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DISCRETE MORSE THEORY FOR THE BARYCENTRIC SUBDIVISION

Abstract. Let F be a discrete Morse function on a simplicial complex L. We construct a discrete Morse function $\Delta(F)$ on the barycentric subdivision $\Delta(L)$. The constructed function $\Delta(F)$ "behaves the same way" as F, i.e., has the same number of critical simplices and the same gradient path structure.

§1. Introduction

We work with discrete Morse theory, which is a discrete analog of classical Morse theory. It was developed by R. Forman [1]. This theory can be applied to any simplicial complexes and regular CW-complexes, and, although its definition is quite simple, many classical results analogous to those of the continuous Morse theory arise in its scope.

In 2010, E. Gallais [2] proved that for every smooth Morse function on a smooth manifold M there exists a PL-triangulation of this manifold that admits a discrete Morse function with the same number of critical points (simplices). This work was continued by B. Benedetti [3].

The classical construction of the barycentric subdivision of simplicial complexes can be used to approximate a smooth structure on a triangulated topological manifold. We develop a simple algorithm to "transfer" a discrete Morse function, defined on a simplicial complex, onto the barycentric subdivision of this complex in such a way that all important data about this function (i.e., the number and dimensions of the critical simplices and the gradient path structure) stays unchanged. This can be done in several different ways, and we can produce several different Morse functions. The main result of our work is as follows.

Theorem 1.1. *Let F be a discrete Morse function on a simplicial complex L. Assume that for each critical simplex $\alpha \in \mathrm{Crit}(F)$, an ordering*

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Ord_α *on its vertices is chosen. Then the pairing on the barycentric subdivision* $\Delta(L)$ *constructed in Sec.* 3 *defines a discrete Morse function* $\Delta(F)$ *on* $\Delta(L)$ *such that the following holds*:

(1) *The critical simplices of* $\Delta(F)$ *are exactly those that have the chosen orderings* Ord_* *as their labels. That is, every critical* k*-simplex of* F *contains exactly one critical* k*-simplex of* $\Delta(F)$*, which can be chosen arbitrarily before constructing* $\Delta(F)$*. This defines a bijection* $\mathrm{Crit}(F) \to \mathrm{Crit}(\Delta(F))$.

(2) *There exists a natural bijection* $\mathrm{Gr}(F) \to \mathrm{Gr}(\Delta(F))$ *that respects the bijection* $\mathrm{Crit}(F) \to \mathrm{Crit}(\Delta(F))$ *defined above.*

It is worthy to mention in this respect that E. Babson and P. Hersch [4] introduced a technique that can be (as one particular application) used to build a certain Morse function on the barycentric subdivision of an arbitrary simplicial complex. This Morse function arises from a lexicographic order on the maximal chains of simplices of L, i.e., on the maximal simplices of $\Delta(L)$. The question whether there are connections between our work and [4] remains open.

The structure of this paper is as follows. In Sec. 2, we give definitions of a discrete Morse function and the barycentric subdivision. In Sec. 3, we construct a Morse function on the barycentric subdivision of a simplicial complex, and in Sec. 4, we prove that the constructed function satisfies the required conditions.

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§2. Preliminaries

We start with definitions.

A discrete Morse function on a simplicial complex [1]**.** Let L be a regular simplicial complex. By α^p, β^p in this section we denote its p-dimensional simplices, or, for short, *p-simplices.*

A *discrete vector field* on L is a set of pairs

$$\left(\alpha^p, \beta^{p+1}\right)$$

of its simplices such that

(1) each simplex of the complex occurs in at most one pair;

(2) in each pair, the simplex α^p is a facet of β^{p+1}.

Given a discrete vector field, a *path* of dimension $p+1$, or a $(p+1)$-*path*, is a sequence of simplices

$$\beta_0^{p+1},\ \alpha_1^p,\ \beta_1^{p+1},\ \alpha_2^p,\ \beta_2^{p+1}, ..., \alpha_m^p,\ \beta_m^{p+1},\ \alpha_{m+1}^p$$

that satisfies the following conditions:

(1) each $\left(\alpha_i^p,\ \beta_i^{p+1}\right)$ is a pair in this discrete vector field for every i;
(2) whenever α and β are neighbors in the path, α is a facet of β;
(3) $\alpha_i \neq \alpha_{i+1}$.

Every path consists of "face"-steps (that is, transitions from a simplex β_i^{p+1} to one of its faces α_{i+1}^p) and "pair"-steps (that is, transitions from a simplex α_i^p to the simplex β_i^{p+1}, where these simplices form a pair in the discrete vector field).

A path is *closed* if $\alpha_{m+1}^p = \alpha_0^p$.

A *discrete Morse function on a regular simplicial complex* is a discrete vector field with no closed paths in it.

Assuming that a discrete Morse function is fixed, the *critical simplices* are those simplices of the complex that are not paired. We denote the set of all critical simplices of a discrete Morse function F by $\mathrm{Crit}(F)$.

A *gradient* $(p+1)$-*path* of a discrete Morse function leading from one critical simplex β^{p+1} to another critical simplex α^p is a $(p+1)$-path that leads from β^{p+1} to α^p:

$$\beta^{p+1},\ \ \alpha_1^p,\ \beta_1^{p+1},\ \alpha_2^p,\ \beta_2^{p+1},\ ...,\ \alpha_m^p,\ \beta_m^{p+1},\ \alpha^p.$$

We denote the set of all gradient paths of a discrete Morse function F by $\mathrm{Gr}\,(F)$.

2.1. The barycentric subdivision.

Definition 2.1. Let L be a simplicial complex. Then its barycentric subdivision $\Delta(L)$ is a simplicial complex such that the vertices of $\Delta(L)$ are in a bijective correspondence with the set of all simplices of L and the subset of vertices in $\Delta(L)$ forms a simplex if and only if the corresponding simplices form a chain in the poset of all simplices of L.

For any two simplices $\alpha, \beta \in \Delta(L)$, we have $\alpha \in \beta$ if and only if the chain that corresponds to α is a subchain of the chain that corresponds to β.

A regular realization of $\Delta(L)$ can be constructed for any regular realisation of L as follows. We realize every vertex of $\Delta(L)$ as the barycenter of the realization of the corresponding simplex of L, as depicted in Fig. 1. The realization of a simplex $\alpha \in \Delta(L)$ lies in the interior of the realization of a simplex $\beta \in L$ if and only if β is the last simplex in the chain of simplices that corresponds to α.

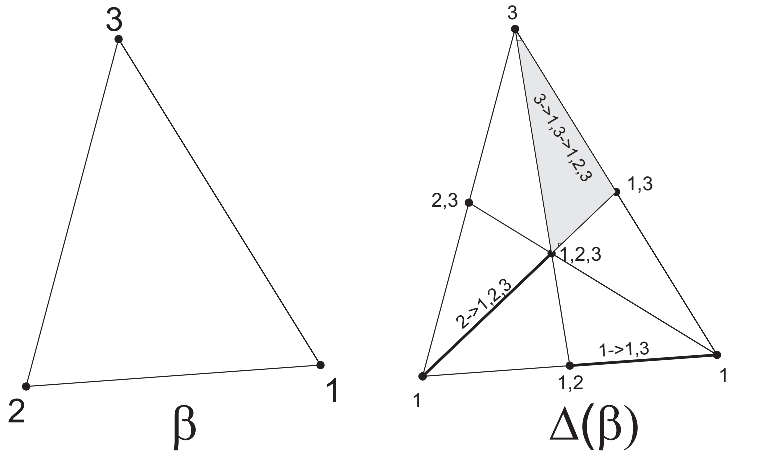

Fig. 1. The barycentric subdivision of a 2-simplex with the chains corresponding to some simplices.

To simplify our construction, we transform chains of simplices of L in the following way. We turn the chain

$$s_1 \to s_2 \to \cdots \to s_{k+1}$$

that corresponds to a simplex γ into the ordered set

$$\lambda(\gamma) = \{s_1, s_2 \setminus s_1, \ldots, s_{k+1} \setminus s_k\},$$

which we call the *label* of γ. It is a linearly ordered partition of the set s_{k+1}.

In this notation, a simplex $\alpha' \in \Delta(L)$ is a face of a simplex $\alpha \in \Delta(L)$ if and only if the partition $\lambda(\alpha)$ can be turned into a refinement of the partition $\lambda(\alpha')$ by deleting some sets from the end of $\lambda(\alpha)$. We will use this geometric picture in our proofs.

Example. Consider the triangle in $\Delta(L)$ with the label

$$(\{a, f\}\ \{d\}\ \{t\}).$$

Its faces have the following labels:

- 1-dimensional (edges): $(\{a, f\}\ \{d, t\})$, $(\{a, f, d\}\ \{t\})$, $(\{a, f\}\ \{d\})$;
- 0-dimensional (vertices): $(\{a, f, d, t\}), (\{a, f, d\}), (\{a, f\})$.

§3. The pairing on the complex $\Delta(L)$

Assume that we have a discrete Morse function F on a regular simplicial complex L. In this section, we define a discrete vector field on the complex $\Delta(L)$. We deal with pieces of critical and noncritical simplices of L in two different ways.

3.1. Noncritical simplices. Let $(\alpha^{n-1}, \beta^n) \in F$. We renumber the vertices so that $\alpha = [n] = \{1, 2, \dots, n\}, \beta = [n+1] = \{1, 2, \dots, n+1\}$. This renumbering is almost arbitrary, except that $n + 1 = \beta \backslash \alpha$. The pairing that we define below does not depend on this renumbering, so it is made only for convenience.

We define pairings on all simplices of $\Delta(L)$ that lie in the interiors of α and β. They are exactly all simplices whose labels are the subdivisions of $[n]$ and $[n+1]$. Let γ be a k-simplex with such a label. Consider four possible cases:

(1) $\lambda(\gamma)$ is an ordered subdivision of $[n]$, i.e., $\gamma \in \Delta(\alpha)$. Then we obtain a pair for γ by adding the singleton $\{n+1\}$ to the right end of $\lambda(\gamma)$. We get a $(k+1)$-simplex that belongs to case 2.

(2) The entry $n+1$ forms a singleton in $\lambda(\gamma)$, and it is the last set in $\lambda(\gamma)$. Then we obtain a pair for γ by deleting $\{n+1\}$ from $\lambda(\gamma)$. We get a $(k-1)$-simplex that belongs to case 1.

(3) The entry $n+1$ forms a singleton in $\lambda(\gamma)$, and it is not the last set in $\lambda(\gamma)$. Then we obtain a pair for γ by uniting the singleton $\{n+1\}$ with the set that follows it. We get a $(k-1)$-simplex that belongs to case 4.

(4) The entry $n+1$ lies in a non-singleton set in $\lambda(\gamma)$. Then we obtain a pair for γ by splitting off the entry $n+1$ to the left of the set containing it and forming a singleton $\{n+1\}$. We get a $(k+1)$-simplex that belongs to case 3.

It is easy to see that every simplex occurs in exactly one pair.

Example. If $n + 1 = 5$, then we will have pairs of simplices such as the two below:

$$((\{1\}\ \{3, 4\}\ \{2, 5\}),\ (\{1\}\ \{3, 4\}\ \{5\}\ \{2\}))$$

and

$$(\{1\}\ \{3,4\}\ \{2\}),\ (\{1\}\ \{3,4\}\ \{2\}\ \{5\}))).$$

Now we prove that this pairing defines a Morse function inside $\Delta(\beta)$.

Lemma 3.1. *There are no cyclic paths in the vector field on $\Delta(\beta)$ defined above.*

Proof. Assume that we have a path Γ in the pairing defined above. Consider the position of the entry $n+1$ in the labels of the simplices in Γ. In every pair-step, this entry leaves some non-singleton set and forms a singleton. Therefore, every face-step, except maybe the first or the last step in the path, is performed by adding this entry to the set next to it. If we add this entry to the right set, then at the next pair-step we will immediately return back to this simplex, which is forbidden. So every face-step is defined uniquely, and the entry $n+1$ travels to the left side of the label during the path. Therefore, no path is cyclic. □

As we will see in Lemma 4.2, we are interested only in n-paths inside $\Delta(\beta)$. The lemma below follows from the construction of the pairing and the definition of the barycentric subdivision.

Lemma 3.2. *Let $\lambda(\gamma) = (I_1\ I_2 \dots I_{n+1})$ be an n-simplex in $\Delta(\beta)$ (note that in this case, all the sets I_i are singletons). Then the following statements hold.*

(1) *There is exactly one $(n-1)$-face of γ that lies on the boundary of β, and it is the simplex with the label $(I_1\ I_2 \dots I_n)$.*
(2) *The simplex γ is paired with the $(n-1)$-simplex given above if and only if $I_{n+1} = \{n+1\}$.*
(3) *If $I_1 \neq \{n+1\}$, then there is exactly one $(n-1)$-face of γ that is paired with another n-simplex of $\Delta(\beta)$. It has the label obtained by uniting in $\lambda(\gamma)$ the singleton $\{n+1\}$ with the singleton that precedes it. This $(n-1)$-face of γ is paired with the simplex γ' whose label can be obtained from $\lambda(\gamma)$ by interchanging the singleton $\{n+1\}$ with the singleton that precedes it.*
(4) *If $I_1 = \{n+1\}$, then there are no $(n-1)$-faces of γ that are paired with other n-simplices of $\Delta(\beta)$.*

This lemma shows that if we construct an n-path that goes through $\Delta(\beta)$, we do not have much choice. We can start from any $(n-1)$-simplex in $\Delta(\alpha)$ and follow the pairings. At each face step, we can either go to the boundary of $\Delta(\beta)$ in a uniquely defined way (Lemma 3.2, Claim 1), or go further in a uniquely defined way (Lemma 3.2, Claim 3), until we arrive at simplices that lie near the vertex $n+1$ and leave $\Delta(\beta)$ (Lemma 3.2, Claim 4).

Informally, these paths form a "flow" from α in the direction of the vertex $p+1$ (see Fig. 2 for an example).

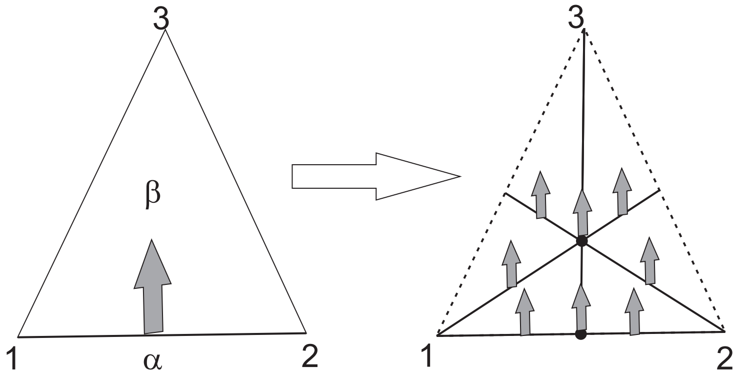

Fig. 2. The pairings on the barycentric subdivision of a pair (α^n, β^{n+1}) for $n+1=3$.

The next lemma follows from the above.

Lemma 3.3. *Assume that a gradient n-path of $\Delta(F)$ goes through $\Delta(\beta)$ and the last simplex of $\Delta(\beta)$ in this path is the $(n-1)$-simplex $\lambda(\gamma) = (I_1\ I_2 \ldots I_n)$ on the boundary of β. Assume that $I_j = \{n+1\}$. Then the first simplex of this path belongs to $\Delta(\alpha)$ and has the label*

$$(I_1\ I_2 \ldots I_{j-1}\ I_{j+1}\ \ldots\ I_n\ [n+1] \setminus \bigcup_{i=1}^{n} I_i).$$

Example. Let $n+1=5$. If an n-path goes through $\Delta(\beta)$ and the last simplex of $\Delta(\beta)$ in this path is the $(n-1)$-simplex labeled by $(\{1\}\ \{3\}\ \{5\}\ \{4\})$, then the first simplex of $\Delta(\beta)$ in this path is the simplex labeled by

$(\{1\}\ \{3\}\ \{4\}\ \{2\})$ and the path looks as follows:

$$\begin{aligned}&(\{1\}\ \{3\}\ \{4\}\ \{2\})\\&(\{1\}\ \{3\}\ \{4\}\ \{2\}\ \{5\})\\&(\{1\}\ \{3\}\ \{4\}\ \{2,5\})\\&(\{1\}\ \{3\}\ \{4\}\ \{5\}\ \{2\})\\&(\{1\}\ \{3\}\ \{4,5\}\ \{2\})\\&(\{1\}\ \{3\}\ \{5\}\ \{4\}\ \{2\})\\&(\{1\}\ \{3\}\ \{5\}\ \{4\}).\end{aligned}$$

3.2. Critical simplices. Let α^n be a critical n-simplex of F. We relabel its vertices so that $\alpha = [n+1] = \{1, 2, \dots, n+1\}$. We will pair the simplices of $\Delta(L)$ that lie in the inner part of α, i.e., those whose labels are the ordered subdivisions of the set $[n+1]$. We leave only one n-simplex non-paired, namely, the simplex with the label

$$\lambda(\alpha') = (\{n+1\}\ \{n\} \dots \{1\}).$$

This simplex depends on our renumbering, and, given an arbitrary n-simplex in $\Delta(\alpha)$ with label Ord_α, we can renumber the vertices in α in the order opposite to Ord_α to make this n-simplex critical in our construction.

Let γ be a k-simplex in $\Delta(\alpha)$ with the label $\lambda(\gamma) = (I_1\ I_1 \dots I_{k+1})$. Let i be the length of the longest common suffix of $\lambda(\gamma)$ and $\lambda(\alpha')$ (i.e., for every $j \leqslant i$, the jth sets from the end in $\lambda(\gamma)$ and $\lambda(\alpha')$ coincide). Three cases are possible:

(1) $i = n$. Then $\gamma = \alpha'$ and we do not pair it.
(2) $i < n$ and the entry $i+1$ forms a singleton in $\lambda(\gamma)$. Then we pair the simplex γ with the $(k-1)$-simplex whose label is obtained from $\lambda(\gamma)$ by uniting this singleton with the set that follows it. This does not change the longest common suffix with $\lambda(\alpha')$.
(3) $i < n$ and the entry $i+1$ lies in a non-singleton set in $\lambda(\gamma)$. Then we pair the simplex γ with the $(k+1)$-simplex whose label is obtained from $\lambda(\gamma)$ by splitting this entry off this set to the left of it and forming a singleton. This does not change the longest common suffix with $\lambda(\alpha')$.

We always pair one simplex of type 2 with a unique simplex of type 3, so this pairing is well defined (see Fig. 3 for an example).

Example. For $n = 5$, we build such pairs as

$$((\{1,2\}\ \{3\}\ \{4,5\}),\ (\{1\}\ \{2\}\ \{3\}\ \{4,5\})),$$

where the length of the longest common suffix with $\lambda(\alpha')$ is 0, and

$$((\{4,5\}\ \{3\}\ \{2\}\ \{1\}),\ (\{4\}\ \{5\}\ \{3\}\ \{2\}\ \{1\})),$$

where the length of the longest common suffix with $\lambda(\alpha')$ is 3.

Now we prove that this pairing forms a Morse function on $\Delta(\alpha)$.

Lemma 3.4. *There are no cyclic paths in the vector field on $\Delta(\alpha)$ defined above.*

Proof. Let Γ be a path in this field. Consider the longest common suffix of the labels of simplices in Γ and $\lambda(\alpha')$. It never grows during the path. If it gets shorter, then the path cannot be cyclic.

Assume that this suffix stays the same during Γ and its length is i. Then all the pair-steps in Γ are performed by splitting the entry $i+1$ off a non-singleton set to the left. Therefore, all the face-steps in Γ, except maybe the first and the last one, are performed by adding this entry to the set to the left of it. So, the entry $i+1$ travels to the left in the label, and Γ cannot be cyclic. $\square$

Lemma 3.5. *For every $(n-1)$-simplex $\gamma \in \Delta(\alpha)$ that lies on the boundary of α there is a unique n-path in $\Delta(\alpha)$ that starts at α' and exits $\Delta(\alpha)$ through γ.*

Proof. As we already know, there is a unique n-simplex $\gamma' \in \Delta(\alpha)$ that has γ on its boundary. Our path has to go through γ'.

The longest common suffix with $\lambda(\alpha')$ can only decrease during the path (at the start, its length is $n+1$). For arbitrary i, the entry i moves inside the label during the path only when this suffix has length $i-1$. So, by the time the length of this suffix becomes smaller than $i-1$, the permutation of the entries $i, i+1, \ldots, n+1$ in the label is fixed and does not change any more. If such a path exists, then all entries appear in decreasing order, and this implies the uniqueness of the path. Knowing that, it is not hard to construct such a path. For example, if $n = 4$ and $\gamma = (\{2\}\ \{4\}\ \{5\}\ \{1\})$, then

$$\gamma' = (\{2\}\ \{4\}\ \{5\}\ \{1\}\ \{3\})$$

and the path looks as follows:

$$
\begin{aligned}
&\alpha' = (\{5\}\ \{4\}\ \{3\}\ \{2\}\ \{1\})\\
&(\{4,5\}\ \{3\}\ \{2\}\ \{1\})\\
&(\{4\}\ \{5\}\ \{3\}\ \{2\}\ \{1\})\\
&(\{4\}\ \{5\}\ \{2,3\}\ \ \{1\})\\
&(\{4\}\ \{5\}\ \{2\}\ \{3\}\ \{1\})\\
&(\{4\}\ \{2,5\}\ \{3\}\ \{1\})\\
&(\{4\}\ \{2\}\ \{5\}\ \{3\}\ \{1\})\\
&(\{2,4\}\ \{5\}\ \{3\}\ \{1\})\\
&(\{2\}\ \{4\}\ \{5\}\ \{3\}\ \{1\})\\
&(\{2\}\ \{4\}\ \{5\}\ \{1,3\}),\\
&\gamma' = (\{2\}\ \{4\}\ \{5\}\ \{1\}\ \{3\}),\\
&\gamma = (\{2\}\ \{4\}\ \{5\}\ \{1\}).
\end{aligned}
$$

□

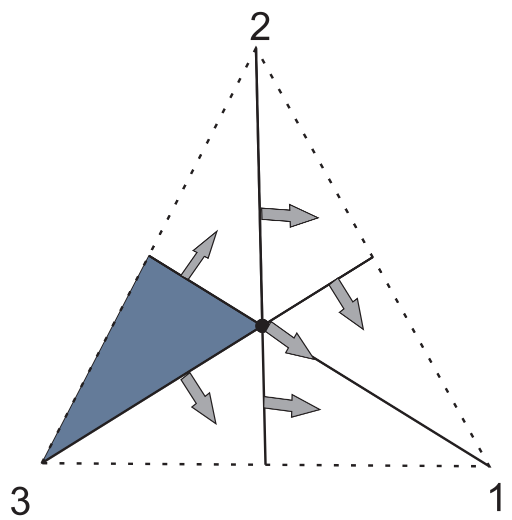

Fig. 3. The pairings on the barycentric subdivision of a simplex α for $n = 2$.

Any gradient path that contains n-simplices from $\Delta(\alpha)$ can look in one of the two following ways.

(1) It starts from α' and makes steps inside $\Delta(\alpha)$. Then at some face-step, this path exits $\Delta(\alpha)$ through an $(n-1)$-simplex on the boundary. According to Lemma 3.5, this path is uniquely determined by its exit simplex.

(2) It enters α' at some pair-step from some $(n+1)$-simplex $\gamma \in \Delta(L)$. From the structure of the barycentric subdivision it follows that the simplex γ lies in the barycentric subdivision of some $(n+1)$-simplex β of L, and the simplex α is a facet of β. This path is an $(n+1)$-path, and it terminates at α'.

§4. Paths on the barycentric subdivision

Claim 1 of Theorem 1.1 follows from the construction of $\Delta(F)$. In this section, we prove the rest of Theorem 1.1.

Lemma 4.1. *Let F be a discrete Morse function on a simplicial complex F. Then the pairing $\Delta(F)$ constructed in Sec. 3 is a Morse function.*

Proof. We need to prove that no path in $\Delta(F)$ is cyclic. By Lemmas 3.1 and 3.4, this is true for a path that stays inside one simplex of L.

Let Γ be a path in $\Delta(F)$, and let Γ include parts of more than one simplex of L. Every simplex $\gamma \in \Gamma$ lies in the inner part of some simplex in L. Take these simplices as a sequence and delete the repetitions. The resulting sequence Γ' is cyclic if Γ is cyclic.

From the definition of $\Delta(F)$ it follows that for any two consecutive simplices in Γ', one is a facet of the other. Moreover, if the simplex with the lower dimension precedes the simplex with the higher dimension, then these two simplices are paired. So, the sequence Γ' consists of face-steps and pair-steps. By the definition of a Morse function, no two pair-steps can be consecutive.

If no two face-steps are consecutive in Γ', then Γ' is a path in F and cannot be cyclic. If there are at least two consecutive face-steps, then the dimension of the simplices decreases during Γ' more times than it increases, and Γ' cannot be cyclic. □

Now we consider, for arbitrary n, how the gradient n-paths behave in $\Delta(F)$. If a gradient n-path starts at a critical simplex α', then it leaves the corresponding simplex α of L through an $(n-1)$-simplex that lies in an $(n-1)$-face of α. If an n-path enters $\Delta(\beta)$, where β is noncritical, through an $(n-1)$-simplex on its boundary, then it leaves this simplex through another $(n-1)$-simplex that lies on the boundary of β as well. So, this path never gets out of the n-simplices of L. We obtain the following lemma.

Lemma 4.2. *Assume that* Γ *is a gradient* n*-path. Then all* n*-simplices in* Γ *lie in the interiors of* n*-simplices of* L.

In other words, a gradient n-path in $\Delta(F)$ never gets inside the simplices of L of dimensions higher than n.

Now we prove that the critical path structure of the function $\Delta(F)$ is isomorphic to the gradient path structure of the function F. We do it by constructing two maps between the set of gradient paths $\mathrm{Gr}\,(F)$ of the function F and the set of gradient paths $\mathrm{Gr}\,(\Delta(F))$ of the function $\Delta(F)$. These maps are one-to-one, they are opposite to each other, and they respect our bijection $\mathrm{Crit}(F) \to \mathrm{Crit}(\Delta(F))$.

The map $\mathrm{Gr}\,(\Delta(F)) \to \mathrm{Gr}\,(F)$**.**

Let Γ be a gradient n-path in $\Delta(F)$. We construct the corresponding sequence Γ' of simplices of L, as we did in the proof of Lemma 4.1. By Lemma 4.2 and by the construction of $\Delta(F)$, this sequence is an n-path in F. Moreover, it starts and ends in critical points, since only critical points of F contain critical points of $\Delta(F)$. These critical simplices correspond to the beginning and the end of Γ.

Therefore, Γ' is a gradient path of F.

The map $\mathrm{Gr}\,(\Delta(F)) \to \mathrm{Gr}\,(F)$**.**

Let Γ be a gradient n-path in F from a simplex β to a simplex α:

$$\beta = \beta_0,\ \alpha_1,\ \beta_1,\ \alpha_2,\ \beta_2,\ \ldots,\ \alpha_k,\ \beta_k,\ \alpha.$$

We construct a corresponding path $\Delta(\Gamma)$ in $\Delta(F)$ that goes from β' to α'. We define the path inside $\Delta(\beta_i)$ for each i successively, starting from the end of Γ. Our path exits $\Delta(\beta_k)$ through α', which, by Lemma 3.1, determines the path in $\Delta(\beta_k)$ uniquely. For every i, $1 \leqslant i \leqslant k$, the first simplex of the path constructed in $\Delta(\beta_i)$ becomes the last simplex of the path in $\Delta(\beta_{i-1})$ and determines the path in $\Delta(\beta_{i-1})$ uniquely. For the simplex β, the same holds by Lemma 3.4. Therefore, the path $\Delta(\Gamma)$ is defined uniquely.

It is easy to see that for every gradient path Γ in F, we have $(\Delta(\Gamma))' = \Gamma$, and from Lemma 4.2 it follows that for every gradient path Γ in $\Delta(F)$, we have $(\Delta(\Gamma')) = \Gamma$. Therefore, our maps define a bijection between the path structure on F and $\Delta(F)$. Theorem 1.1 is proved.

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