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THE WEAK SOLUTIONS OF HOPF TYPE TO 2D MAXWELL FLOWS WITH INFINITE NUMBER OF RELAXATION TIMES

Abstract. The system of equations, describing motion of fluids of Maxwell type is considered

$$\frac{\partial}{\partial t}v + v \cdot \nabla v - \int\limits_0^t K(t-\tau)\Delta v(x,\tau)\ d\tau + \nabla p = f(x,t), \quad \mathrm{div}\ v = 0.$$

Here $K(t)$ is exponential series $K(t) = \sum\limits_{s=1}^{\infty} \beta_s e^{-\alpha_s t}$. The existence of weak solution for initial boundary value problem

$$v(x,0) = v_0(x), \quad v \cdot n|_{\partial\Omega} = 0, \quad \mathrm{rot}\ v|_{\partial\Omega} = 0$$

is proved.

§1. Introduction

Consider the system of equations which describes Maxwell flows. This system has a form

$$\frac{\partial}{\partial t}v + v \cdot \nabla v - \int\limits_0^t K(t-\tau)\Delta v(x,\tau)\ d\tau + \nabla p = f(x,t) \tag{1}$$

$$\mathrm{div}\ v = 0 \tag{2}$$

The kernel $K(\tau)$ is represented here as a sum

$$K(\tau) = \sum_{s=1}^{\infty} \beta_s e^{-\alpha_s \tau}, \qquad \alpha_s, \beta_s > 0. \tag{3}$$

The system is considered in a domain $\Omega \subset \mathbf{R^2}$ with smooth boundary $\partial\Omega$ from class C^1. Here $v(x,t)$ is a velocity, $p(x,t)$ is a pressure, $f(x,t)$ is a vector of external forces.

Such systems describe motion of fluids with infinite numbers of relaxation and retardation times. These fluids have property: when the stress

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equals zero then the velocity decreases like exponent. And conversely when the velocity is zero then the stress vanishes with respect to exponential low.

System (1,2) with the kernel K represented in the form of finite sum

$$K(\tau) = \sum_{s=1}^{m} \beta_s e^{-\alpha_s \tau}, \qquad \alpha_s, \beta_s > 0 \tag{4}$$

was studied in the papers of Oskolkov, Cotsiolis and other authors [2,3]. The authors introduced m new variables (where m is the number of the summands in sum (4)). In the case when the number of the summands is infinite such method can not be applied.

If we omit the nonlinear term, then the system may be reduced to the Gurtin-Pipkin equation in the space of solenoidal vectors. Such equations were investigated in the papers of Vlasov, Ivanov and some other [8,10].

For the sake of simplicity we introduce the notation

$$\mathbf{K}v(x,t) = \int_0^t K(t-\tau)v(x,\tau)\ d\tau \tag{5}$$

We consider boundary value problem for arbitrary bounded domain $\Omega \subset \mathbf{R}^2$, $\partial\Omega \in C^1$

$$v \cdot n|_{\partial\Omega} = 0, \quad \text{rot } v|_{\partial\Omega} = 0 \tag{6}$$

Here by the operator *rot* we mean the scalar operator in two-dimensional space

$$\text{rot } v = \frac{\partial}{\partial x_1} v_2 - \frac{\partial}{\partial x_2} v_1$$

Let the function $v(x,t)$ satisfy initial condition

$$v|_{t=0} = v_0 \tag{7}$$

Use the notation. Symbol $\|\,.\,\|_{2,\Omega}$ is used for the norm in the space $L_2(\Omega)$, and $\|\,.\,\|_{p,q}$ is the norm in the space $L_q(0,T;L_p(\Omega))$. The scalar product in the space $L_2(\Omega)$ is denoted $(u,v)_\Omega$. The expression $(u_x, v_x)_\Omega$ is used for the sum

$$\sum_{i=1}^{2} (u_{x_i}, v_{x_i})_\Omega = (u_x, v_x)_\Omega. \tag{8}$$

By $J^{\cdot}(\Omega)$ we denote the space of infinite differentiable finite solenoidal functions

$$J^{\cdot}(\Omega) = \{u \in \overset{\circ}{C}{}^{\infty}(\Omega) | \text{div } u = 0\} \tag{9}$$

$\overset{\circ}{J}(\Omega)$ is a supplement of $J^{\cdot}(\Omega)$ in L_2-norm. The symbol $\overset{\circ}{J}{}^1_{(n)}$ is used for the space

$$\overset{\circ}{J}{}^1_{(n)}(\Omega) = \{u | u \in W_2^1(\Omega), \text{div } u = 0, u \cdot n|_{\partial\Omega} = 0\}. \tag{10}$$

And by $J^1(\Omega)$ we denote a supplement $J^{\cdot}(\Omega)$ in W_2^1-norm.

Hopf introduced the notion of weak solution for the Navier–Stockes equations [5, 6]. We introduce similar notion for Maxwell flows. Function $v(x,t)$ is a weak solution for of problem (1), (2), (6),(7), if v satisfies the integral identity

$$\int\limits_{Q_T} (-v\Phi_t - v_k v \Phi_{x_k} + (\mathbf{K}v_x)\Phi_x)\, dx\, dt = \int\limits_{Q_T} f\Phi \tag{11}$$

for any solenoidal $\Phi(x,t) \in L_2(0,T; \overset{\circ}{J}{}^1_{(n)})$ which equals zero on the ends of cylinder Q_T, $\Phi(x,0) = \Phi(x,T) = 0$, and such that $\Phi_x, \Phi_t \in L_2(Q_T)$. Moreover v should satisfy initial conditions (7) in the following sense

$$\|v(x,t) - v_0\|_{2,\Omega} \to 0, \quad t \to +0. \tag{12}$$

The main result of the paper is the following theorem

Theorem 1. *Let $f \in L_2(Q_T)$, $f_x \in L_2(Q_T)$, $v_0 \in \overset{\circ}{J}{}^1_{(n)}$. Let the coefficients α_s,β_s satisfy*

$$\sum_{s=1}^{\infty} \beta_s \alpha_s < \infty, \qquad \sum_{s=1}^{\infty} \beta_s < \infty \tag{13}$$

Then initial boundary value problem has a weak solution in $Q_T = \Omega \times [0,T]$. This solution $v \in L_\infty(0,T; \overset{\circ}{J}{}^1_{(n)}(\Omega))$, furthermore $v_x \in L_\infty(0,T; \overset{\circ}{J}(\Omega))$.

§2. A priori estimates

In this section we obtain some a priori estimates for the solutions of initial boundary value problem (6), (7).

At first we multiply equation (1) by v in the space$L_2(\Omega)$ (By equation (2) the nonlinear term vanishes)

$$\frac{1}{2}\frac{d}{dt}\|v\|_{2,\Omega}^2 + (\mathbf{K}v_x, v_x)_\Omega = (f, v)_\Omega. \tag{14}$$

Then transform the term with the operator $\mathbf{K}$. This operator commutates with the differential operator $\frac{\partial}{\partial x}$. Moreover for solenoidal vectors it holds

$$(\mathbf{K}v_x, v_x)_\Omega = \left(\frac{\partial}{\partial x}(\mathbf{K}v), v_x\right)_\Omega = (\mathrm{rot}\ (\mathbf{K}v), \mathrm{rot}\ v)_\Omega = (\mathbf{K}\mathrm{rot}\ v, \mathrm{rot}\ v)_\Omega. \tag{15}$$

Let us introduce new function η^s

$$\eta^s = \int_0^t e^{-\alpha_s(t-\tau)} v(x,\tau)\ d\tau \tag{16}$$

Then we have

$$\frac{\partial}{\partial t}\eta^s = v - \alpha_s \eta^s \tag{17}$$

Substitution of function (16) into (14) yields

$$\frac{1}{2}\|v\|^2_{2,\Omega} + \left(\int_0^t \sum_{s=1}^{\infty} \beta_s e^{-\alpha_s(t-\tau)} \mathrm{rot}\ v(x,\tau)\ d\tau, \mathrm{rot}\ v(x,t)\right)_\Omega = (f,v)_\Omega \tag{18}$$

With the help of the new function η^s we transform the second term in (14)

$$\begin{aligned}(\mathbf{K}\mathrm{rot}\ v, \mathrm{rot}\ v)_\Omega &= \left(\int_0^t \sum_{s=1}^{\infty} \beta_s e^{-\alpha_s(t-\tau)} \mathrm{rot}\ v(x,\tau)\ d\tau, \mathrm{rot}\ v(x,t)\right)_\Omega \\ &= (\sum_{s=1}^{\infty} \beta_s \mathrm{rot}\ \eta^s, \frac{\partial}{\partial t}\mathrm{rot}\ \eta^s + \alpha_s \mathrm{rot}\ \eta^s)_\Omega \\ &= \frac{1}{2}\frac{d}{dt}\sum_{s=1}^{\infty} \beta_s \|\mathrm{rot}\ \eta^s\|^2_{2,\Omega} + \sum_{s=1}^{\infty} \beta_s \alpha_s \|\mathrm{rot}\ \eta^s\|^2_{2,\Omega}\end{aligned} \tag{19}$$

Thus by (19) and (14) we get

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{2,\Omega} + \frac{1}{2}\frac{d}{dt}\sum_{s=1}^{\infty} \beta_s \|\mathrm{rot}\,\eta^s\|^2_{2,\Omega} + \sum_{s=1}^{\infty} \beta_s \alpha_s \|\mathrm{rot}\,\eta^s\|^2_{2,\Omega} = (f,v)_\Omega \tag{20}$$

Identity (20) implies estimates of the norms $\|v(t)\|_{2,\Omega}$ and $\|\mathrm{rot}\ \eta^s\|_{2,\Omega}$. Indeed let us introduce the function ξ

$$\xi(t) = \|v(t)\|^2_{2,\Omega} + \sum_{s=1}^{\infty} \beta_s \|\mathrm{rot}\ \eta^s\|^2_{2,\Omega} \tag{21}$$

With the help of the function $\xi(t)$ identity (20) may be re-written in this way

$$\frac{1}{2}\frac{d}{dt}\xi + \sum_{s=1}^{\infty}\beta_s\alpha_s\|\mathrm{rot}\ \eta^s\|_{2,\Omega}^2 \leqslant \|f(t)\|_{2,\Omega}^2 + \xi \tag{22}$$

By (22) and by Gronwall lemma we have

$$\sup_{t\in[0,T]}[\|v(t)\|_{2,\Omega}^2 + \sum_{s=1}^{\infty}\beta_s\|\mathrm{rot}\ \eta^s\|_{2,\Omega}^2] \leqslant C_1, \tag{23}$$

where $C_1 = e^t(\xi(0) + \int\limits_0^t \|f\|_{2,\Omega}^2)$. Since $\mathrm{rot}\ \eta_s(0) = 0$, then

$$C_1 = e^t\left(\|v_0\|_{2,\Omega}^2 + \int\limits_0^t \|f\|_{2,\Omega}^2\right). \tag{24}$$

Now let us apply the operator rot to both sides of equation (1) and multiply the obtained expression by $\mathrm{rot}\ v$

$$\frac{1}{2}\frac{d}{dt}\|\mathrm{rot}\ v\|_{2,\Omega}^2 - (\mathbf{K}\Delta\mathrm{rot}\ v, \mathrm{rot}\ v)_\Omega = (\mathrm{rot}\ f, \mathrm{rot}\ v)_\Omega \tag{25}$$

Proceeding in the similar way we transform the second term in (25)

$$\begin{aligned}
-(\mathbf{K}\Delta\mathrm{rot}\ v, \mathrm{rot}\ v)_\Omega &= (\mathbf{K}\Delta v, \Delta v)_\Omega \\
= \Big(\sum_{s=1}^{\infty}\beta_s\Delta\eta^s, \Delta v\Big)_\Omega &= \Big(\sum_{s=1}^{\infty}\beta_s\Delta\eta^s, \frac{\partial}{\partial t}\Delta\eta^s + \alpha_s\Delta\eta^s\Big)_\Omega \\
&= \frac{1}{2}\frac{d}{dt}\Big(\sum_{s=1}^{\infty}\beta_s\|\Delta\eta^s\|_{2,\Omega}^2\Big) + \sum_{s=1}^{\infty}\beta_s\alpha_s\|\Delta\eta^s\|_{2,\Omega}^2
\end{aligned} \tag{26}$$

Thus the following inequality is true

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\mathrm{rot}\ v\|_{2,\Omega}^2 + \frac{1}{2}\frac{d}{dt}\left(\sum_{s=1}^{\infty}\beta_s\|\Delta\eta^s\|_{2,\Omega}^2\right) + \sum_{s=1}^{\infty}\beta_s\alpha_s\|\Delta\eta^s\|_{2,\Omega}^2 \\
\leqslant \|\mathrm{rot}\ v\|_{2,\Omega}^2 + \|\mathrm{rot}\ f\|_{2,\Omega}^2
\end{aligned} \tag{27}$$

Define a new function $\zeta(t)$

$$\zeta(t) = \|\mathrm{rot}\ v(t)\|_{2,\Omega}^2 + \sum_{s=1}^{\infty}\beta_s\|\Delta\eta^s(t)\|_{2,\Omega}^2 \tag{28}$$

With the help of the function ζ we reduce (25) to the relation

$$\frac{d}{dt}\zeta + \sum_{s=1}^{\infty} \beta_s \alpha_s \|\Delta\eta^s\|_{2,\Omega}^2 \leqslant \|\mathrm{rot}\ f\|_{2,\Omega}^2 + \zeta \tag{29}$$

By virtue of Gronwall lemma inequality (29) yields the estimate

$$\sup_{t\in[0,T]} [\|\mathrm{rot}\ v\|_{2,\Omega}^2 + \sum_{s=1}^{\infty} \beta_s \|\Delta\eta^s\|_{2,\Omega}^2] \leqslant C_2 \tag{30}$$

§3. Proof of the theorem

The theorem is proved by the Galerkin method. Let $h_l(x)$ be fundamental system of functions in the space $\overset{\circ}{J}{}^1_{(n)}$. Then initial data $v_0(x)$ may be expressed as a limit of finite sums in L_2-topology

$$v^{0N} = \sum_{l=1}^{N} C_{lN}^0 h_l \to v^0, \qquad N \to \infty \tag{31}$$

We shall seek the approximations v^N for the solution of problem (1), (2),(6), (7) like

$$v^N(x,t) = \sum_{l=1}^{N} C_{lN}(t) h_l(x) \qquad N = 1, 2, \dots \tag{32}$$

Then the functions $C_{lN}(t)$ should satisfy the initial conditions

$$C_{lN}(0) = C_{lN}^0, \qquad l = 1, 2, \dots, N. \tag{33}$$

We shall try to find v^N which satisfy the following integral identity

$$(v_t^N, h_l)_\Omega + (v_k^N v_{x_k}^N, h_l)_\Omega + (\mathbf{K} v_x^N, h_{lx})_\Omega = (f, h_l)_\Omega, \quad l = 1, 2, \dots, N. \tag{34}$$

Substitution (32) into (34) yields

$$\frac{d}{dt}C_{lN}(t) + \sum_{i,j=1}^{N} a_{lij} C_{iN}(t) C_{jN}(t) + \sum_{i=1}^{N} (\mathbf{K}[C_{il}(t) h_{lx}], h_{lx}) = (f, h_l), \tag{35}$$

where a_{lij} are constants.

System (35) is the system of ordinary integro-differential equations. It can be solved locally by method of succesive approximations of Picard. The global existence of a solution is deduced from boundedness of C_{lN}. Indeed the functions $C_{lN}(t)$ are bounded because v^N are bounded. The last conclusion may be proved in the following way. Let us multiply identity

(34) by $C_{lN}(t)$ and summarize with respect to $l = 1, \ldots, N$.The functions v^N satisfy (34) so v^N satisfy integral identity (11). Thus (23) is true for v^N and the norms $\|v^N\|_{2,\Omega}$ and the coefficients $C_{lN}(t)$ can be estimated by the constant C_1.

As a result of this we have constructed the sequence v^N, $N = 1, \ldots$, such that the norms $\|v^N\|_{2,\infty}$ and $\|v_x^N\|_{2,\infty}$ are bounded.Then for $T < \infty$ we get the estimates for the norms $\|v^N\|_{2,Q_T}$ and $\|v_x^N\|_{2,Q_T}$. They will be estimated by the constant $\sqrt{T}C_1$. Since a bounded set in the Hilbert space is weakly compact then we may choose a weakly converging subsequence v^{N_k} from the sequence v^N. (Let us denote this sequence also by v^N.) Reasoning in similar way we may affirm that v_x^N converges to v_x weakly in $L_2(Q_T)$. Moreover the sequence of products $\{v^N v_x^N\}$ is also bounded and we may choose the subsequence such that $v^N v_x^N$ converges weakly to vv_x in the space $L_2(Q_T)$.

Furthermore the set $\{\mathbf{K}v^N\}$, $N = 1, \ldots$ is also bounded because the operator $\mathbf{K}$ is continuous in $L_2(Q_T)$. Proceeding in similar way we prove that $\mathbf{K}v^N$ converges weakly to $\mathbf{K}v$ in $L_2(Q_T)$.

This convergence is enough to come to the limit with respect to $N \to \infty$ in the relation

$$\int\limits_{Q_T} (-v^N \Phi_t - v_k^N v^N \Phi_{x_k} + (\mathbf{K}v_x^N)\Phi_x)\, dx\, dt = \int\limits_{Q_T} f\Phi \tag{36}$$

for any $\Phi(x,t) \in L_2(0,T; \overset{\circ}{J}{}^1_{(n)}(\Omega))$ such that $\Phi_x, \Phi_t \in L_2(Q_T)$ and $\Phi(x,0) = \Phi(x,T) = 0$. Boundary condition (6) and incompressibility condition (2) are fulfilled for the limit function v.

Now we shall prove that v satisfies initial condition (12). We knew that $\|v^N(x,t) - v_0(x)\|_{2,\Omega} \to 0$ when $N \to \infty$. Moreover we proved that $v^N(x,t) - v(x,t)$ converges weakly in $L_2(\Omega)$ for any t. Thus we have weak convergence of $v(x,t) - v_0(x)$ to zero when $t \to 0$. So we get that $\|v_0\|_{2,\Omega} \leqslant \underline{\lim}_{t\to 0} \|v(x,t)\|_{2,\Omega}$. On the other hand v^N satisfies inequality (23). Passage to the limit in (23) when $N \to \infty$ gives

$$\|v(x,t)\|^2_{2,\Omega} \leqslant \|v_0\|^2_{2,\Omega} + \int\limits_0^t \|f\|^2_{2,\Omega}, \tag{37}$$

and so $\overline{\lim}_{t\to 0} \|v(x,t)\|_{2,\Omega} \leqslant \|v_0\|_{2,\Omega}$. Consequently we get the existence of limit $\|v(x,t)\|_{2,\Omega}$ when $t \to 0$ and $\lim_{t\to 0} \|v(x,t)\|_{2,\Omega} = \|v_0\|_{2,\Omega}$. Weak convergence and convergence of norms yield (12).

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