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ON AN INVERSE DYNAMIC PROBLEM FOR THE WAVE EQUATION WITH A POTENTIAL ON A REAL LINE

Abstract. We consider the inverse dynamic problem for the wave equation with a potential on a real line. The forward initial-boundary value problem is set up with a help of boundary triplets. As an inverse data we use an analog of a response operator (dynamic Dirichlet-to-Neumann map). We derive equations of inverse problem and also point out the relationship between dynamic inverse problem and spectral inverse problem from a matrix-valued measure.

§1. Introduction

For a potential $q \in C^2(R) \cap L_1(\mathbb{R})$ we consider an operator H in $L_2(\mathbb{R})$ given by

$$(Hf)(x) = -f''(x) + q(x)f(x), \quad x \in \mathbb{R},$$
$$\operatorname{dom} H = \{f \in H^2(\mathbb{R}) \,|\, f(0) = f'(0) = 0\}.$$

Then

$$(H^*f)(x) = -f''(x) + q(x)f(x), \quad x \in \mathbb{R},$$
$$\operatorname{dom} H^* = \{f \in L_2(\mathbb{R}) \,|\, f \in H^2(-\infty, 0),\ f \in H^2(-\infty, 0)\}.$$

For a continuous function g we denote

$$g_\pm := \lim_{\varepsilon \to 0} g(0 \pm \varepsilon).$$

Let $B := \mathbb{R}^2$. The *boundary operators* $\Gamma_{0,1} : \operatorname{dom} H^* \mapsto B$ are introduced by the rules

$$\Gamma_0 w := \begin{pmatrix} w_+ - w_- \\ w'_+ - w'_- \end{pmatrix}, \quad \Gamma_1 w := \frac{1}{2}\begin{pmatrix} w'_+ + w'_- \\ -w_+ - w_- \end{pmatrix}.$$

Key words and phrases: inverse problem, Schrödinger operator, wave equation, Boundary Control method, boundary triplets.

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Integrating by parts for $u, v \in \operatorname{dom} H^*$ shows that the abstract second Green identity holds:

$$(H^*u, v)_{L_2(\mathbb{R})} - (u, H^*v)_{L_2(\mathbb{R})} = (\Gamma_1 u, \Gamma_0 v)_B - (\Gamma_0 u, \Gamma_1 v)_B\,.$$

The mapping

$$\Gamma := \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} : \operatorname{dom} H^* \mapsto B \times B$$

evidently is surjective. Then a triplet $\{B, \Gamma_0, \Gamma_1\}$ is a *boundary triplet* for H^* (see [9]). With the help of boundary triplets one can describe self-adjoint extensions of H, see [10,12,16]. In [6] the authors used the concept of boundary triplets to set up and study a boundary value problem for abstract dynamical system with boundary control in Hilbert space, they also used it for the purpose of describing the special (wave) model of the one-dimensional Schrödinger operator on an interval [8].

Let $T > 0$ be fixed. We use the triplet $\{B, \Gamma_0, \Gamma_1\}$ to set up the dynamical system with special boundary control (acting in the origin) for a wave equation with a potential on a real line:

$$u_{tt} + H^*u = 0, \quad t > 0, \tag{1.1}$$

$$(\Gamma_0 u)(t) = \begin{pmatrix} f_1(t) \\ f_2'(t) \end{pmatrix}, \quad t > 0, \tag{1.2}$$

$$u(\,\cdot\,, 0) = u_t(\,\cdot\,, 0) = 0. \tag{1.3}$$

Here the function $F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, $f_1\ f_2 \in L_2(0,T)$, is interpreted as a *boundary control*. The solution to (1.1)–(1.3) is denoted by u^F. The *response operator*, the analog of a Dirichlet-to-Neumann map is introduced by the rule

$$\left(R^T F\right)(t) := \left(\Gamma_1 u^F\right)(t), \quad t > 0.$$

The speed of the wave propagation in the system (1.1)–(1.3) equal to one, that is why the natural set up of the dynamic inverse problem is to find a potential $q(x)$, $x \in (-T, T)$ from the knowledge of a response operator R^{2T} (cf. [1,3,7]).

In the second section we derive the representation formula for the solution u^F and introduce the operators of the Boundary Control method. In the third section we derive Krein and Gelfand–Levitan equations of the dynamic inverse problem and point out the the relationship between dynamic and spectral inverse problems.

§2. Forward problem, operators of the Boundary Control method

It is straightforward to check that when $q = 0$, the solution to (1.1)–(1.3) is given by:

$$u^F(x,t) = \begin{cases} \frac{1}{2}f_1(t-x) - \frac{1}{2}f_2(t-x), & x > 0, \\ -\frac{1}{2}f_1(t+x) - \frac{1}{2}f_2(t+x), & x < 0, \\ 0, & 0 < t < |x|. \end{cases}$$

Everywhere we consider operators acting in L_2-spaces, that is why it is reasonable to introduce the *outer space* of the system (1.1)–(1.3), the space of controls as $\mathcal{F}^T := L_2(0,T;\mathbb{R}^2)$, $F \in \mathcal{F}^T$, $F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$.

Theorem 1. *The solution to* (1.1)–(1.3) *with a control* $F \in \mathcal{F}^T \cap C_0^\infty(\mathbb{R}_+)$, *admits the following representation:*

$$u^F(x,t) = \begin{cases} \frac{1}{2}f_1(t-x) - \frac{1}{2}f_2(t-x) \\ \quad + \int_x^t w_1(x,s)f_1(t-s) \\ \quad + w_2(x,s)f_2(t-s)\,ds, & 0 < x < t, \\ -\frac{1}{2}f_1(t+x) - \frac{1}{2}f_2(t+x) \\ \quad + \int_{-x}^t w_1(x,s)f_1(t-s) \\ \quad + w_2(x,s)f_2(t-s)\,ds, & 0 < -x < t, \\ 0, & 0 < t < |x|. \end{cases} \tag{2.1}$$

where kernels $w_1(x,t)$ *and* $w_2(x,t)$ *satisfy the following Goursat problems:*

$$\begin{cases} w_{1tt}(x,t) - w_{1xx}(x,t) + q(x)w_1(x,t), & 0 < |x| < t, \\ \frac{d}{dx}w_1(x,x) = -\frac{q(x)}{4}, & x > 0, \\ \frac{d}{dx}w_1(x,-x) = -\frac{q(x)}{4}, & x < 0, \end{cases} \tag{2.2}$$

$$\begin{cases} w_{2tt}(x,t) - w_{2xx}(x,t) + q(x)w_2(x,t), & 0 < |x| < t, \\ \frac{d}{dx}w_2(x,x) = \frac{q(x)}{4}, & x > 0, \\ \frac{d}{dx}w_2(x,-x) = -\frac{q(x)}{4}, & x < 0. \end{cases} \tag{2.3}$$

Proof. Take arbitrary $F \in \mathcal{F}^T \cap C_0^\infty(0,T;\mathbb{R}^2)$ and look for u^F in the form (2.1). Then for $x > 0$ we have:

$$
\begin{aligned}
u_{xx}(x,t) = &\frac{1}{2}f_1''(t-x) - \frac{1}{2}f_2''(t-x) - \frac{d}{dx}w_1(x,x)f_1(t-x)\\
&+ w_1(x,x)f_1'(t-x) - \frac{d}{dx}w_2(x,x)f_2(t-x) + w_2(x,x)f_2'(t-x)\\
&- w_{1\,x}(x,x)f_1(t-x) - w_{2x}(x,x)f_2(t-x)\\
&+ \int_x^t w_{1\,xx}(x,s)f_1(t-s) + w_{2\,xx}(x,s)f_2(t-s)\,ds,
\end{aligned}
$$

$$
\begin{aligned}
u_{tt}(x,t) = &\frac{1}{2}f_1''(t-x) - \frac{1}{2}f_2''(t-x) + w_1(x,x)f_1'(t-x)\\
&+ w_2(x,x)f_2'(t-x) + w_{1\,s}(x,x)f_1(t-x) + w_{2\,s}(x,x)f_2(t-x)\\
&+ \int_x^t \left(w_{1\,ss}(x,s)f_1(t-s) + w_{2\,ss}(x,s)f_2(t-s)\right)\,ds,
\end{aligned}
$$

Plugging these expressions into (1.1), we obtain that for $x > 0$ the following relation holds true:

$$
\begin{aligned}
0 = &\int_x^t ((w_{1\,ss}(x,s) - w_{1\,xx}(x,s) + q(x)w_1(x,s))f_1(t-s)\\
&+ (w_{2\,ss}(x,s) - w_{2\,xx}(x,s) + q(x)w_2(x,s))f_2(t-s))\,ds\\
&+ f_1(t-x)\Big[2\frac{d}{dx}w_1(x,x) + \frac{q(x)}{2}\Big]\\
&+ f_2(t-x)\Big[2\frac{d}{dx}w_2(x,x) - \frac{q(x)}{2}\Big].
\end{aligned}
\tag{2.4}
$$

Similarly, for $x < 0$:

$$
\begin{aligned}
u_{xx}(x,t) = &-\frac{1}{2}f_1''(t+x) - \frac{1}{2}f_2''(t+x)\\
&+ \frac{d}{dx}w_1(x,-x)f_1(t+x) + w_1(x,-x)f_1'(t+x)\\
&+ \frac{d}{dx}w_2(x,-x)f_2(t+x) + w_2(x,-x)f_2'(t+x)\\
&+ w_{1\,x}(x,-x)f_1(t+x) + w_{2x}(x,-x)f_2(t+x)
\end{aligned}
$$

$$+\int_{-x}^{t} w_{1\,xx}(x,s)f_1(t-s)+w_{2\,xx}(x,s)f_2(t-s)\,ds,$$

$$\begin{aligned}u_{tt}(x,t)=&-\frac{1}{2}f_1''(t+x)-\frac{1}{2}f_2''(t+x)\\&+w_1(x,-x)f_1'(t+x)+w_2(x,-x)f_2'(t+x)\\&+w_{1\,s}(x,-x)f_1(t+x)+w_{2\,s}(x,-x)f_2(t+x)\\&+\int_{-x}^{t}\big(w_{1\,ss}(x,s)f_1(t-s)+w_{2\,ss}(x,s)f_2(t-s)\big)\,ds.\end{aligned}$$

Then for $x<0$ we have the equality:

$$\begin{aligned}0=&\int_{-x}^{t}\big((w_{1\,ss}(x,s)-w_{1\,xx}(x,s)+q(x)w_1(x,s))f_1(t-s)\\&+(w_{2\,ss}(x,s)-w_{2\,xx}(x,s)+q(x)w_2(x,s))f_2(t-s)\big)\,ds\\&+f_1(t+x)\Big[-2\frac{d}{dx}w_1(x,-x)-\frac{q(x)}{2}\Big]\\&+f_2(t+x)\Big[-2\frac{d}{dx}w_2(x,-x)-\frac{q(x)}{2}\Big].\end{aligned}\tag{2.5}$$

The condition $\Gamma_0 u=F$ at $x=0$ yields that

$$\begin{aligned}u^+(\,\cdot\,,t)-u^-(\,\cdot\,,t)=&f_1(t)\\&+\int_0^t(w_1{}^+(0,s)-w_1{}^-(0,s))f_1(t-s)\\&+(w_2{}^+(0,s)-w_2{}^-(0,s))f_2(t-s)\,ds,\\u_x^+(\,\cdot\,,t)-u_x^-(\,\cdot\,,t)=&f_2'(t)\\&+\int_0^t(w_1{}_x^+(0,s)-w_1{}_x^-(0,s))f_1(t-s)\\&+(w_2{}_x^+(0,s)-w_2{}_x^-(0,s))f_2(t-s)\,ds,\end{aligned}$$

The above equalities imply the continuity of kernels w_1, w_2 at $x = 0$:

$$w_1{}^+(0,s) = w_1{}^-(0,s), \quad w_2{}^+(0,s) = w_2{}^-(0,s), \tag{2.6}$$

$$w_1{}_x^+(0,s) = w_1{}_x^-(0,s), \quad w_2{}_x^+(0,s) = w_2{}_x^-(0,s). \tag{2.7}$$

Using the arbitrariness of $F \in \mathcal{F}^T \cap C_0^\infty(0,T;\mathbb{R}^2)$ in (2.4), (2.5) and continuity conditions (2.6), (2.6), we obtain that w_1, w_2 satisfy (2.2), (2.3). $\square$

Remark 1. When $F \in \mathcal{F}^T$, the function u^F defined by (2.1) is a generalized solution to (1.1)–(1.3).

The *response operator* $R^T : \mathcal{F}^T \mapsto \mathcal{F}^T$ with the domain

$$D_R = \left\{\mathcal{F}^T \cap C_0^\infty(0,T;\mathbb{R}^2)\right\}$$

is defined by

$$(R^T F)(t) := \left(\Gamma_1 u^F\right)(t), \quad 0 < t < T.$$

Representation (2.1) implies that the response operator has a form:

$$(R^T F)(t) = \begin{pmatrix} (R_1 F)(t) \\ (R_2 F)(t) \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} f_1'(t) \\ -f_2(t) \end{pmatrix} + R * \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} f_1'(t) + \int\limits_0^t \left(w_1{}_x(0,s) f_1(t-s) + w_2{}_x(0,s) f_2(t-s)\right) ds \\ \frac{1}{2} f_2(t) - \int\limits_0^t \left(w_1(0,s) f_1(t-s) + w_2(0,s) f_2(t-s)\right) ds \end{pmatrix}, \tag{2.8}$$

where

$$R(t) := \begin{pmatrix} r_{11}(t) & r_{12}(t) \\ r_{21}(t) & r_{22}(t) \end{pmatrix} = \begin{pmatrix} w_1{}_x(0,t) & w_2{}_x(0,t) \\ -w_1(0,t) & -w_2(0,t) \end{pmatrix}$$

is a *response matrix*. We introduce the *inner space*, the space of states of system (1.1)–(1.3) as $\mathcal{H}^T := L_2(-T,T)$. The representation (2.1) implies that $u^F(\,\cdot\,,T) \in \mathcal{H}^T$.

A *control operator* $W^T : \mathcal{F}^T \mapsto \mathcal{H}^T$ is defined by the formula $W^T F := u^F(\,\cdot\,,T)$. The *reachable set* is defined by the rule

$$U^T := W^T \mathcal{F}^T = \left\{u^F(\,\cdot\,,T) \,\middle|\, F \in \mathcal{F}^T\right\}.$$

We introduce the notations:

$$S := \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}, \quad J^T : \mathcal{F}^T \mapsto \mathcal{F}^T, \quad \left(J^T F\right)(t) = F(T-t),$$

and note that

$$S = S^*,\ SS = \frac{1}{2} I.$$

It will be convenient for us to associate the outer space $\mathcal{H}^T = L_2(-T,T)$ with a vector space $L_2(0,T;\mathbb{R}^2)$ by setting for $a \in L_2(-T,T)$ (we keep the same notation for a function)

$$a = \begin{pmatrix} a_1(x) \\ a_2(x) \end{pmatrix} \in L_2(0,T;\mathbb{R}^2), \quad a_1(x) := a(x), \quad a_2(x) := a(-x), \quad x \in (0,T).$$

Thus, bearing in mind this association, we consider the control operator W^T, which maps $\mathcal{F}^T$ to $\mathcal{H}^T = L_2(0,T;\mathbb{R}^2)$, acting (cf. (2.1)) by the rule:

$$\left(W^T F\right)(x) = \begin{pmatrix} \frac{1}{2} f_1(T-x) - \frac{1}{2} f_2(T-x) \\ -\frac{1}{2} f_1(T-x) - \frac{1}{2} f_2(T-x) \end{pmatrix} + \begin{pmatrix} \int_x^T w_1(x,s) f_1(T-s) + w_2(x,s) f_2(T-s)\, ds \\ \int_x^T w_1(-x,s) f_1(T-s) + w_2(-x,s) f_2(T-s)\, ds \end{pmatrix}.$$

On introducing the operator $W : \mathcal{F}^T \mapsto \mathcal{H}^T = L_2(0,T;\mathbb{R}^2)$ defined by the formula

$$(WF)(x) = \begin{pmatrix} \int_x^T w_1(x,s) f_1(s) + w_2(x,s) f_2(s)\, ds \\ \int_x^T w_1(-x,s) f_1(s) + w_2(-x,s) f_2(s)\, ds \end{pmatrix}$$

and noting that $\mathcal{F}^T = \mathcal{H}^T$, we can without abusing the notations rewrite W^T in a form:

$$W^T F = S\,(I + 2SW)\,J^T F = S\,(I + K)\,J^T F, \tag{2.9}$$

where

$$K = 2SW, \quad (KF)(x) = \begin{pmatrix} \int_x^T k_{11}(x,s) f_1(s) + k_{12}(x,s) f_2(s)\, ds \\ \int_x^T k_{21}(x,s) f_1(s) + k_{22}(x,s) f_2(s)\, ds \end{pmatrix}. \tag{2.10}$$

Theorem 2. *The control operator is a boundedly invertible isomorphism between $\mathcal{F}^T$ and $\mathcal{H}^T$, and $U^T = \mathcal{H}^T$.*

Proof. It is clear that in representation (2.9) each of the operators $S : \mathcal{H}^T \mapsto \mathcal{H}^T$, $I + K : \mathcal{F}^T \mapsto \mathcal{H}^T$, $J^T : \mathcal{F}^T \mapsto \mathcal{F}^T$ is boundedly invertible isomorphism. $\square$

The *connecting operator* $C^T : \mathcal{F}^T \mapsto \mathcal{F}^T$ is introduced via the quadratic form:

$$\left(C^T F_1, F_2\right)_{\mathcal{F}^T} = \left(u^{F_1}(\,\cdot\,,T), u^{F_2}(\,\cdot\,,T)\right)_{\mathcal{H}^T}.$$

The crucial fact in the Boundary Control method is that the connecting operator is expressed in terms of inverse dynamic data:

Theorem 3. *The connecting operator C^T admits the following representation:*

$$\left(C^T F\right)(t) = \frac{1}{2}\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} + \int_0^T C(t,s)\begin{pmatrix} f_1(s) \\ f_2(s) \end{pmatrix} ds,$$

where

$$C_{1,1}(t,s) = p_1(2T-t-s) - p_1(|t-s|), \quad p_1(s) = \int_0^s r_{11}(\alpha)\,d\alpha,$$

$$C_{1,2}(t,s) = \widetilde{p}_1(2T-t-s) - \widetilde{p}_1(t-s), \quad \widetilde{p}_1(s) = \begin{cases} \int_0^s r_{12}(\alpha)\,d\alpha, & s > 0, \\ -\int_0^{-s} r_{12}(\alpha)\,d\alpha, & s < 0, \end{cases}$$

$$C_{2,1}(t,s) = -\widetilde{r}_{21}(t-s) - \widetilde{r}_{21}(2T-t-s), \quad \widetilde{r}_{21}(s) = \begin{cases} r_{21}(s), & s > 0, \\ -r_{21}(-s), & s < 0, \end{cases}$$

$$C_{2,2}(t,s) = -r_{22}(|t-s|) - r_{22}(2T-t-s).$$

Proof. We take $F, G \in \mathcal{F}^T \cap C_0^\infty(0,T;\mathbb{R}^2)$ and introduce the Blagoveschenskii function by setting

$$\Psi(t,s) = \left(u^F(\,\cdot\,,t), u^G(\,\cdot\,,s)\right)_{\mathcal{H}^T}, \quad s,t > 0.$$

Our aim is to show that Ψ satisfy the wave equation. Indeed, using that $u^F_{tt} = -H^* u^F$ and the Green identity, we can evaluate:

$$\begin{aligned}\Psi_{tt}(t,s) - \Psi_{ss}(t,s) &= \left(-H^* u^F(\,\cdot\,,t), u^G(\,\cdot\,,s)\right)_{\mathcal{H}^T} + \left(u^F(\,\cdot\,,t), H^* u^G(\,\cdot\,,s)\right)_{\mathcal{H}^T} \\ &= \left((\Gamma_0 u^F)(t), (\Gamma_1 u^G)(s)\right)_B - \left((\Gamma_1 u^F)(t), (\Gamma_0 u^G)(s)\right)_B \\ &=: P(t,s).\end{aligned}$$

Note that Ψ satisfy $\Psi(0,s) = \Psi_t(0,s) = 0$, and that

$$\Psi(T,T) = \left(u^F(\,\cdot\,,T), u^G(\,\cdot\,,T)\right)_{\mathcal{H}^T} = \left(C^T F, G\right)_{\mathcal{F}^T}.$$

So, by d'Alembert formula:

$$\left(C^T F, G\right)_{\mathcal{F}^T} = \int_0^T \int_\tau^{2T-\tau} P(\tau,\sigma)\, d\sigma\, d\tau. \tag{2.11}$$

We rewrite the right hand side:

$$P(t,s) = \left(\begin{pmatrix} f_1(t) \\ f_2'(t) \end{pmatrix}, (RG)(s)\right)_B - \left((RF)(t), \begin{pmatrix} g_1(s) \\ g_2'(s) \end{pmatrix}\right)_B, \tag{2.12}$$

and continue the functions g_1, g_2 (we keep the same notations) from $(0,T)$ to the interval $(0,2T)$ by the rule:

$$\begin{aligned} g_1(s) &= \begin{cases} g_1(s), & 0 < s < T, \\ -g_1(2T-s), & T < s < 2T, \end{cases} \\ g_2(s) &= \begin{cases} g_2(s), & 0 < s < T, \\ g_2(2T-s), & T < s < 2T. \end{cases} \end{aligned} \tag{2.13}$$

After such a continuation the second term in (2.12) become odd in s with respect to $s = T$ and disappears after integration in (2.11), so we come to the following expression for the quadratic form:

$$\left(C^T F, G\right)_{\mathcal{F}^T} = \int_0^T \int_\tau^{2T-\tau} \left(\begin{pmatrix} f_1(\tau) \\ f_2'(\tau) \end{pmatrix}, (RG)(\sigma)\right)_B d\sigma\, d\tau. \tag{2.14}$$

Integrating by parts in (2.14) and using that $C^T = \left(C^T\right)^*$ and arbitrariness of F yields

$$\left(C^T G\right)(\tau) = \begin{pmatrix} \int_\tau^{2T-\tau} (R_1 G)(\sigma)\, d\sigma \\ (R_2 G)(\tau) + (R_2 G)(2T-\tau) \end{pmatrix}. \tag{2.15}$$

Evaluating (2.15) making use of (2.8) and continuation of g_1, g_2 (2.13), we obtain that

$$
\begin{aligned}
(C^T G)(\tau) = {} & \frac{1}{2}\begin{pmatrix} g_1(\tau) \\ g_2(\tau) \end{pmatrix} \\
& + \frac{1}{2}\begin{pmatrix} \int\limits_{\tau}^{2T-\tau}\int\limits_{0}^{\sigma}(r_{11}(s)g_1(\sigma-s)+r_{12}(s)g_2(\sigma-s))\,ds \\ -\int\limits_{0}^{\tau}(r_{21}(s)g_1(\tau-s)+r_{22}(s)g_2(\tau-s))\,ds \end{pmatrix} \\
& + \begin{pmatrix} 0 \\ \int\limits_{0}^{2T-\tau}(r_{21}(s)g_1(2T-\tau-s)+r_{22}(s)g_2(2T-\tau-s))\,ds \end{pmatrix}.
\end{aligned}
\tag{2.16}
$$

Consider the term

$$
\int\limits_{\tau}^{2T-\tau}\int\limits_{0}^{\sigma} r_{11}(s)g_1(\sigma-s)\,ds\,d\sigma = I(2T-\tau) - I(\tau), \tag{2.17}
$$

where

$$
I(\tau) = \int\limits_{0}^{\tau}\int\limits_{\alpha}^{\tau} r_{11}(\sigma-\alpha)g_1(\alpha)\,d\sigma\,d\alpha.
$$

We evaluate (2.17) using that g_1 is odd with respect to T:

$$
I(\tau) = \int\limits_{0}^{\tau}\int\limits_{0}^{|\tau-\alpha|} r_{11}(\sigma)\,d\sigma g_1(\alpha)\,d\alpha = \int\limits_{0}^{\tau} p_1(|\tau-\alpha|)g_1(\alpha)\,d\alpha, \tag{2.18}
$$

where $p_1(s) = \int\limits_0^s r_{11}(\alpha)\,d\alpha$. We can rewrite the first term in (2.17) in a form:

$$
\begin{aligned}
I(2T-\tau) &= \left(\int\limits_{0}^{T} + \int\limits_{\tau}^{2T-\tau}\right)\int\limits_{0}^{2T-\tau-\alpha} r_{11}(\sigma)\,d\sigma g_1(\alpha)\,d\alpha \\
&= \int\limits_{0}^{T} p_1(2T-\tau-\alpha)g_1(\alpha)\,d\alpha - \int\limits_{\tau}^{T} p_1(\alpha-\tau)g_1(\alpha)\,d\alpha.
\end{aligned}
\tag{2.19}
$$

Then from (2.18) and (2.19) we obtain that

$$\int_{\tau}^{2T-\tau}\int_0^{\sigma} r_{11}(s)g_1(\sigma-s)\,ds\,d\sigma = \int_0^T \left(p_1(2T-\tau-\alpha)-p_1(|\alpha-\tau|)g_1(\alpha)\right)\,d\alpha,$$

which proves the formula for C_{11}. Now we consider the term

$$\int_{\tau}^{2T-\tau}\int_0^{\sigma} r_{12}(s)g_2(\sigma-s)\,ds\,d\sigma. \tag{2.20}$$

Note that it has the same structure as (2.17), but we should take into account that $g_2(s)$ is odd with respect to $s=T$. Counting this, we have that:

$$I(2T-\tau)=\int_0^T p_2(2T-\tau-\alpha)g_2(\alpha)\,d\alpha+\int_{\tau}^T p_2(\alpha-\tau)g_2(\alpha)\,d\alpha,$$

where $p_2(s)=\int_0^s r_{12}(\alpha)\,d\alpha$. Then

$$\begin{aligned} I(2T-\tau)-I(\tau) &= \int_0^T p_2(2T-\tau-\alpha)g_2(\alpha)\,d\alpha \\ &\quad+\int_{\tau}^T p_2(\alpha-\tau)g_2(\alpha)\,d\alpha-\int_0^T p_2(|\alpha-\tau|)g_2(\alpha)\,d\alpha, \end{aligned} \tag{2.21}$$

After we introduce the notation

$$\widetilde{p}_1(s)=\begin{cases}\int_0^s r_{12}(\alpha)\,d\alpha, & s>0,\\ -\int_0^{-s} r_{12}(\alpha)\,d\alpha, & s<0,\end{cases} = \begin{cases} p_2(s), & s>0,\\ -p_2(-s), & s<0,\end{cases}$$

we can rewrite (2.20), taking into account (2.21), as

$$\int_{\tau}^{2T-\tau}\int_0^{\sigma} r_{12}(s)g_2(\sigma-s)\,ds\,d\sigma=\int_0^T \left(\widetilde{p}_1(2T-\tau-\alpha)-\widetilde{p}_1(\tau-\alpha)\right)g_2(\alpha)\,d\alpha,$$

which proves the formula for C_{12}. Similarly one can prove formulas for C_{21}, C_{22}. □

We note that the symmetry of C^T implies the restriction on the entries, specifically, the following relation should hold:

$$C_{2,1}(t,s) = C_{1,2}(t,s).$$

This equality is equivalent to

$$-\widetilde{r}_{21}(t-s) - \widetilde{r}_{21}(2T-t-s) = \widetilde{p}_1(2T-t-s) - \widetilde{p}_1(s-t),$$

which yields:

$$-\widetilde{r}_{21}(s) = \widetilde{p}_1(s).$$

Remark 2. The components of the response matrix have to be connected by the relation:

$$r_{21}'(s) = -r_{12}(s), \quad s > 0.$$

§3. Dynamic inverse problem

In this section we derive equations of inverse dynamic problem, using them we answer the question on recovering a potential $q(x)$, $x \in (-T,T)$ from the response operator R^{2T}.

3.1. Krein equations. Let $y(x)$ be a solution to the following Cauchy problem:

$$\begin{cases} -y'' + qy = 0, & x \in (-T,T), \\ y(0) = 0, & y'(0) = 1. \end{cases} \tag{3.1}$$

We set up the *special control problem*: to find $F \in \mathcal{F}^T$ such that $W^T F = y$ in $\mathcal{H}^T$. By Theorem 2, such a control F exists, but we can say even more:

Theorem 4. *The solution to a special control problem is a unique solution to the following equation:*

$$\left(C^T F\right)(t) = (T-t)\begin{pmatrix}1\\0\end{pmatrix}, \quad t \in (0,T). \tag{3.2}$$

Proof. We observe that if $G \in \mathcal{F}^T \cap C_0^\infty(0,T;\mathbb{R}^2)$, then integration by parts shows that

$$u^G(x,T) = \int_0^T (T-t) u_{tt}^G(x,t)\, dt.$$

Using this observation, we can evaluate the quadratic form:

$$\begin{aligned}(C^T F, G)_{\mathcal{F}^T} &= (W^T F, W^T G)_{\mathcal{H}^T} = \big(y(\,\cdot\,), u^G(\,\cdot\,,T)\big)_{\mathcal{H}^T} \\ &= \int_{-T}^{T} y(x) \int_0^T (T-t) u^G_{tt}(x,t)\,dt\,dx \\ &= \int_0^T (T-t)\,\big(y(\,\cdot\,), -H^* u^G(\,\cdot\,,t)\big)_{\mathcal{H}^T}\,dx\,dt \\ &= \int_0^T (t-T)\Big[\big((\Gamma_0 y(\,\cdot\,))(t), (\Gamma_1 u^G)(t)\big)_B \\ &\qquad - \big((\Gamma_1 y(\,\cdot\,))(t), \big(\Gamma_0 u^G\big)(t)\big)_B\Big]\,dt \\ &= \int_0^T (T-t)\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix} g_1(t)\\ g_2'(t)\end{pmatrix}\right)dt,\end{aligned}$$

from where (3.2) follows due to the arbitrariness of G. □

Representation formulas (2.1) imply that that the solution F to a special control problem satisfies relations:

$$\begin{aligned} y(T) &= u^F(T,T) = \frac{1}{2} f_1(0) - \frac{1}{2} f_2(0), \\ y(-T) &= u^F(-T,T) = -\frac{1}{2} f_1(0) - \frac{1}{2} f_2(0).\end{aligned}$$

Thus solving (3.2) for all $T \in (0,T)$, we recover the solution $y(x)$ to (3.1) on the interval $(-T,T)$. Then the potential $q(x)$, $x \in (-T,T)$ can be recovered as $q(x) = \frac{y''(x)}{y(x)}$, $x \in (-T,T)$.

3.2. Gelfand–Levitan equations. We introduce the notation:

$$C^T = \frac{1}{2}(I + C), \quad (CF)(t) = 2\int_0^T C(t,s)\begin{pmatrix} f_1\\ f_2\end{pmatrix} ds. \tag{3.3}$$

For $F, G \in \mathcal{F}^T$ we set $W^T F = a$, $W^T G = b$, where $a, b \in \mathcal{H}^T$, on using the controllability (Theorem 2), we have that (see (2.9))

$$F = J^T(I+K)^{-1}S^{-1}a = 2J^T(I+K)^{-1}Sa,$$
$$G = J^T(I+K)^{-1}S^{-1}b = 2J^T(I+K)^{-1}Sb.$$

Using above representations we can rewrite the quadratic form as:

$$\begin{aligned}(C^T F, G)_{\mathcal{H}^T} &= \Big(\frac{1}{2}(I+C)2J^T(I+K)^{-1}Sa, 2J^T(I+K)^{-1}Sb\Big)_{\mathcal{H}^T} \\ &= \Big(2\big((I+K)^{-1}\big)^* J^T(I+C)J^T(I+K)^{-1}Sa, Sb\Big)_{\mathcal{H}^T}\end{aligned} \tag{3.4}$$

On the other hand:

$$\big(C^T F, G\big)_{\mathcal{H}^T} = \big(W^T F, W^T G\big)_{\mathcal{H}^T} = (a, b)_{\mathcal{H}^T} = (2Sa, Sb)_{\mathcal{H}^T}. \tag{3.5}$$

On comparing (3.4) and (3.5), we obtain the following operator identity:

$$\big((I+K)^{-1}\big)^* J^T(I+C)J^T(I+K)^{-1} = I. \tag{3.6}$$

We introduce the following notations

$$I + M = (I+K)^{-1}, \tag{3.7}$$

$$(MF)(x) = \begin{pmatrix} \int_x^T m_{11}(x,s)f_1(s) + m_{12}(x,s)f_2(s)\,ds \\ \int_x^T m_{21}(x,s)f_1(s) + m_{22}(x,s)f_2(s)\,ds \end{pmatrix}$$

$$(M^*a)(t) = \begin{pmatrix} \int_0^t m_{11}(x,t)a_1(x) + m_{21}(x,t)a_2(s)\,dx \\ \int_0^t m_{12}(x,t)a_1(s) + m_{22}(x,t)a_2(x)\,dx \end{pmatrix}.$$

It is easy to check that on a diagonal the kernels of operators K and M satisfy a relation

$$m_{ij}(x,x) = -k_{ij}(x,x), \quad i,j = \{1,2\}, \quad x \in (0,T). \tag{3.8}$$

Rewritten in new notations, the operator equality (3.6), has a form:

$$(I+M)^*(I+\widetilde{C})(I+M) = I, \tag{3.9}$$

where

$$\widetilde{C} = J^T C J^T, \quad \Big(\widetilde{C}F\Big)(t) = \int_0^T \widetilde{C}(t,s)F(s)\,ds. \tag{3.10}$$

The relation (3.9) is equivalent to the equality

$$M^* + (I+M)^* \left(M + \widetilde{C} + \widetilde{C}M\right) = 0. \tag{3.11}$$

On introducing a function

$$\Phi(x,s) = m(x,s) + \widetilde{C}(x,s) + \int_0^T \widetilde{C}(x,\alpha)m(\alpha,s)\,d\alpha, \quad x,s \in (0,T),$$

we can write down an equality on the kernel for the operator in the left hand side in (3.11) $M^* + \Phi + M^*\Phi = 0$:

$$m(s,x) + \Phi(x,s) + \int_0^t m(\alpha,x)\Phi(\alpha,s)\,d\alpha = 0, \quad x,s \in (0,T).$$

Since $m(s,x) = 0$ when $x < s$, we obtain that Φ satisfies the relation:

$$\Phi(x,s) + \int_0^t m(\alpha,x)\Phi(\alpha,s)\,d\alpha = 0, \quad x < s.$$

Thus the function Φ satisfies a Volterra equation of a second kind, and due to this we obtain that $\Phi(x,s) = 0$ for $x < s$, which immediately yields the following equation on the matrix function m:

$$m(x,s) + \widetilde{C}(x,s) + \int_0^T \widetilde{C}(x,\alpha)m(\alpha,s)\,d\alpha = 0, \quad 0 < x < s < T. \tag{3.12}$$

As a result we can formulate the following

Theorem 5. *The matrix kernel of the operator M* (3.7) *satisfy the Gelfand–Levitan equation* (3.12)*, where the kernel $\widetilde{C}$ is defined by* (3.3), (3.10). *Solving this equation, one can recover the potential using relations between kernels* (2.10), (3.8) *and relations on diagonals $\{x = t\}$, $\{-x = t\}$ in* (2.2), (2.3):

$$q(x) = 2\frac{d}{dx}\left(m_{11}(x,x) - m_{12}(x,x)\right), \quad x \in (0,T),$$

$$q(-x) = -2\frac{d}{dx}\left(m_{11}(x,x) + m_{12}(x,x)\right), \quad x \in (0,T).$$

3.3. Relationship between dynamic and spectral inverse data. The problem of finding relationships between different types of inverse data is very important in inverse problems theory. We can mention [2,4,5,14,15] on some recent results in this direction. Below we show the relationship between the dynamic response function and matrix spectral measure.

Consider two solution to the equation

$$-\phi'' + q(x)\phi = \lambda\phi, \quad -\infty < x < \infty, \tag{3.13}$$

satisfying the Cauchy data:

$$\varphi(0,\lambda) = 0,\ \varphi'(0,\lambda) = 1,\ \theta(0,\lambda) = -1,\ \theta'(0,\lambda) = 0.$$

Note that

$$\Gamma_0\varphi = 0,\ \Gamma_0\theta = 0,\ \Gamma_1\varphi = \begin{pmatrix}1\\0\end{pmatrix},\ \Gamma_1\theta = \begin{pmatrix}0\\1\end{pmatrix}.$$

We fix some $N > 0$ and prescribe self-adjoint boundary conditions at $x = \pm N$:

$$a_1\phi(-N,\lambda) + b_1\phi'(-N,\lambda) = 0, \quad a_1^2 + b_1^2 \neq 0, \tag{3.14}$$

$$a_2\phi(N,\lambda) + b_2\phi'(N,\lambda) = 0, \quad a_2^2 + b_2^2 \neq 0. \tag{3.15}$$

Eigenvalues and normalized eigenfunctions of (3.13), (3.14), (3.15) are denoted by $\{\lambda_n, y_n\}_{n=1}^{\infty}$. Let $\beta_n, \gamma_n \in \mathbb{R}$ be such that

$$y_n(x) = \beta_n\varphi(x,\lambda_n) + \gamma_n\theta(x,\lambda_n), \quad \text{then} \quad \Gamma_1 y_n = \begin{pmatrix}\beta_n\\ \gamma_n\end{pmatrix}.$$

Let $F \in \mathcal{F}^T \cap C_0^\infty(0,T;\mathbb{R}^2)$, and v^F be a solution to (1.1)–(1.3), (3.14), (3.15), i.e., a solution to the initial boundary value problem for a wave equation on the interval $(-N,N)$. Multiplying the wave equation for v^F by y_n and integrating by parts, we get the following relation:

$$\begin{aligned}
0 &= \int_{-T}^{T} v_{tt}^F y_n\,dx - \int_{-N}^{N} v_{xx}^F y_n\,dx + \int_{-N}^{N} q(x)v^F y_n\,dx \\
&= \int_{-N}^{N} v_{tt}^F y_n\,dx + (v^F, Hy_n) + (\Gamma_1 v^F, \Gamma_0 y_n)_B - (\Gamma_0 v^F, \Gamma_1 y_n)_B \\
&= \int_{-T}^{T} v_{tt}^F y_n\,dx + \lambda_n(v^F, y_n) - \left(\begin{pmatrix} f_1(t)\\ f_2'(t)\end{pmatrix}, \begin{pmatrix}\beta_n\\ \gamma_n\end{pmatrix} \right)_B .
\end{aligned}$$

Looking for the solution to (1.1)–(1.3) in a form

$$v^F = \sum_{k=1}^{\infty} c_k(t) y_k(x), \tag{3.16}$$

we plug (3.16) into (1.1) and multiply by y_n to get:

$$\int_{-N}^{N} \sum_{k=1}^{\infty} c_k''(t) y_k(x) y_n(x)\, dx + \int_{-N}^{N} \sum_{k=1}^{\infty} c_k(t) y_k(x) \lambda_n y_n(x)\, dx = \left(\begin{pmatrix} f_1(t) \\ f_2'(t) \end{pmatrix}, \begin{pmatrix} \beta_n \\ \gamma_n \end{pmatrix} \right)_B .$$

Thus we obtain that $c_n(t)$, $n \geqslant 1$, satisfies the following Cauchy problem:

$$\begin{cases} c_n''(t) + \lambda_n c_n(t) & = \left(\begin{pmatrix} f_1(t) \\ f_2'(t) \end{pmatrix}, \begin{pmatrix} \beta_n \\ \gamma_n \end{pmatrix} \right)_B, \\ c_n(0) = 0, & c_n'(0) = 0. \end{cases}$$

the solution of which is given by the formula

$$c_n(t) = \int_0^t \frac{\sin \sqrt{\lambda_n}(t-s)}{\sqrt{\lambda_n}} \left(f_1(s)\beta_n + f_2'(s)\gamma_n \right) ds.$$

Then for v^F (3.16) we have the expansion:

$$\begin{aligned} v^F(x,t) &= \sum_{k=1}^{\infty} \int_0^t \frac{\sin \sqrt{\lambda_n}(t-s)}{\sqrt{\lambda_n}} \left(f_1(s)\beta_n + f_2'\gamma_n \right) ds \left(\beta_n \varphi(x,\lambda_n) + \gamma_n \theta(x,\lambda_n) \right) \\ &= \sum_{k=1}^{\infty} \int_0^t \frac{\sin \sqrt{\lambda_n}(t-s)}{\sqrt{\lambda_n}} \left(\begin{pmatrix} \beta_n \\ \gamma_n \end{pmatrix} \otimes \begin{pmatrix} \beta_n \\ \gamma_n \end{pmatrix} \begin{pmatrix} f_1(s) \\ f_2'(s) \end{pmatrix}, \begin{pmatrix} \varphi(x,\lambda_n) \\ \theta(x,\lambda_n) \end{pmatrix} \right) \\ &= \int_{-\infty}^{\infty} \int_0^t \frac{\sin \sqrt{\lambda}(t-s)}{\sqrt{\lambda}} \left(d\Sigma_N(\lambda) \begin{pmatrix} f_1(s) \\ f_2'(s) \end{pmatrix}, \begin{pmatrix} \varphi(x,\lambda) \\ \theta(x,\lambda) \end{pmatrix} \right). \end{aligned} \tag{3.17}$$

Where $d\Sigma_N(\lambda)$ is a matrix measure (see [13]). Due to the finite speed of the wave propagation in system (1.1)–(1.3) (equal to one), we have the relation

$$v^F(\,\cdot\,,t) = u^F(\,\cdot\,,t), \quad \text{for}\, t < N, \tag{3.18}$$

and for $T < N$ holds that $R^{2T}F = \Gamma_1 v^F$. Thus the response operator R^T for $T < 2N$, is given by

$$\begin{aligned}(RF)(t) = \Gamma_1 v^F &= \sum_{k=1}^{\infty} c_k(t)\Gamma_1 y_k = \sum c_k(t)\begin{pmatrix}\beta_k\\ \gamma_k\end{pmatrix}\\ &= \sum_{k=1}^{\infty}\int_0^t \frac{\sin\sqrt{\lambda_k}(t-s)}{\sqrt{\lambda_k}}\,(f_1(s)\beta_k + f_2'\gamma_k)\;ds\begin{pmatrix}\beta_k\\ \gamma_k\end{pmatrix} \qquad (3.19)\\ &= \int_{-\infty}^{\infty}\int_0^t \frac{\sin\sqrt{\lambda}(t-s)}{\sqrt{\lambda}} d\Sigma_N(\lambda)\begin{pmatrix}f_1(s)\\ f_2'(s)\end{pmatrix}\,ds, \quad 0 < t < 2N.\end{aligned}$$

Taking $F, G \in \mathcal{F}^T \cap C_0^\infty(0,T;\mathbb{R}^2)$, for $T < N$ we evaluate the quadratic form using (3.17) and (3.18):

$$(C^T F, G)_{\mathcal{F}^T} = (u^F, u^G)_{\mathcal{H}^T} = (v^F, v^G)_{\mathcal{H}^T} \qquad (3.20)$$

$$= \sum_{k=1}^{\infty}\int_0^T\int_0^T \frac{\sin\sqrt{\lambda_n}(t-s)}{\sqrt{\lambda_n}}(f_1\beta_n + f_2'\gamma_n)\,ds\frac{\sin\sqrt{\lambda_n}(t-\tau)}{\sqrt{\lambda_n}}(g_1\beta_n + g_2'\gamma_n)\,d\tau$$

$$= \int_0^T\int_0^T\int_{-\infty}^{\infty} \frac{\sin\sqrt{\lambda}(t-s)}{\sqrt{\lambda}}\frac{\sin\sqrt{\lambda}(t-\tau)}{\sqrt{\lambda}}\left(d\Sigma_N(\lambda)\begin{pmatrix}f_1(s)\\ f_2'(s)\end{pmatrix}, \begin{pmatrix}g_1(\tau)\\ g_2'(\tau)\end{pmatrix}\right)ds\,d\tau$$

We observe that in view of the unite speed of wave propagation in system (1.1)–(1.3), in representation formulas for response operator (3.19) and for connecting operator (3.20), we can substitute $d\Sigma_N(\lambda)$ by any $d\Sigma_M(\lambda)$, $M > N$, where $d\Sigma_M(\lambda)$ corresponds to some selfadjoint boundary conditions at $\pm M$, or we can let N go to infinity, and substitute $d\Sigma_N(\lambda)$ by a limit measure $d\Sigma(\lambda)$ (see [13]).

The inverse problem for a Schrödinger operator on a half-line from a spectral measure is solved in [11], in [13] the inverse spectral problem for a Schrödinger operator on a real line from a matrix measure is discussed, but some questions remain open. At the same time, in the case of a half-line in [1, 2, 14] the authors established the relationships between the dynamic and spectral inverse problems.

Remark 3. The control, response and connecting operators admit representations in terms of spectral inverse data (matrix measure $d\Sigma(\lambda)$), see (3.17), (3.19) and (3.20). This circumstance makes it possible to assume that the progress in studying the inverse spectral problem from a matrix measure will be greatly stimulated by the progress in studying the inverse dynamic problem in the spirit of [1, 2, 14].

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