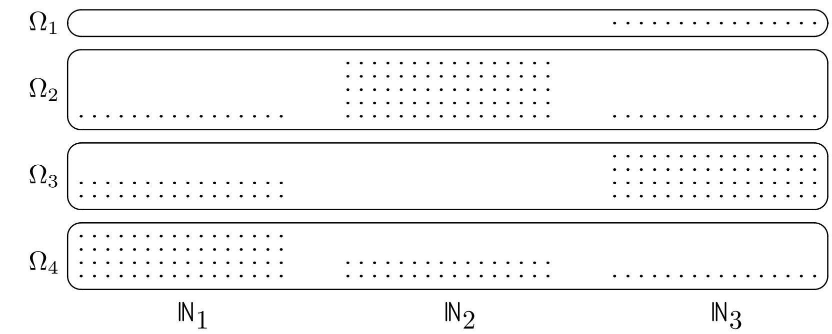
$$G := G[\mathcal{Z}, \Lambda] = \prod_{j=1}^{q} \mathbb{S}_\infty(\Omega_j), \qquad \mathbf{G} := \mathbf{G}[\mathcal{Z}, \Lambda] = \prod_{j=1}^{q} \mathbf{S}_\infty(\Omega_j).$$

Next, we define the following subgroup $K^\circ \subset G[\mathcal{Z}, \Lambda]$:

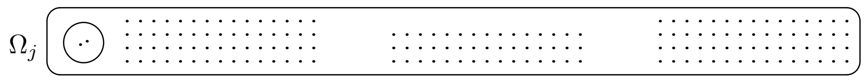
$$K^\circ = K^\circ[\mathcal{Z}] := \prod_{i=1}^{p} \Bigl(\mathbb{S}_\infty(\mathbb{N}_i) \ltimes \Bigl(\prod_j S_{\zeta_{ji}} \Bigr)^\infty \Bigr),$$

and its completion $\mathbf{K}^\circ \subset \mathbf{G}[\mathcal{Z}, \Lambda]$:

$$\mathbf{K}^\circ = \mathbf{K}^\circ[\mathcal{Z}] := \prod_{i=1}^{p} \Bigl(\mathbf{S}_\infty(\mathbb{N}_i) \ltimes \Bigl(\prod_j \mathbf{S}_{\zeta_{ji}} \Bigr)^\infty \Bigr).$$



a) The set $\cup_{i=1}^{p} \cup_{j=1}^{q} (\mathbb{N}_i \times \mathbb{I}(\zeta_{ji}))$. Here $q = 4$, $p = 3$, $\Lambda = 0$, and $\mathcal{Z} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 5 & 1 \\ 2 & 0 & 4 \\ 4 & 2 & 1 \end{pmatrix}$. The groups $\mathbb{S}_\infty(\Omega_1)$, $\mathbb{S}_\infty(\Omega_2)$, $\mathbb{S}_\infty(\Omega_3)$, $\mathbb{S}_\infty(\Omega_4)$ consist of permutations of the sets Ω_j. The group G is the product of $\mathbb{S}_\infty(\Omega_j)$. The groups $\mathbb{S}_\infty(\mathbb{N}_1)$, $\mathbb{S}_\infty(\mathbb{N}_2)$, $\mathbb{S}_\infty(\mathbb{N}_3)$ act by permutations of columns (inside the given $\mathbb{N}_j$). The subgroup $K^{\circledast} \subset G$ is the product $\mathbb{S}_\infty(\mathbb{N}_1) \times \mathbb{S}_\infty(\mathbb{N}_2) \times \mathbb{S}_\infty(\mathbb{N}_3)$. For each column we have the group of all permutations inside the column preserving subcolumns. The group K° is generated by $K^{\circledast}$ and all such subgroups. In our case, $K^\circ = \bigl(\mathbb{S}_\infty(\mathbb{N}_1) \ltimes (S_2 \times S_4)^\infty\bigr) \times \bigl(\mathbb{S}_\infty(\mathbb{N}_2) \ltimes (S_5 \times S_2)^\infty\bigr) \times \bigl(\mathbb{S}_\infty(\mathbb{N}_2) \ltimes (S_4)^\infty\bigr)$.



b) A picture with a nonzero L_j (surrounded by a circle).

Fig. 2. Reference to Sec. 1.

Also, we define the following subgroup in $G[\mathcal{Z}, \Lambda]$:

$$K^{\circledast}[\mathcal{Z}] := \prod_{i=1}^{p} \mathbb{S}_\infty(\mathbb{N}_i) \subset K^\circ[\mathcal{Z}],$$

and its completion

$$\mathbf{K}^{\circledast}[\mathcal{Z}] := \prod_{i=1}^{p} \mathbf{S}_\infty(\mathbb{N}_i) \subset \mathbf{K}^\circ[\mathcal{Z}].$$

Below, (G, K) denotes a pair (group, subgroup) of the form

$$(G,K) = (G[\mathcal{Z},\Lambda], K^{\circ}[\mathcal{Z}]) \qquad \text{or} \qquad (G,K) = (G[\mathcal{Z},\Lambda], K^{\circledast}[\mathcal{Z}]).$$

REMARK. One can also consider intermediate wreath products K between $K^{\circ}[\mathcal{Z}]$ and $K^{\circledast}[\mathcal{Z}]$; below we consider one example from this zoo.

1.8. Colors, smells, melodies. We wish to draw figures, also we want to have a more flexible language.

a) To each Ω_j we assign a *color*, say red, blue, white, red, green, etc. We also think that a color is an attribute of all points of Ω_j. We denote colors by $\daleth_j$.

b) Next, to each $\mathbb{N}_i$ we assign a *smell* $\aleph_i$, say Magnolia, Matricana, Pinus, Ledum, Rafflesia, etc. In figures, we denote smells by $\blacktriangle, \times, \square, \dots$. We also think that a smell $\aleph_i$ is an attribute of all points of $(\mathbb{N}_i \times \mathbb{I}(\zeta_{ji})) \subset \Omega_j$.

c) Orbits of the group $\mathbb{S}_\infty(\mathbb{N}_i)$ on Ω_j are one-point orbits or countable homogeneous spaces $\mathbb{S}_\infty/\mathbb{S}^1_\infty \simeq \mathbb{N}$. To each countable orbit we assign a *melody*, say, violin, harp, tomtom, flute, drum, ... In figures, we denote melodies symbols $\heartsuit$, $\succ$, ∇, $\sharp$, $\ddagger$, etc. Note that a melody makes sense after fixing a smell and a color.

Example. See Fig. 2c. We have a 4×3 table. Boxes are distinguished by colors, columns (corresponding to $\mathbb{N}_1$, $\mathbb{N}_2$, $\dots$) are by distinguished smells. Rows inside the intersection of a box and a column are indexed by melodies.

1.9. Admissible representations. Let ρ be a unitary representation of G. We say that ρ is a *K-admissible representation* if the restriction of ρ to K is admissible. Equivalently, we say that ρ is a *representation of the pair* (G, K).

1.10. Reformulation of admissibility in terms of continuity. The embedding $K \to G$ admits an extension to a map $\mathbf{K} \to \mathbf{G}$ of the corresponding completions. Consider the group

$$G_{\mathbf{K}} := G \cdot \mathbf{K} \subset \mathbf{G}$$

generated by G and $\mathbf{K}$. Any element of $G_{\mathbf{K}}$ admits a (nonunique) representation as $g\mathbf{k}$ where $g \in G$, $\mathbf{k} \in K$.

We consider the natural topology on the subgroup $\mathbf{K}$ and assume that $\mathbf{K}$ is an open-closed subgroup in $G_{\mathbf{K}}$.

Proposition 1.3. *A unitary representation of G is K-admissible if and only if it is continuous in the above sense.*

1.11. Lemma on admissibility.

Lemma 1.4. *Let ρ be an irreducible unitary representation of G. If $H^\alpha \neq 0$ for some α, then the representation ρ is K-admissible.*

Proof. Consider the subspace $\mathcal{H} := \cup H^\alpha$. Fix $g \in G$. For sufficiently large β, the element g commutes with K^β. Therefore, $\mathcal{H}$ is g-invariant. The closure of $\mathcal{H}$ is a subrepresentation. □

Corollary 1.5. *If an irreducible unitary representation of G has a K-fixed vector, then it is K-admissible.*

1.12. The existing representation theory. A well-developed existing theory is related to the pair $G = \mathbb{S}_\infty \times \mathbb{S}_\infty$ and the diagonal subgroup $K = \mathbb{S}_\infty$, see [4, 13, 15]. In our notation, $\mathcal{Z} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The representation theory of this pair includes also earlier works on the Thoma characters [17], see [18].

Olshanski [15] also considered the following pairs:

– $G = \mathbb{S}_{\infty+1} \times \mathbb{S}_\infty$, $K = \mathbb{S}_\infty$; in our notation, $\mathcal{Z} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\Lambda = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

– $G = \mathbb{S}_{2\infty}$, $K = \mathbb{S}_\infty \times \mathbb{S}_\infty$; in our notation, $\mathcal{Z} = \begin{pmatrix} 1 & 1 \end{pmatrix}$.

– $G = \mathbb{S}_{2\infty}$, $K = \mathbb{S}_\infty \ltimes \mathbb{Z}_2^\infty$, and also $G = \mathbb{S}_{2\infty+1}$ with the same subgroup K. In our notation, $\mathcal{Z} = (2)$ and $\Lambda = 0$ or 1.

In all these cases, the pairs (G, K) are limits of spherical pairs of finite groups.

Nessonov [12] considered the case $\mathcal{Z} = \begin{pmatrix} 1 & \dots & 1 \end{pmatrix}$ and described all $K^\circledast$-spherical representations of $G(\mathcal{Z}; 0)$. Note that this result has no finite-dimensional counterpart.

1.13. Trains. Consider the following $(\alpha+N+N+\infty)\times(\alpha+N+N+\infty)$ matrix $\Theta_N^{[\alpha]} \in \mathbb{S}_\infty$:

$$\Theta_N^{[\alpha]} = \begin{pmatrix} 1_\alpha & 0 & 0 & 0 \\ 0 & 0 & 1_N & 0 \\ 0 & 1_N & 0 & 0 \\ 0 & 0 & 0 & 1_\infty \end{pmatrix} \in \mathbb{S}_\infty.$$

In fact, $\Theta_N^{[\alpha]}$ is contained in $\mathbb{S}_\infty^\alpha$.

Consider a pair (G, K). For a multi-index $\alpha = (\alpha_1, \dots, \alpha_p)$, we denote by $\Theta_N^{[\alpha]}$ the element

$$\Theta_N^{[\alpha]} = \left(\Theta_N^{[\alpha_1]}, \dots, \Theta_N^{[\alpha_p]}\right) \in K.$$

Again, $\Theta_N^{[\alpha]} \in K^\alpha$.

Fix multi-indices α, β, γ. Consider double cosets

$$\mathfrak{h} \in K^\beta \setminus G/K^\alpha, \qquad \mathfrak{g} \in K^\gamma \setminus G/K^\beta,$$

and choose their representatives $g \in \mathfrak{g}$, $h \in \mathfrak{h}$. Consider the sequence

$$f_N = g\Theta_N^{[\alpha]} h \in G.$$

Consider the double coset $\mathfrak{f}_N$ containing f_N,

$$\mathfrak{f}_N \in K^\gamma \setminus G/K^\alpha.$$

Theorem 1.6. a) *The sequence $\mathfrak{f}_N$ is eventually constant.*

b) *The limit*

$$\mathfrak{g} \circ \mathfrak{h} := \lim_{N \to \infty} \mathfrak{f}_N$$

does not depend on the choice of representatives g, h.

c) *The product $\circ$ obtained in this way is associative, i.e., for any*

$$\mathfrak{g} \in K^\delta \setminus G/K^\gamma, \qquad \mathfrak{h} \in K^\gamma \setminus G/K^\beta, \qquad \mathfrak{f} \in K^\beta \setminus G/K^\alpha,$$

we have

$$(\mathfrak{g} \circ \mathfrak{h}) \circ \mathfrak{f} = \mathfrak{g} \circ (\mathfrak{h} \circ \mathfrak{f}).$$

Thus we obtain a category $\mathbb{T}(G, K)$, whose objects are multi-indices α and morphisms $\alpha \to \beta$ are elements of $K^\beta \setminus G/K^\alpha$. We say that $\mathbb{T}(G, K)$ is the *train* of the pair (G, K).

1.14. The involution in the train. The map $g \mapsto g^{-1}$ induces a map of the quotient spaces $K^\alpha \setminus G/K^\beta \to K^\beta \setminus G/K^\alpha$, we denote it by

$$\mathfrak{g} \mapsto \mathfrak{g}^\square.$$

Obviously,

$$(\mathfrak{g} \circ \mathfrak{h})^\square = \mathfrak{h}^\square \mathfrak{g}^\square.$$

1.15. Representations of the train. Now let ρ be a unitary representation of the pair (G,K). We define subspaces H^α as above and denote by P^α the orthogonal projection to H^α. For $\mathfrak{g} \in K^\beta \setminus G/K^\alpha$, choose a representative $g \in \mathfrak{g}$. Consider the operator

$$\overline{\rho}_{\alpha,\beta}(g) := P^\beta \rho(g) : \ H^\alpha \to H^\beta.$$

By definition, we have

$$\|\overline{\rho}_{\alpha,\beta}(g)\| \leqslant 1. \tag{1}$$

Theorem 1.7. a) *The operator $\overline{\rho}_{\alpha,\beta}(g)$ depends only on the double coset $\mathfrak{g}$ containing g.*

b) *We obtain a representation of the category $\mathbb{T}(G,K)$, i.e., for any*

$$\mathfrak{g} \in K^\gamma \setminus G/K^\beta, \qquad \mathfrak{h} \in K^\beta \setminus G/K^\alpha,$$

the following identity holds:

$$\overline{\rho}_{\beta,\gamma}(\mathfrak{g})\,\overline{\rho}_{\alpha,\beta}(\mathfrak{h}) = \overline{\rho}_{\alpha,\gamma}(\mathfrak{g} \circ \mathfrak{h}).$$

c) *We obtain a $*$-representation, i.e.,*

$$\big(\overline{\rho}_{\alpha,\beta}(\mathfrak{g})\big)^* = \overline{\rho}_{\beta,\alpha}(\mathfrak{g}^\square).$$

d) *The operator $\rho(\Theta_N^\alpha)$ weakly converges to the projection P^α.*

Theorem 1.8. *Our construction provides a bijection between the set of all unitary representations of the pair (G,K) and the set of all $*$-representations of the category $\mathbb{T}(G,K)$ satisfying condition* (1).

We omit the proofs of Theorems 1.6–1.8 and Theorem 1.9 formulated below, because they are literal copies of the proofs in [10].

Our main purpose is to give an explicit description of trains; also, we give some constructions of representations of groups.

1.16. Sphericity.

Theorem 1.9. *Let $(G,K) = (G[\mathcal{Z},\Lambda], K^\circ[\mathcal{Z}])$ or $(G[\mathcal{Z},\Lambda], K^\circledast[\mathcal{Z}])$, as above. If $\Lambda = 0$, then the pair (G,K) is spherical. In other words, for every irreducible unitary representation of (G,K), the dimension of the space of K-fixed vectors is $\leqslant 1$.*

1.17. The structure of the rest of the paper. In the next section, we recall a definition of tensor products of Hilbert spaces. In Sec. 3, we consider three examples. In Secs. 4, 5, we present a description of trains for arbitrary pairs

$$(G(\mathcal{Z},\Lambda), K^{\circ}[\mathcal{Z}]), \qquad (G(\mathcal{Z},\Lambda), K^{\circledast}[\mathcal{Z}]).$$

§2. Tensor products of Hilbert spaces

Here we recall a definition of an infinite tensor product of Hilbert spaces, see the detailed von Neumann's paper [16] or a short introduction in the addendum to [2].

2.1. Definition of tensor products. Let H_1, H_2, $\dots$ be a countable collection of Hilbert spaces (they can be finite-dimensional or infinite-dimensional). Fix a unit vector $\xi_k \in H_k$ in each space. The tensor product

$$(H_1, \xi_1) \otimes (H_2, \xi_2) \otimes (H_3, \xi_3) \otimes \dots$$

is defined in the following way. We choose an orthonormal basis $e_j[k]$ in each H_k, assuming that $e_1[k] = \xi_k$. Next, we consider the Hilbert space with the orthonormal basis

$$e_{\alpha_1}[1] \otimes e_{\alpha_2}[2] \otimes \dots$$

such that $e_{\alpha_N}[N] = \xi_N$ for sufficiently large N (note that this basis is countable).

The construction substantially depends on the choice of distinguished vectors. The spaces $\otimes(H_k, \xi_k)$ and $\otimes(H_k, \eta_k)$ are canonically isomorphic if and only if

$$\sum |\langle \xi_j, \eta_j \rangle - 1| < \infty.$$

In particular, we can omit distinguished vectors in a finite number of factors (more precisely, we can choose them in an arbitrary way).

2.2. The action of symmetric groups in tensor products. The symmetric groups S_n act in the tensor powers $H^{\otimes n}$ by permutations of factors. This phenomenon has a straightforward analog.

We denote by

$$(H, \xi)^{\otimes \infty} := (H, \xi) \otimes (H, \xi) \otimes \dots$$

the infinite symmetric power of (H, ξ).

Proposition 2.1. a) *The complete symmetric group* $\mathbf{S}_\infty$ *acts in* $(H,\xi)^{\otimes\infty}$ *by permutations of factors. The representation is continuous with respect to the topology of* $\mathbf{S}_\infty$.

b) *The vector* $\xi^{\otimes\infty}$ *is a unique* $\mathbf{S}_\infty$*-fixed vector in* $(H,\xi)^{\otimes\infty}$.

c) *The subspace of* $\mathbf{S}_\infty^\alpha$*-fixed vectors is*

$$H^{\otimes\alpha}\otimes\xi^{\otimes\infty}.$$

Proposition 2.2. *Fix a sequence* ξ_k *of unit vectors in a Hilbert space* H. *The symmetric group* $\mathbb{S}_\infty$ *acts in the tensor product* $\otimes_k(H,\xi_k)$ *by permutations of factors.*

We emphasize that in this case there is no action of the complete symmetric group $\mathbf{S}_\infty$.

§3. An example: triangulated surfaces

3.1. The group. Let $\mathcal{Z}=\bigl(3\bigr)$ and $\lambda\geqslant 0$ be arbitrary. First, we consider the pair

$$(G(\mathcal{Z},\Lambda),K^\circ(\mathcal{Z}))=(\mathbb{S}_{\lambda+3\infty},\mathbb{S}_\infty\ltimes(S_3)^\infty)\,.$$

We reduce the subgroup and set

$$K:=\mathbb{S}_\infty\ltimes\bigl(\mathbb{Z}_3\bigr)^\infty,$$

where $\mathbb{Z}_3\subset S_3$ is the group of cyclic permutations (or, equivalently, the group of even permutations).

Now

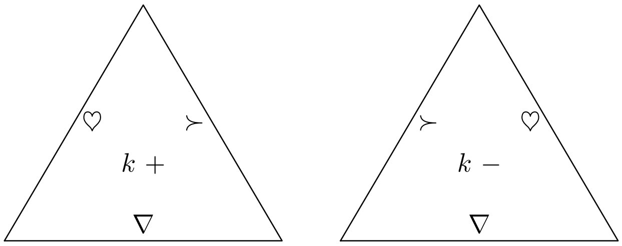
$$\Omega=L\sqcup\bigl(\mathbb{N}\times\mathbb{I}(3)\bigr).$$

Let $\alpha\geqslant 0$; we define a set $\Omega_{[\alpha]}$ as the set of all fixed points of K^α,

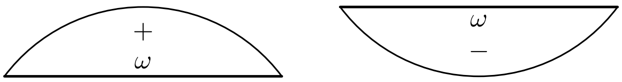
$$\Omega_{[\alpha]}=L\sqcup\bigl(\mathbb{M}(\alpha)\times\mathbb{I}(3)\bigr).$$

Thus, we have only one color, only one smell, but three melodies, say harp (∇), violin ($\heartsuit$), tube ($\succ$).

Remark. Let $\lambda\geqslant 3$. Then the operation $\lambda\mapsto\lambda-3$ does not change the topological group $G_{\mathbf{K}}$. Indeed, we can add one point to the set $\mathbb{N}$ and exclude three points from L. Therefore, we may consider only the cases $\lambda=0$, 1, 2.



a) A plus-triangle and a minus-triangle.



b) A plus-tag and a minus-tag.

Fig. 3. Reference to Sec. 3.2. Items for a complex.

3.2. The encoding of elements of the symmetric group. Fix $\alpha, \beta \geqslant 0$.

First, we take the following collection of items (see Fig. 3).

A. *Plus-triangles and minus-triangles.* To each element $k \in \mathbb{N}$ we assign a pair of oriented triangles $T_{\pm}(k)$ with label k. We write the labels ∇, $\heartsuit$, $\succ$ on the interiors of the sides of $T_+(k)$ (respectively, $T_-(k)$) clockwise (respectively, counter-clockwise).

An important remark: a number k and a melody determine some element $\omega \in \Omega$.

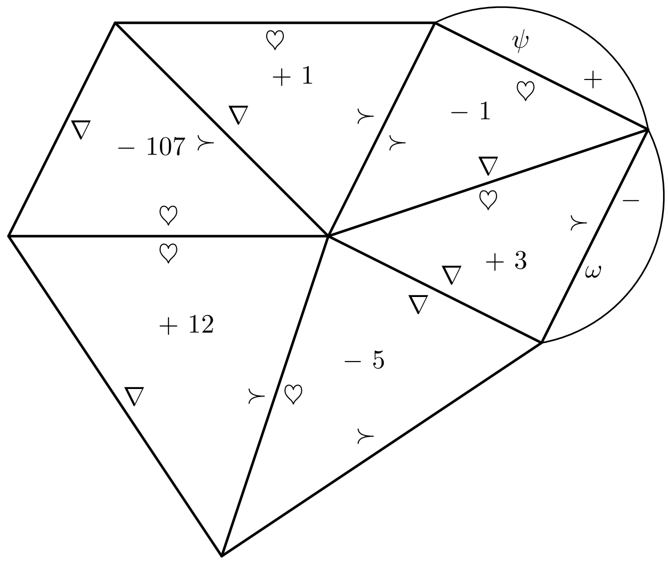
B. *Plus-tags and minus-tags.* To each element $\omega \in \Omega$ we assign two oriented segments $D_{\pm}(\omega)$ with tags, see Fig. 3. We write the label ω and the label "+" (respectively, "−") on the segment $T_+(\omega)$ (respectively, $T_-(\omega)$).

We take the following collection of items:

$$T_+(k) \quad \text{where } k > \alpha; \qquad T_-(l) \quad \text{where } l > \beta;$$

$$D_+(\omega) \quad \text{where } \omega \in \Omega[\alpha]; \qquad D_-(\omega) \quad \text{where } \omega \in \Omega[\beta].$$

Each element of Ω is present on precisely one edge of one item $T_+(k)$ or $D_+(\omega)$ (and, respectively, on one item $T_-(k)$ or $D_-(\omega)$).



A piece of a complex. Removing the numbers and the melodies $\succ$, ∇, $\heartsuit$ (and leaving the signs and the labels on the tags), we pass to double cosets.

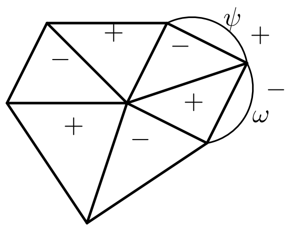
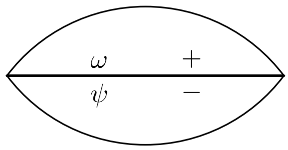
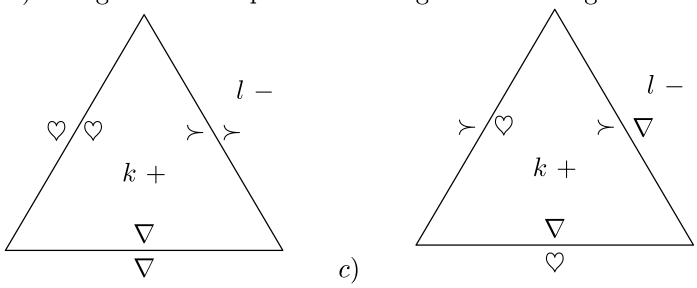


Fig. 4. Reference to Secs. 3.2–3.3.

Fix an element $g \in \mathbb{S}_\infty(\Omega)$. For each $\omega \in \Omega$, we identify (keeping in mind the orientations) the edge of $T_+(\cdot)$ or $D_+(\cdot)$ labeled by ω with the edge of $T_-(\cdot)$ or $D_-(\cdot)$ labeled by $g\omega$.



a) A degenerate component. An edge with two tags.



b) c)

This is the stereographic projection of a sphere and a graph on the sphere. A pure envelope (b) and an envelope (c).

Fig. 5. Reference to Secs. 3.2–3.3.

In this way, we obtain a two-dimensional oriented triangulated surface $\Xi(g)$ with tags on the boundary. Our picture satisfies the following properties:

(i) The surface consists of a countable number of compact components.

(ii) Each component is a two-dimensional oriented triangulated surface with tags on the boundary (we allow also a segment with two tags, see Fig. 5a).

(iii) All triangles and tags have labels "+" or "−", neighboring objects have different signs.

(iv) The plus-triangles (respectively, minus-triangles) are indexed by $\alpha+1$, $\alpha+2$, $\alpha+3$, $\dots$ (respectively, $\beta+1$, $\beta+2$, $\beta+3$, $\dots$).

(v) The sides of plus-triangles are labeled (from the interior) by ∇, $\heartsuit$, $\succ$ clockwise. The sides of minus-triangles are labeled by the same symbols counter-clockwise.

(vi) The plus-tags are indexed by the elements of $\Omega_{[\alpha]}$; the minus-tags, by the elements of $\Omega_{[\beta]}$.

(vii) Almost all components are spheres composed of two triangles, and the melodies on the sides of the edges coincide.

We regard such surfaces up to combinatorial equivalence.

Lemma 3.1. *Every surface equipped with the data described above has the form $\Xi[g]$. Different elements $g \in \mathbb{S}_\infty(\Omega)$ produce different equipped surfaces.*

Proof. We present the inverse construction. Above, we have assigned two elements of Ω to each edge. Let μ correspond to the plus-side, ν correspond to the minus-side. Then g sends μ to ν. $\square$

Thus we obtain a bijection.

Now we need a technical definition. We say that an *envelope* is a component consisting of two triangles. We say that an envelope is *pure* if the melodies on both sides of each edge coincide (see Fig. 5).

3.3. The projection to double cosets.

Lemma 3.2. *The right multiplications $g \mapsto gh$ by elements of $\mathbb{S}^\alpha_\infty(\mathbb{N})$ correspond to permutations of the labels $\alpha+1$, $\alpha+2$, $\alpha+3$, ... on plus-triangles. Correspondingly, the left multiplications by elements of $\mathbb{S}^\beta_\infty(\mathbb{N})$ correspond to permutations of the labels on minus-triangles.*

Lemma 3.3. *The right multiplications by elements of $(\mathbb{Z}_3)^{-\alpha+\infty} \subset K^\alpha$ correspond to cyclic permutations of the symbols ∇, $\heartsuit$, $\succ$ inside each plus-triangle.*

Corollary 3.4. *Passing to double cosets $K^\beta \setminus G/K^\alpha$ corresponds to forgetting the numbers of triangles and the melodies on the interior sides of edges of triangles, and removing all envelopes.*

Thus, we remember only labels on tags and signs.

3.4. The construction of the train. Objects of the category are integers $\alpha \geqslant 0$. Fix indices α and β. A *morphism* $\alpha \to \beta$ is a compact (in general, disconnected) triangulated surface without envelopes equipped with data (ii), (iii), (vi) from the list above (labels $\pm$ and labels on tags).

To multiply $\mathfrak{G} : \alpha \to \beta$ and $\mathfrak{H} : \beta \to \gamma$, we glue (according to the orientations) the minus-segments of the boundary of $\mathfrak{G}$ with the plus-segments of the boundary of $\mathfrak{H}$ having the same labels on their tags. We remove the corresponding tags, forget their labels, forget the contour of gluing. Some envelopes can appear, we remove them.

We obtain a morphism $\alpha \to \gamma$.

Theorem 3.5. *The multiplication described above is the multiplication in the train of the pair* (G, K).

Proof. Fix $h, g \in G$. Consider the corresponding surfaces $\Xi[h]$, $\Xi[g]$. Take a very large number N. Then the set of labels k on the minus-triangles of $\Xi[\theta_\beta^N h]$ and the set of labels l on the plus-triangles of $\Xi[g]$ are disjoint. Therefore, the plus-triangles of $\Xi[\theta_\beta^N h]$ preserve their neighbors after the multiplication $\Xi[\theta_\beta^N h] \to g\Xi[\theta_\beta^N h]$. Also, the minus-triangles of $\Xi[g]$ preserve their neighbors after the multiplication $g \mapsto g\Xi[\theta_\beta^N h]$. Therefore, both surfaces $\Xi[h]$, $\Xi[g]$ are pieces of the surface $\Xi[g\theta_\beta^N h]$. $\square$

3.5. The involution on the train. We reverse the signs and reverse the orientation.

3.6. Examples of representations. Let V be a Hilbert space. Fix a unit vector

$$\xi \in V \otimes V \otimes V \tag{2}$$

invariant with respect to the cyclic permutations of elements of the tensor product.

Consider the tensor product

$$\bigotimes_{l \in L} V \otimes (V \otimes V \otimes V, \xi_i)^{\otimes \infty}.$$

The group $G_{\mathbf{K}}$ acts on this product by permutations of factors. Namely, $\mathbf{S}_\infty$ acts by permutations of the factors $(V \otimes V \otimes V, \xi)$. The copies of $\mathbb{Z}_3$ act by cyclic permutations of factors of the products $V \otimes V \otimes V$. The group G acts by permutations of the factors V.

3.7. Another pair. The subgroup $K^{\circledast}$. Let $\mathcal{Z}$, λ, $G = G[\mathcal{Z}, \lambda]$ be as above and consider the pair

$$(G, K) := (G(\mathcal{Z}, \lambda), K^{\circledast}(\mathcal{Z})) = (\mathbb{S}_{\lambda + 3\infty}, \mathbb{S}_\infty).$$

Return to Lemma 3.2. Now we remove the numbers of triangles but preserve the melodies. We also remove all pure envelopes.

In the construction of the representation, we can replace (2) by an arbitrary unit vector

$$\xi \in V \otimes V \otimes V.$$

3.8. Another pair. The subgroup K°. Let $\mathcal{Z}$, λ, $G = G[\mathcal{Z}, \lambda]$ be as above. Consider the pair

$$(G, K) := (G(\mathcal{Z}, \lambda), K^\circ(\mathcal{Z})) = (\mathbb{S}_{\lambda+3\infty}, \mathbb{S}_\infty \ltimes (S_3)^\infty).$$

First, we construct representations. In the construction of Sec. 3.6, we take

$$\xi \in \mathsf{S}^3 V \subset V \otimes V \otimes V,$$

where $\mathsf{S}^3 V$ is the third symmetric power of V.

An attempt to repeat the construction of the train meets an obvious difficulty: permutations of the melodies change the orientations of the triangles. However, we can pass from triangulations to dual graphs. Now we can enumerate the double cosets by trivalent graphs. See the next section.

§4. The general case, $K = K^\circ$ is a wreath product

4.1. The group. Here we consider an arbitrary matrix $\mathcal{Z}$ and an arbitrary vector Λ. Now

$$\Omega_j = L_j \sqcup \coprod_{i=1}^{p} \big(\mathbb{N}_i \times \mathbb{I}(\zeta_{ji})\big),$$

$$G := G[\mathcal{Z}, \Lambda] = \prod_{j=1}^{q} \mathbb{S}_\infty(\Omega_j),$$

$$K := K^\circ[\mathcal{Z}] = \prod_{i=1}^{p} \Big(\mathbb{S}_\infty(\mathbb{N}_i) \ltimes \Big(\prod_j S_{\zeta_{ji}}\Big)^\infty\Big).$$

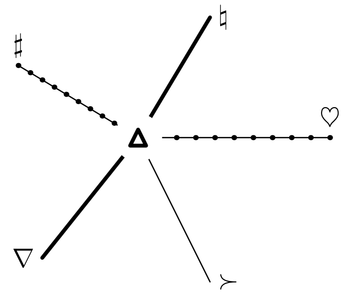
Recall that we attributed a color to each Ω_j, a smell to each $\mathbb{N}_i$, and a melody to each infinite orbit of $\mathbb{S}_\infty(\mathbb{N}_i)$ on Ω_j.

We denote

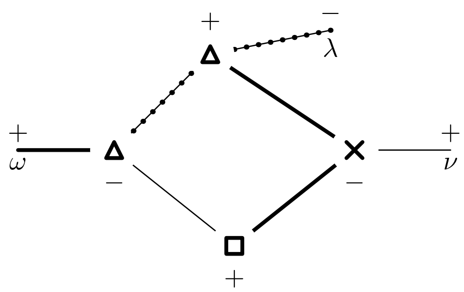
$$\Omega := \coprod_{j \leqslant q} \Omega_i$$

and regard $G[\mathcal{Z}, \Lambda]$ as a subgroup in $\mathbb{S}_\infty(\Omega)$. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_p)$, we denote by $\Omega_{[\alpha]}$ the set of all K^α-fixed points of Ω,

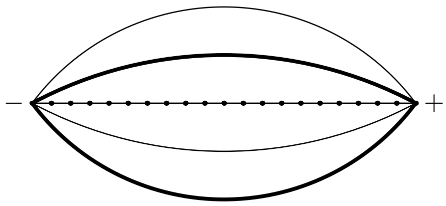
$$\Omega_{[\alpha]} = \coprod_{j=1}^{q} \bigg(L_j \sqcup \coprod_{i=1}^{p} \big(\mathbb{M}(\alpha_i) \times \mathbb{I}(\zeta_{ji})\big)\bigg).$$



a) A node.



b) A double coset.



c) A *trivial* component of a graph.

Fig. 6. Reference to Sec. 4.2.

4.2. The encoding of elements of symmetric groups. For each element of $G[\mathcal{Z}, \Lambda]$, we construct a graph equipped with some additional data.

For each smell i, we draw a *node* $T[\aleph_i]$ (see Fig. 6). It contains a vertex of smell $\aleph_i$ and $\sum_j \zeta_{ji}$ semi-edges. The edges are colored, each color $\daleth_j$ is used for coloring ζ_{ji} edges. To each edge of a given color we attribute a melody from the corresponding list of ζ_{ji} melodies (different edges have different melodies). Thus, the semi-edges of $T[\aleph_i]$ are in a one-to-one correspondence with the orbits of $\mathbb{S}_\infty(\mathbb{N}_i)$ on Ω.

Now we prepare the following collection of items.

a) For each smell $\aleph_i$ and each $k \in \mathbb{N}$, we draw two copies $T_\pm[\aleph_i; k]$ of the node $T[\aleph_i]$, their vertices are labeled by k and $\pm$. We throw out the nodes $T_+[\aleph_i; k]$ with $k \leqslant \alpha_i$ and the nodes $T_-[\aleph_i; m]$ with $m \leqslant \beta_i$

b) For each color $\daleth_j$ and each element ω of $\Omega_j \cap \Omega_{[\alpha]}$, we draw a tag $D_+(\omega)$ and mark this tag by ω, the color of ω, and "+". We draw similar tags $D_-(\omega)$ for the elements $\omega \in \Omega_{[\beta]}$. We imagine a tag as a vertex and a semi-edge.

Thus, the set Ω is in a one-to-one correspondence with the sets

$$\mathcal{E}_+ = \left\{ \begin{matrix} \text{all semi-edges of} \\ \text{all nodes } T_+[\aleph_i, k] \end{matrix} \right\} \bigcup \Omega_{[\alpha]}$$

and

$$\mathcal{E}_- = \left\{ \begin{matrix} \text{all semi-edges of} \\ \text{all nodes } T_-[\aleph_i, k] \end{matrix} \right\} \bigcup \Omega_{[\beta]}.$$

Denote the bijections $\Omega \to \mathcal{E}_\pm$ by $H_\pm$.

Now for each $\omega \in \Omega$ we connect a semi-edge $H_+(\omega) \in \mathcal{E}_+$ with the semi-edge $H_-(g\omega) \in \mathcal{E}_-$. We obtain a graph with the following properties.

(i) The graph consists of a countable number of compact components.

(ii) There are two types of vertices, interior vertices[2] and terminal vertices (ends of semi-edges).

(iii) Each interior vertex has a smell $\aleph_i$ and a sign "+" or "−".

(iv) The interior plus-vertices are indexed by the set $\{\alpha_i+1, \alpha_i+2, \dots\}$; the interior minus-vertices, by $\{\beta_i+1, \beta_i+2, \dots\}$.

(v) The terminal vertices fall into two classes, entries and exits. The entries are indexed by the elements of $\Omega_{[\alpha]}$ and the labels "+". The exits are indexed by the elements of $\Omega_{[\beta]}$ and the labels "−".

(v) Neighboring vertices have different signs.

[2]The case $\sum_j \zeta_{ji} = 1$ is admissible.

(vi) The edges are colored, the number of edges of color $\daleth_j$ coming to an interior vertex of smell $\aleph_i$ is ζ_{ji}. The edge adjacent to a terminal vertex with label ω has the color of ω.

(vii) To each semi-edge adjacent to an interior vertex, a melody is attributed compatible with its color. At each vertex of smell $\aleph_i$, each melody compatible with the smell is present precisely once.

(viii) All but a finite number of components consist of two interior vertices and edges connecting these vertices.

We call components described in (viii) *trivial*. We say that a component is *completely trivial* if for every edge the smells of both semi-edges coincide.

Theorem 4.1. *There is a one-to-one correspondence between the set of all graphs satisfying* (i)–(viii) *and the infinite symmetric group* $\mathbb{S}_\infty(\Omega)$.

Proof. Consider an edge. It has a plus-semi-edge and a minus-semi-edge. Consider the corresponding elements $\phi \in \mathcal{E}_+$ and $\psi \in \mathcal{E}_-$. Set $g\phi = \psi$. $\square$

4.3. The projection to double cosets.

Proposition 4.2. a) *The right multiplications by elements of* $\mathbb{S}_\infty^\alpha(\mathbb{N}_i)$ *correspond to permutations of the labels* $\{\alpha+1, \alpha+2, \dots\}$ *on plus-vertices of smell* $\aleph_i$.

b) *The right multiplications by elements of* $S_{\zeta_{ji}}^{-\alpha_i+\infty} \subset \mathbb{S}_\infty(\Omega_j)$ *correspond to permutations of the melodies of semi-edges of color* $\daleth_j$ *adjacent to fixed vertices of smell* $\aleph_i$.

Corollary 4.3. *Passing to double cosets corresponds to forgetting the labels* $\in \mathbb{N}$ *and melodies.*

The colors, smells, signs, and also labels on tags are preserved.

4.4. The multiplication of double cosets. Given two morphisms $\mathfrak{G} : \alpha \to \beta$, $\mathfrak{H} : \beta \to \gamma$, we glue the exits of $\mathfrak{g}$ with the corresponding entries of $\mathfrak{h}$ (and forget the vertices of gluings).

The involution is the inversion of the signs and also the entries/exits.

Theorem 4.4. *This product coincides with the product in the train* $\mathbb{T}(G, K)$.

4.5. Some representations of (G,K). We consider a collection of Hilbert spaces $W_1, \dots, W_q$ indexed by colors. Fix i. Consider the tensor product

$$\mathcal{H}_i = \bigotimes_{j=1}^{p} W_j^{\otimes \zeta_{ji}}. \tag{3}$$

Note that the factors of the product are in a one-to-one correspondence with the semi-edges of $T[\aleph_i]$.

Fix a unit vector

$$\xi_i \in \bigotimes_{j=1}^{p} \mathsf{S}^{\zeta_{ji}} W_j \subset \mathcal{H}_i. \tag{4}$$

Consider the tensor product

$$\mathfrak{W} := \bigotimes_{j=1}^{q} W_j^{\otimes \lambda_j} \otimes \bigotimes_{i=1}^{q} (\mathcal{H}_i, \xi_i)^{\otimes\infty} = \bigotimes_{j=1}^{q} W_j^{\otimes \lambda_j} \otimes \bigotimes_{i=1}^{q} \Bigl(\bigotimes_{j=1}^{p} W_j^{\otimes \zeta_{ji}}, \xi_i \Bigr)^{\otimes\infty}.$$

Note that the factors W of this tensor product are indexed by the elements of $\sqcup \Omega_j$. Formally, we can write

$$\bigotimes_{j=1}^{q} \bigotimes_{\omega \in \Omega_j} W_j.$$

However, this makes no sense without distinguished vectors.

Each group $\mathbb{S}_\infty(\Omega_j)$ acts by permutations of the factors W_j. This determines an action of G. Each group $\mathbf{S}_\infty(\mathbb{N}_i)$ acts by permutations of the factors $(\mathcal{H}_i, \xi_i)$. For each copy of $(\mathcal{H}_i, \xi_i)$ we have an action of $\prod_j S_{\zeta_{ji}}$, namely, the symmetric group $S_{\zeta_{ji}}$ permutes the factors W_j in (4).

Thus we obtain an action of $G_{\mathbf{K}}$.

§5. The general case, $K = K^{\circledast}$ is a product of symmetric groups

The construction of this section is more or less a version of the previous construction. For the smaller group $K = K^{\circledast}$, we can replace a graph by a fat graph, and after this draw a two-dimensional surface. We repeat the construction independently.

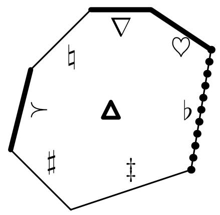


Fig. 7. Reference to Sec. 5.2. A polygon $T_+[\aleph_i]$.

5.1. The group. We consider an arbitrary matrix $\mathcal{Z}$ and an arbitrary vector Λ. Now

$$\Omega_j = L_j \sqcup \coprod_{i=1}^{p} \bigl(\mathbb{N}_i \times \mathbb{I}(\zeta_{ji})\bigr),$$

$$G := G[\mathcal{Z}, \Lambda] = \prod_{j=1}^{q} \mathbb{S}_\infty(\Omega_j), \qquad K := K^{\circledast}[\mathcal{Z}] = \prod_{i=1}^{p} \mathbb{S}_\infty(\mathbb{N}_i).$$

We denote

$$\Omega := \coprod_{j \leqslant q} \Omega_i.$$

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_p)$, we denote by $\Omega_{[\alpha]}$ the set of all K^α-fixed points of Ω,

$$\Omega_{[\alpha]} = \coprod_{j=1}^{q} \biggl(L_j \sqcup \coprod_{i=1}^{p} \bigl(\mathbb{M}(\alpha_i) \times \mathbb{I}(\zeta_{ji})\bigr)\biggr).$$

5.2. Constructions of the train. Fix a smell $\aleph_i$. The nontrivial orbits of $\mathbb{S}_\infty(\mathbb{N}_i)$ on $\sqcup \Omega_j$ are indexed by pairs (color, melody). The total number of such orbits is

$$\sum_j \zeta_{ij}.$$

We choose an arbitrary cyclic order on the set of such pairs (the construction below depends on this choice). Next, we draw a polygon $T_+[\aleph_i]$ of smell $\aleph_i$ whose sides are marked by pairs (color, melody) according to the cyclic order. We also define the polygon $T_-[\aleph_i]$, whose sides are marked according to the reversed cyclic order.

Consider the following types of items.

– *Plus-polygons and minus-polygons.* For each $i \leqslant p$, for each $k \in \mathbb{N}_i$, we draw the pair of oriented polygons $T_\pm[\aleph_i, k]$ described above; they are additionally labeled by $k \in \mathbb{N}$. Every side of every polygon $T_\pm[\aleph_i, k]$ has a smell (the smell of the polygon), a color, and a melody; therefore, a side determines an element of Ω.

– *Plus-tags and minus-tags.* For each element $\omega \in \Omega$, we draw two tags $D_\pm(\omega)$ labeled by ω and $\pm$, see Fig. 3. The side of a tag is colored by the color of ω.

Fix multi-indices α, β.

We take the following collection:

– the polygons $T_+[\aleph_i, k]$ if $k > \alpha_i$;

– the polygons $T_-[\aleph_i, m]$ if $m > \beta_i$;

– the tags $T_+[\omega]$ if $\omega \in \Omega_+[\alpha]$;

– the tags $T_-[\omega]$ if $\omega \in \Omega_-[\beta]$.

Now we have one-to-one correspondences between the set Ω and the set of all edges of all plus-triangles and plus-tags. Also, we have one-to-one correspondences between the set Ω and the set of all edges of all minus-triangles and minus-tags.

For each $g \in G$, we glue a polygonal complex. For each $\omega \in \Omega$, we identify the (oriented) edge of a plus-polygon or a plus-tag corresponding to ω with the (oriented) edge of a minus-polygon or a minus-tag corresponding to $g\omega$.

Thus we obtain a polygonal two-dimensional oriented surface with tags on the boundary satisfying the following properties:

(i) The surface consists of a countable number of compact components.

(ii) Each component is tiled by polygons of types $T_\pm[\aleph_i]$ and has tags $D_\pm$ on the boundary.

(iii) Each polygon is labeled by "+" or "−", neighboring polygons have different signs.

(iv) Each edge has a color, which is common for both (plus and minus) sides of the edge.

(v) Each edge has two melodies, on the plus-side and on the minus-side.

(vi) A cyclic order of pairs (color, melody) around the perimeter of each polygon $T_\pm[\aleph_i]$ is fixed.

(vii) The plus-polygons (respectively, minus-polygons) of a fixed smell $\aleph_i$ are indexed by $\alpha_i + 1$, $\alpha_i + 2 \dots$ (respectively $\beta_i + 1$, $\beta_i + 2, \dots$).

(viii) The plus-tags are indexed by the points of $\Omega_{[\alpha]}$, and the minus-tags, by points of $\Omega_{[\beta]}$.

(ix) All but a finite number of components of the surface are unions of pairs $T_+[\aleph_j, k]$ and $T_-[\aleph_j, l]$. We call such components "*envelopes*." *A pure envelope* is an envelope such that the melodies on the plus and minus sides of each edge coincide.

Theorem 5.1. *The data of this type are in a one-to-one correspondence with the group G.*

The inverse construction. For each $g\omega$, we find ω inside the pairs (plus-polygon, side). This side is also a side of a minus-polygon and encodes the element $g\omega$.

5.3. Passing to double cosets $K^\beta \setminus G/K^\alpha$. The literal analog of Lemma 3.2 holds.

In order to pass to double cosets $K^\alpha \setminus G/K^\beta$, we forget the labels $k \in \mathbb{N}$ and remove all envelopes.

We obtain a compact surface tiled by polygons.

– Polygons are equipped with signs $\pm$ and smells.

– Each edge is equipped with a color and a pair of melodies on the negative side and the positive side of the edge (the coloring and melodization of the edges of each polygon is fixed up to a cyclic permutation of sides, as above).[3]

– The boundary edges of the surface are equipped with signs, the positive edges are indexed by the points of $\Omega_{[\alpha]}$, the negative edges are indexed by the points of $\Omega_{[\beta]}$.

We say that such a surface is a morphism $\alpha \to \beta$.

Let $\mathfrak{G} : \alpha \to \beta$, $\mathfrak{H} : \beta \to \gamma$ be two surfaces. For each $\omega \in \Omega[\beta]$, we glue the ω-exit of $\mathfrak{G}$ with the ω-entry of $\mathfrak{H}$ (according to the orientation). Removing the envelopes, we arrive at a complex of the same type.

Theorem 5.2. *This multiplication coincides with the multiplication in the train.*

Proof is the same as for Theorem 3.5.

[3]Recall that in many cases melodies can be uniquely reconstructed from colors and may be forgotten.

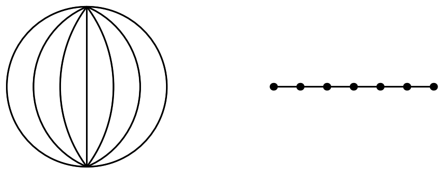


Fig. 8. Reference to Sec. 5.5. A digonal complex and the corresponding one-dimensional chain.

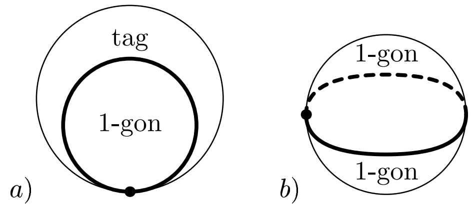


Fig. 9. Reference to Sec. 5.5. The only possible connected 1-gonal complexes.

5.4. The involution in the train. We reverse the signs $\pm$ and reverse the orientation.

5.5. Simple cases. Note that our construction admits 2-gons and 1-gons, see Figs. 8, 9. Let the matrix $\mathcal{Z}$ satisfy $\sum_j \zeta_{ij} = 2$ for all i. Then all our polygons are 2-gons. A digonal complex can be regarded as a union of chains (see Fig. 8), and we can use the language of chips, see [15] and also [10].

If $\mathcal{Z} = \begin{pmatrix} 1 & \dots & 1 \end{pmatrix}$, then our complex consists of 1-gons. This corresponds to Nessonov's case, see [10].

5.6. Constructions of representations. In the construction of a tensor product from Sec. 4.5, we can choose arbitrary unit vectors

$$\xi_i \in \bigotimes_{j=1}^{p} W_j^{\otimes \zeta_{ji}} =: \mathcal{H}_i$$

instead of (4).

References

1. A. A. Gaifullin, Yu. A. Neretin, *Infinite symmetric group and bordisms of pseudo-manifolds.* — J. Topol. Anal., `doi.org/10.1142/S179352531850022X`.
2. A. Guichardet, *Symmetric Hilbert Spaces and Related Topics,* Lect. Notes Math. **261**, Springer-Verlag, Berlin–New York, 1972.
3. A. Kechris, Ch. Rosendal, *Turbulence, amalgamation, and generic automorphisms of homogeneous structures.* — Proc. Lond. Math. Soc. (3) **94**, No. 2 (2007), 302–350.
4. S. Kerov, G. Olshanski, A. Vershik, *Harmonic analysis on the infinite symmetric group.* — Invent. Math. **158**, No. 3 (2004), 551–642.
5. A. A. Kirillov, *Elements of the Theory of Representations,* Springer-Verlag, Berlin–New York, 1976.
6. A. Lieberman, *The structure of certain unitary representations of infinite symmetric groups.* — Trans. Amer. Math. Soc. **164** (1972), 189–198.
7. Yu. A. Neretin, *Categories of Symmetries and Infinite-Dimensional Groups,* Oxford Univ. Press, New York, 1996.
8. Yu. A. Neretin, *Infinite symmetric group and combinatorial descriptions of semi-groups of double cosets*, `arXiv:1106.1161`.
9. Yu. A. Neretin, *Infinite tri-symmetric group, multiplication of double cosets, and checker topological field theories.* — Int. Math. Res. Not. **2012**, No. 3 (2012), 501–523.
10. Yu. A. Neretin, *Infinite symmetric groups and combinatorial constructions of topological field theory type.* — Russian Math. Surv. **70**, No. 4 (2015), 715–773.
11. Yu. A. Neretin, *Symmetric groups and checker triangulated surfaces,* preprint (to appear).
12. N. I. Nessonov, *Representations of $\mathfrak{S}_\infty$ admissible with respect to Young subgroups.* — Sb. Math. **203**, No. 3 (2012), 424–458.
13. A. Okounkov, *Thoma's theorem and representations of the infinite bisymmetric group.* — Funct. Anal. Appl. **28**, No. 2 (1994), 100–107.
14. G. I. Olshanski, *Unitary representations of the infinite symmetric group: a semi-group approach.* — In: *Representations of Lie Groups and Lie Algebras,* Akad. Kiadó, Budapest, 1985, pp. 181–197.
15. G. I. Olshanski, *Unitary representations of (G, K)-pairs connected with the infinite symmetric group $S(\infty)$.* — Leningrad Math. J. **1**, No. 4 (1990), 983–1014.
16. J. von Neumann, *On infinite direct products.* — Compos. Math. **6** (1938), 1–77. Reprinted in: J. von Neumann, *Collected Works.*

17. E. Thoma, *Die unzerlegbaren, positiv-definiten Klassenfunktionen der abzählbar unendlichen, symmetrischen Gruppe.* — Math. Z. **85** (1964), 40–61.
18. A. M. Vershik, S. V. Kerov, *Characters and factor representations of the infinite symmetric group.* — Sov. Math. Dokl. **23**, No. 2 (1981), 389–392.

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