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N. Gordeev, U. Rehmann

DOUBLE COSETS OF STABILIZERS OF TOTALLY ISOTROPIC SUBSPACES IN A SPECIAL UNITARY GROUP II

Abstract. In the article (N.Gordeev and U. Rehmann. Double cosets of stabilizers of totally isotropic subspaces in a special unitary group I., Zapiski Nauch. Sem. POMI, v. 452 (2016), 86–107) we have considered the decomposition $\mathrm{SU}(D,h) = \cup_i P_u\gamma_i P_v$ where $\mathrm{SU}(D,h)$ is a special unitary group over a division algebra D with an involution, h is a symmetric or skew symmetric non-degenerated Hermitian form, and P_u, P_v are stabilizers of totally isotropic subspaces of the unitary space. Since $\Gamma = \mathrm{SU}(D,h)$ is a point group of a classical algebraic group $\widetilde{\Gamma}$ there is the "order of adherence" on the set of double cosets $\{P_u\gamma_i P_v\}$ which is induced by the Zariski topology on $\widetilde{\Gamma}$. In the current paper we describe the adherence of such double cosets for the cases when $\widetilde{\Gamma}$ is an orthogonal or a symplectic group (that is, for groups of types B_r, C_r, D_r).

Introduction

Let K be a field and let F/K be a separable extension of degree $\leqslant 2$. Further, let D be a division algebra over the center F of index c with a fixed anti-automorphism $x \to x^\star$ which is trivial on K (here we include the cases $D = K$ when $\star$ is trivial and $D = F$). If $K = F$, the involution $\star$ is called of *the first kind* and it is called of *the second kind* if $\deg F/K = 2$.

Key words and phrases: classical algebraic groups, double cosets of closed subgroups, the order of adherence.

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Let V be a (left) linear space over D of dimension d with a non-degenerate Hermitian or skew-Hermitian form $h = (\ ,\ )$ with respect to the involution $\star$ (here we also consider symmetric and skew-symmetric forms over K as Hermitian or skew-Hermitian forms with $D = F = K$). We will call any linear space over D with a form h (not necessarily non-degenerate) an h-space.

In [4] we have assumed that the h-space V satisfies the T-condition: ([1, IX, §4]):

For every $x \in V$ there is an element $\alpha \in D$ such that $\alpha + \epsilon\alpha^\star = h(x, x)$ where $\epsilon = 1$ if h is a Hermitian form and $\epsilon = -1$ if h is a skew Hermitian form.

In particular, the T-condition holds if h is skew-Hermitian or $\mathrm{char} K \neq 2$.

In this paper we assume that $\mathrm{char} K \neq 2$.

We denote by n the dimension of a maximal totally isotropic subspace of V (Witt index). *Below we assume $n \geqslant 1$.*

We denote by $\mathrm{U}(D, h)$ the group of isometries of the h-space V (the unitary group). The special unitary group is the subgroup of $\mathrm{U}(D, h)$

$$\mathrm{SU}(D, h) := \{g \in \mathrm{U}(D, h) \mid \mathrm{Nrd}\, g = 1\}$$

(here Nrd is the reduced norm; see [1, VIII, §12]). We assume

$$\dim_D V = d < \infty.$$

There exists a simple algebraic group $\widetilde{\Gamma}$ which is defined over the field K such that $\Gamma = \widetilde{\Gamma}(K) = \mathrm{SU}(D, h)$ (see [6, 8]). The group $\widetilde{\Gamma}$ is a group of the type A_r, B_r, C_r, or D_r. In [4] we divided all possible groups into two sets: we say that we are in the Special Case if $\widetilde{\Gamma}$ is a group of the type D_n which is completely split over a field K (that is, $\Gamma = \mathrm{SO}_{2n}(K)$); in all other possibilities we say that we are in the General Case.

Let $k \leqslant l$ be positive integers and let $\mathcal{I}_k, \mathcal{I}_l$ be the sets of all totally isotropic subspaces of dimensions k and l, respectively, and let $v \in \mathcal{I}_k$, $u \in \mathcal{I}_l$. Further, let P_u, P_v be the stabilizers of the subspaces u, v, respectively. Then there exist maximal K-defined parabolic subgroups $\widetilde{P}_u, \widetilde{P}_v$ such that $\widetilde{P}_u(K) = P_u$, $\widetilde{P}_v(K) = P_v$. In [4] we described the decomposition $\Gamma = \cup_i P_u \gamma_i P_v$ in terms of the intersection distance

$$\mathrm{d_{in}}(u, g(v)) = \dim u - \dim u \cap g(v)$$

(see, [3, 4]) and the Witt index of the unitary space $u + g(v)$ where g is a representative of $P_u \gamma_i P_v$.

Namely, let $I(U)$ be the codimension of a maximal isotropic subspace $U_0 \leqslant U$ and let

$$X_{pq} = \{(p, q) \mid 0 \leqslant p \leqslant \min\{k, n-l\},\ 0 \leqslant q \leqslant k-p\}$$

be the set of integers. Then for the General Case we have (see [4])

Theorem 1. *The double cosets $P_u \gamma P_v$ can be enumerated as follows:*

(i) $\Gamma = \bigcup_{(p,q) \in X_{pq}} P_u \gamma_{pq} P_v$;

(ii) $g \in P_u \gamma_{pq} P_v \Leftrightarrow \mathrm{d_{in}}(g(v), u) = l-k+p+q$ *and* $I(u+g(v)) = q$.

To formulate the result for the Special Case we need to describe the difference between this Case and the General Case. In the General Case all totally isotropic subspaces of the same dimension belong to a single Γ-orbit. In the Special Case we have two Γ-orbits of totally isotropic spaces of the maximal dimension n. We fix a maximal totally isotropic subspace u_0 and denote its Γ-orbit by $\mathcal{I}_n^+$. By $\mathcal{I}_n^-$, we denote the other Γ-orbit. Two maximal isotropic subspases u', v' belong to the same Γ-orbit if and only if $\mathrm{sign}\,(u', v') = 1$ where $\mathrm{sign}\,(u', v') = (-1)^{\mathrm{d_{in}}(u', v')}$ (see [4]). Further, let $v_g := u + g(v)$ where $g \in \Gamma$. There exists a unique maximal totally isotropic subspace v_g^u in v_g which contains the space u ([4]). In the Special Case when $\dim u < n$ and $v_g^u = n$, the double coset of g is defined not only by $\dim v_g, I(v_g)$ but also by the "orientation" of v_g^u , that is, either $v_g^u \in \mathcal{I}_n^+$ or $v_g^u \in \mathcal{I}_n^-$ (see i1) below).

Theorem 2. *Let $\Gamma = \mathrm{SO}(V) = \mathrm{SO}_{2n}(K)$ be a completely split orthogonal group of the dimension $2n$. The double cosets $P_u \gamma P_v$ can be enumerated in the following way:*

(i1) *If* $0 < n - l \leqslant k$ *then*

$$\Gamma = \Big(\bigcup_{\substack{0\leqslant p\leqslant n-l,\\ 0\leqslant q\leqslant k-p}} P_u\gamma_{pq}P_v \Big) \cup \Big(\bigcup_{q\leqslant k+l-n} P_u\gamma^-_{n-l\,q}P_v \Big).$$

(i2) *If* $k < n - l$ *then*

$$\Gamma = \bigcup_{\substack{0\leqslant p\leqslant k,\\ 0\leqslant q\leqslant k-p}} P_u\gamma_{pq}P_v.$$

(i3) *If* $l = n, k < n$ *then*

$$\Gamma = \bigcup_{q\leqslant k} P_u\gamma_q P_v.$$

(i4) *If* $k = l = n$ *then*

$$\begin{cases} \Gamma = \bigcup\limits_{0\leqslant q=2m\leqslant n} P_u\gamma_q P_v & \text{if} \quad \operatorname{sign}(v,u) = 1, \\ \\ \Gamma = \bigcup\limits_{1\leqslant q=2m+1\leqslant n} P_u\gamma_q P_v & \text{if} \quad \operatorname{sign}(v,u) = -1. \end{cases}$$

(ii1) $g \in P_u\gamma_{pq}P_v$, $p \neq n - l \Leftrightarrow \mathrm{d_{in}}(g(v),u) = l - k + p + q$ *and* $I(u + g(v)) = q$, $g \in P_u\gamma_{n-l\,q}P_v \Leftrightarrow \mathrm{d_{in}}(g(v),u) = n - k + q$ *and* $I(u + g(v)) = q$, *and* $v^u_g \in \mathcal{I}^+_n$,
$g \in P_u\gamma^-_{n-l\,q}P_v \Leftrightarrow \mathrm{d_{in}}(g(v),u) = n-k+q$ *and* $I(u + g(v)) = q$ *and* $v^u_g \in \mathcal{I}^-_n$,
(ii2) $g \in P_u\gamma_{pq}P_v \Leftrightarrow \mathrm{d_{in}}(g(v),u) = l-k+p+q$ *and* $I(u + g(v)) = q$,
(ii3) $g \in P_u\gamma_q P_v \Leftrightarrow \mathrm{d_{in}}(g(v),u) = n - k + q$ *and* $I(u + g(v)) = q$,
(ii4) $g \in P_u\gamma_q P_v \Leftrightarrow \mathrm{d_{in}}(g(v),u) = q = I(u + g(v))$.

The main result. It is a well known fact that double cosets of parabolic subgroups of a simple algebraic group are locally open subsets with respect to the Zariski topology and their closures are unions of double cosets of the same parabolic subgroups (see [2]). It gives an order on the set of such double cosets which, in the case of standard parabolic subgroups, is called the "Bruhat order." This order can be described in this case in terms of the decomposition of the elements of the Weyl group which corresponds to given cosets ([2]). Here we describe the "Bruhat order" for double cosets $P_u\gamma_i P_v$ of $\Gamma = \mathrm{SU}(D,h)$ in terms of the dimension and the Witt index of unitary subspaces $u + \gamma_i(v)$. We may define the "Bruhat order"$\preceq$ on the set $\{P_u\gamma_i P_v\}$:

$$P_u\gamma P_v \preceq P_u\gamma' P_v \Leftrightarrow \widetilde{P}_u\gamma\widetilde{P}_v \subset \overline{\widetilde{P}_u\gamma'\widetilde{P}_v}$$

(here $\overline{X}$ is the Zariski closure of X). In this paper we consider only those cases where the group $\widetilde{\Gamma}$ is of the type B_r, C_r, D_r (that is, the cases where $\star$ is an involution of the first kind). The case where the group $\widetilde{\Gamma}$ is of the type A_r will be considered in the next paper.

Since in the Special Case there appear the double cosets $P_u\gamma^{\pm}_{pq}P_v$ and we will call any double coset of the form $P_u\gamma^{+}_{pq}P_v := P_u\gamma_{pq}P_v$ *positive* and $P_u\gamma^{-}_{pq}P_v$ *negative*. Note, that most of all double cosets are positive except the Special Case i1), when $p = n - l$ (note, that positivity or negativity here depends on our choice of the "positive" orbit $\mathcal{I}^{+}$). However, when considering the whole massive of double cosets we will write $P_u\gamma^{\pm}_{pq}P_v$.

The main result of this paper is the following

Theorem. *Let $\widetilde{\Gamma}$ be a simple algebraic group of type B_r, C_r,* or *D_r defined over a field $K, \mathrm{char}K \neq 2$, and let $\Gamma = \widetilde{\Gamma}(K) = \mathrm{SU}(D,h)$ be the group of special unitary transformations of a unitary space V over a division algebra D with an involution. Further, let $k \leqslant l$ be integers and let $v \in \mathcal{I}_k$, $u \in \mathcal{I}_l$. Let P_u, P_v be the stabilizers of u, v. Then $P_u = \widetilde{P}_u(K), P_v = \widetilde{P}_v(K)$ for some maximal K-defined parabolic subgroups $\widetilde{P}_u, \widetilde{P}_v$ of the group $\widetilde{\Gamma}$ and*

$P_u\gamma^{\pm}_{pq}P_v \preceq P_u\gamma^{\pm}_{p'q'}P_v \Leftrightarrow p+q \leqslant p'+q'$ *and* $q \leqslant q'$ *and* $(p,q) \neq (p',q')$

if $P_u\gamma^{\pm}_{pq}P_v \neq P_u\gamma^{\pm}_{p'q'}P_v$.

The proof of the Theorem is contained in Sec. 3. In Sections 1, 2 we collect more or less known facts which we use in the proof of the Theorem.

Notations and terminology.

Unitary transformations.

K is a field with char$K \neq 2$;

$K_s, \overline{K}$ are, respectively, the separable or algebraic closure of K;

D is a division algebra over the center K with an involution $\star$ of the first kind (possibly, $D = K$ and $\star$ is a trivial involution);

V is a left linear space over D of the dimension $d = \dim_D V < \infty$ with a non-degenerated symmetric or skew-symmetric Hermitian form $h = (\ ,\ )$;

$\mathrm{U}(D,h)$ is the group of unitary transformations of V and

$$\Gamma := \mathrm{SU}(D,h) := \{g \in \mathrm{U}(D,h) \mid \mathrm{Nrd}\, g = 1\}$$

is a special unitary group ($\mathrm{Nrd}\, g$ is a reduced norm of $g \in \mathrm{GL}_d(D)$ (see [6], [1, VIII, §12]));

n is the dimension of a maximal totally isotropic subspace of V (Witt index);

$\mathcal{I}_k$ is the set of all totally isotropic subspaces of V of dimension $k \leqslant n$;

$\mathcal{I}_n^+, \mathcal{I}_n^-$ are different Γ-orbits of maximal totally isotropic subspaces of V in the Special Case (see [4]); moreover, we assume below that $\mathcal{I}_n^+$ is the Γ-orbit of $V_n^+ = \langle e_1, \dots, e_n\rangle$ and $\mathcal{I}_n^-$ is the Γ-orbit of $V_n^- = \langle e_1, \dots, e_{n-1}, f_n\rangle$ where $\{e_i, f_j\}$ is a fixed basis of V such that $(e_i, e_j) = (f_i, f_j) = 0$ for every i, j and $(e_i, f_j) = \delta_{ij}$ (see [4]);

here $k \leqslant l \leqslant n, u \in \mathcal{I}_l, v \in \mathcal{I}_k$ and P_v, P_u are the stabilizers of v, u in Γ;

if $U \leqslant V$, then $I(U)$ is the codimension of a maximal totally isotropic subspace $U_0 \leqslant U$;

$H_2 = \langle e, f\rangle, (e,e) = (f,f) = 0, (e,f) = 1$ is the hyperbolic plane;

$H_{2s} = \underbrace{H_2 + \cdots + H_2}_{s-\text{times}}$ is the $2s$ dimensional hyperbolic space.

Algebraic groups.

Here $\widetilde{\Gamma}$ is a K-defined simple algebraic group of the type B_r, C_r, D_r such that $\widetilde{\Gamma}(K) = \Gamma$ (see [6, 8]); we use terminology of [7], in particular, we identify the algebraic group $\widetilde{\Gamma}$ with $\widetilde{\Gamma}(\overline{K})$;

for every subset $X \subset \widetilde{\Gamma}$ we denote by $\overline{X}$ the Zariski closure of X;

G_a is the one dimensional additive group: $G_a(K) = K^+$.

§1. Unitary groups over division algebras with an involution

Here we use the notations and assumptions that have been made above.

Division algebras with involutions.

There is a map $i : D \to \mathrm{M}_c(K_s)$ which is induced by the isomorphism $D \otimes_K K_s \approx \mathrm{M}_c(K_s)$. Since $\star$ is an involution of the first kind we may choose the latter isomorphism and extend $\star$ on $\mathrm{M}_c(K_s)$ so that

$$X^\star = X^t \tag{1.1}$$

where X^t is the transposed matrix of X (in this case the involution $\star$ is called of the first type (*orthogonal* type)), or

$$X^\star = J_c X^t J_c^{-1} \tag{1.2}$$

where

$$J_c = \begin{pmatrix} 0_{\frac{c}{2}} & E_{\frac{c}{2}} \\ -E_{\frac{c}{2}} & 0_{\frac{c}{2}} \end{pmatrix},$$

and $0_{\frac{c}{2}}, E_{\frac{c}{2}} \in \mathrm{M}_{\frac{c}{2}}(K_s)$ is, respectively, the zero matrix and the identity matrix (in this case the involution $\star$ is called of the second type (*symplectic* type) (see [6, 5]). Note that

$$J_c^t = J_c^{-1}. \tag{1.3}$$

If $\star$ is an involution of the first kind then $c = 2^s$ for some $s \geqslant 1$ (see [5, I. §2, Corollary 2.8]).

Unitary spaces over division algebras with involutions.

The h-space V has the Witt decomposition

$$V = V_\flat \perp A_V \perp V^\flat$$

(here $X \perp Y$ denotes the direct sum of the orthogonal subspaces $X, Y \leqslant V$) where $V_\flat, V^\flat$ are maximal totally isotropic subspaces such that $V_\flat \perp V^\flat \approx H_{2n}$ (here H_{2n} is a hyperbolic space of dimension $2n$ and A_V is an anisotropic space). Let $\{e_1, \dots, e_n\}$, $\{l_1, \dots l_{d-2n}\}$,

$\{f_1, \dots, f_n\}$ be a basis of $V_\flat, A_V, V^\flat$ such that $(e_i, f_j) = \delta_{ij}$. Then the Gram matrix of h for this basis $e_1, \dots, e_n, l_1, \dots l_{d-2n}, f_n, \dots, f_1$ of V has the form

$$S_h = \begin{pmatrix} 0_{n\times n} & 0_{n\times(d-2n)} & I_n \\ 0_{(d-2n)\times n} & A & 0_{(d-2n)\times n} \\ \pm I_n & 0_{n\times(d-2n)} & 0_{n\times n} \end{pmatrix} \in \mathrm{GL}_d(D) \qquad (1.4)$$

where

$$I_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ & & \cdots & & \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \in \mathrm{GL}_n(D),$$

$$A \in \mathrm{GL}_{(d-2n)\times(d-2n)}(D), \quad (A^\star)^t = \pm A, \quad 0_{a\times b} \in \mathrm{M}_{a\times b}(D)$$

is the zero matrix.

The chosen injection $i : D \to \mathrm{M}_c(K_s)$ induces an injection $i^* : \mathrm{M}_d(D) \to \mathrm{M}_{cd}(K_s)$ where the entry $m_{pq} \in D$ of a matrix $M \in \mathrm{GL}_d(D)$ is replaced by the matrix $i(m_{pq})$. (We may extend the embedding $i : D \to \mathrm{M}_c(K_s)$ not only for the map $i^* : \mathrm{M}_d(D) \to \mathrm{M}_{cd}(K_s)$ but for every embedding $\mathrm{M}_{p\times q}(D) \to \mathrm{M}_{cp\times cq}(K_s)$ by replacing any entry a_{ij} of a matrix over D by the matrix $i(a_{ij}.)$ We also denote this embedding by i^*. Since i is a ring homomorphism we have $i^*(XYZ) = i^*(X)i^*(Y)i^*(Z)$ for any matrices X, Y, Z over D such that the multiplication XYZ is defined.)

Then

$$i^*(S_h) = \begin{pmatrix} 0_{cn\times cn} & 0_{cn\times c(d-2n)} & i^*(I_n) \\ 0_{c(d-2n)\times cn} & i^*(A) & 0_{c(d-2n)\times cn} \\ i^*(\pm I_n) & 0_{cn\times c(d-2n)} & 0_{cn\times cn} \end{pmatrix} \in \mathrm{GL}_{cd}(K_s)$$

where

$$i^*(I_n) = \begin{pmatrix} 0 & 0 & \cdots & 0 & E_c \\ 0 & 0 & \cdots & E_c & 0 \\ & & \cdots & & \\ E_c & 0 & \cdots & 0 & 0 \end{pmatrix} \in \mathrm{GL}_{cn}(K_s),$$

$E_c \in \mathrm{GL}_c(K_s)$ is the identity matrix.

The adjoint involution $\circledast$ *on* $M_{cd}(K_s)$ *and the group* $\widetilde{\Gamma}$.

We may extend the involution $\star$ from D on a simple algebra $M_{cd}(K_s)$ according to formulas (1.1) or (1.2) on each block. Consider the cases:

i. $\star$ is an involution of the first type and h is a symmetric form. Then $A^{\star t} = A$ and in the down-left corner of the matrix S_h we have I_n (see (1.4)). Since the involution $\star$ is of the first type its extension on $\mathrm{M}_c(K_s)$ is a transposition of matrices. Every matrix of $\mathrm{M}_{cd}(K_s)$ may be considered as $d \times d$-block-matrix $X = (X_{ij})$ where each entry X_{ij} is a matrix from $\mathrm{M}_c(K_s)$. Put

$$X^{\circledast} = (X_{ij})^{\circledast} := (X_{ij}^{\star})^t.$$

Then if we consider X as a matrix of $\mathrm{M}_{cd}(K_s)$ we will have $X^{\circledast} = X^t$. In particular,

$$i^*(S_h)^{\circledast} = i^*(S_h)^t = i^*(S_h).$$

Thus,

$$\begin{aligned}\{X \in \mathrm{SL}_{dc}(K_s) \mid X i^*(S_h) X^{\circledast} = X i^*(S_h) X^t = i^*(S_h)\} \\ = \mathrm{SO}_{dc}(K_s, i^*(S_h)).\end{aligned}$$

ii. $\star$ is an involution of the second type and h is a skew-symmetric form. Then $A^{\star t} = -A$ and in the down-left corner of the matrix S_h we have $-I_n$ (see (1.4)). Since the involution $\star$ is of the second type, its extension on $\mathrm{M}_c(K_s)$ is a transformation (1.2). If $X = (X_{ij}) \in \mathrm{M}_d(\mathrm{M}_c(K_s))$ with $X_{ij} \in \mathrm{M}_c(K_s)$ then $X_{ij}^{\star} = J_c X_{ij}^t J_c^{-1}$. Let us put

$$\begin{aligned}X^{\circledast} = (X_{ij})^{\circledast} := (X_{ij}^{\star})^t &= (J_{cd}(X_{ij}^t) J_{cd}^{-1})^t \\ &= (J_{cd}^{-1})^t X^t J_{cd}^t = J_{cd} X^t J_{cd}^{-1},\end{aligned}$$

where

$$J_{cd} = \begin{pmatrix} J_c & 0_c & \cdots & 0_c \\ 0_c & J_c & 0_c & \cdots \\ \cdots & & & \\ 0_c & \cdots & 0_c & J_c \end{pmatrix}$$

(see (1.2)). Since h is a skew-symmetric form we have

$$i^*(S_h)^{\circledast} = J_{cd}(i^*(S_h)^t) J_{cd}^{-1} = -(i^*(S_h))$$

and therefore

$$J_{cd}(i^*(S_h))^t = -(i^*(S_h)J_{cd}.$$

The matrix $\mathcal{J} = J_{cd}$ is skew-symmetric. Hence

$$\begin{aligned}(i^*(S_h)\mathcal{J})^t = \mathcal{J}^t(i^*(S_h))^t &= -\mathcal{J}(i^*(S_h))^t \\ &= -(-(i^*(S_h)\mathcal{J}) = i^*(S_h)\mathcal{J}\end{aligned}$$

and therefore $i^*(S_h)\mathcal{J} \in \mathrm{M}_{cd}(K_s)$ is a symmetric matrix. Now we have

$$\begin{aligned}&\{X \in \mathrm{SL}_{dc}(K_s) \mid Xi^*(S_h)X^{\circledast} = i^*(S_h)\} \\ &\quad = \{X \in \mathrm{SL}_{dc}(K_s) \mid Xi^*(S_h)\mathcal{J}X^t = i^*(S_h)\mathcal{J}\} \\ &\qquad = \mathrm{SO}_{dc}(K_s, i^*(S_h)\mathcal{J}).\end{aligned}$$

iii. $\star$ is an involution of the first type and h is a skew-symmetric form. The same arguments as above imply that $i^*(S_h)$ is a skew-symmetric matrix and

$$\{X \in \mathrm{SL}_{dc}(K_s) \mid Xi^*(S_h)X^{\circledast} = Xi^*(S_h)X^t = i^*(S_h)\} = \mathrm{Sp}_{dc}(K_s).$$

iv. $\star$ is an involution of the second type and h is a symmetric form. The same arguments as above imply that $i^*(S_h)\mathcal{J}$ is a skew-symmetric matrix and

$$\{X \in \mathrm{SL}_{dc}(K_s) \mid Xi^*(S_h)X^{\circledast} = Xi^*(S_h)\mathcal{J}X^t = i^*(S_h)\mathcal{J}\} = \mathrm{Sp}_{dc}(K_s).$$

The adjoint involution $\circledast$ defines a simple algebraic group $\widetilde{\Gamma}$ of the type B_r, C_r, D_r which is defined by equations over K_s. Since $\mathrm{char} K \neq 2$ the group $\widetilde{\Gamma}$ is defined and completely split over K_s. Also, $\widetilde{\Gamma}(K_s)$ is dense in $\widetilde{\Gamma}$ and is $\mathrm{Gal}(K_s/K)$-stable. Hence $\widetilde{\Gamma}$ is a K-defined group ([7, 11.2.8]). The Galois group $\mathcal{G} = \mathrm{Gal}(K_s/K)$ acts on $\widetilde{\Gamma}(K_s)$ in the following way. The isomorphism $D \otimes_K K_s \approx \mathrm{M}_c(K_s)$ induces the action of $\mathcal{G}$ (namely, $\mathcal{G}$ acts on the right arguments of $D \otimes_K K_s$) such that $\mathrm{M}_c(K_s)^{\mathcal{G}} = D$. The group $\widetilde{\Gamma}(K_s)$ is presented by c-block matrices (see i.-iv.) and the group $\mathcal{G}$ operates on each block. Thus, the invariant subgroup $\Gamma(K_s)^{\mathcal{G}}$ is the group $\mathrm{SU}(D, h)$.

The comparison of the unitary space V and orthogonal or symplectic spaces V^s and $\overline{V}$.

We fix a basis $\mathfrak{B} = \{e_1, \dots, e_n, l_1, \dots l_{d-2n}, f_n, \dots, f_1\}$ of the linear space V such that the Gram matrix of h in this basis is S_h (see (1.4)). Let V_s (respectively $\overline{V}$) be the linear space over the field K_s (respectively $\overline{K}$) of the dimension cd with the fixed basis :

$$\begin{aligned} &\mathfrak{B}_s (= \overline{\mathfrak{B}}) \\ &= \{e_{11}, e_{12}, \dots e_{1c}, \quad e_{21}, e_{22} \dots e_{2c}, \dots, e_{n1}, e_{n2} \dots, e_{nc}, \\ &l_{11}, l_{12}, \dots l_{1c}, \quad l_{21}, l_{22} \dots l_{2c}, \dots, l_{d-2n\,1}, l_{d-2n\,2} \dots, l_{d-2n\,c}, \\ &f_{n1}, f_{n2}, \dots f_{nc}, \quad f_{n-1\,1}, f_{n-1\,2} \dots f_{n-1\,c}, \dots, f_{11}, f_{12} \dots, f_{1c}\}. \end{aligned} \tag{1.5}$$

We consider the space V^s (resp. $\overline{V}$) as an orthogonal or a symplectic space which corresponds to the matrix $i^*(S_h)$ or $i^*(S_h)\mathcal{J}$ (see i.-iv.) in the basis $\mathfrak{B}_s$ (resp. $\overline{\mathfrak{B}}$). We denote the corresponding bilinear form on V_s (resp. $\overline{V}$) by $(\ ,\ )_s$ (resp. $(\ ,\ )_{alg}$).

For every vector

$$x = \sum_{i=1}^{n} a_i e_i + \sum_{i=1}^{d-2n} b_i l_i + \sum_{i=n}^{1} c_i f_i \in V,$$

where $a_i, b_i, c_i \in D$, we put

$$[x] = (i^*(a_1), \dots, i^*(a_n), i^*(b_1), \dots i^*(b_{d-2n}), i^*(c_1), \dots, i^*(c_n)) \in M_{c \times cd}(K_s). \tag{1.6}$$

Thus, $[x]$ is the matrix that corresponds to the row of coordinates

$$(x) = (a_1, \dots, a_n, b_1, \dots, b_{d-2n}, c_1, \dots, c_n) \tag{1.7}$$

of the vector x in the basis $\mathfrak{B}$. Let $[x]_j$ be the j^{th}-row of the matrix $[x]$. Then the row $[x]_j^s$ defines a vector in V_s

$$x_j^s = [x]_j \mathfrak{B}_s^t.$$

(Here we multiply the row of length cd of elements of K_s with the column of elements of the basis $\mathfrak{B}_s$. Thus we get a vector in V_s).

We define the vector $x^s = (x_1^s, x_2^s, \dots, x_c^s) \in V_s^c$ where

$$x_j^s = \text{the } j\text{-s row of the matrix} \quad [x].$$

Denote the "Gram" matrix by

$$G(x^s, y^s) := ((x_i^s, y_j^s)_s) \in M_{c\times c}(K_s).$$

Proposition 3. *If $x, y \in V$, then the matrix $i^*((x,y))$ coincides with the Gram matrix $G(x^s, y^s)$ up to permutations and changing the signs of columns.*

Proof. Let $(x), (y)$ be rows of coordinates of x, y in the basis $\mathfrak{B}$ (see (1.7)) and $[x], [y] \in M_{c\times cd}(K_s)$ be the corresponding matrices of the form (1.6). Then

$$(x, y) = (x)S_h(y^\star)^t.$$

Thus

$$i^*((x,y)) = i^*(x)i^*(S_h)i^*((y)^\star)^t). \tag{1.8}$$

Now we consider the cases i. or iii. Then $\mathcal{S} = i^*(S_h)$ is a symmetric matrix or a skew symmetric matrix over K_s and $i^*((y)^\star)^t) = i^*((y))^t$ (recall that, in these cases, for every component $y_i \in D$ of (y) we have $i(y_i^\star) = i(y_i)^t$). Then from (1.8) we have

$$i^*((x,y)) = [x]\mathcal{S}[y]^t. \tag{1.9}$$

The definition of the Gram matrix $G(x^s, y^s)$ implies that its ij-entry is equal to $[x]_i\mathcal{S}[y]_j^t$. Now we have the statement from (1.9).

Now we consider the cases ii. or iv. Then $\mathcal{S}\mathcal{J} = i^*(S_h)\mathcal{J}$ is a symmetric matrix or a skew symmetric matrix over K_s and

$$i^*((y)^\star)^t) = \mathcal{J}i^*((y))^t J_c^{-1}$$

(recall that in these cases, for every component $y_i \in D$ of (y), we have $i(y_i^\star) = J_c i(y_i)^t J_c^{-1}$). Then from (1.8) we have

$$i^*((x,y)) = [x]\mathcal{S}\mathcal{J}[y]^t J_c^{-1}. \tag{1.10}$$

The definition of the Gram matrix $G(x^s, y^s)$ implies that its ij-entry is equal to $[x]_i\mathcal{S}\mathcal{J}[y]_j^t$. Now we have the statement from (1.10) because the right multiplication by J_c^{-1} is equivalent to a permutation of columns and their multiplication for ± 1. □

Corollary 4.

$$(x,y)=0 \Leftrightarrow (x_i^s, y_j^s)_s = 0 \textit{ for every } \quad i,j=1,\dots,c$$
$$\Leftrightarrow (x_i^s, y_j^s)_{alg} = 0 \textit{ for every } \quad i,j=1,\dots,c.$$

Proof. From Proposition 3

$$(x,y)=0 \Leftrightarrow i^*((x,y)) = \text{ zero matrix} \Leftrightarrow G(x^s,y^s) = \text{ zero matrix.}$$

Now the statement follows from the definition of $G(x^s,y^s)$. □

The correspondence between totally isotropic subspaces in V and in V^s. Let $U \leqslant V$ and let $U_s \leqslant V_s$ be the subspaces generated (over K_s) by all $x_j^s, j=1,\dots,c$, where $x \in V$. The subspace of $\overline{V}$ which is generated (over $\overline{K}$) by the same vectors will be denoted by $\overline{U}$. If $U = \langle l_1,\dots,l_k\rangle$ then U_s is generated by vectors l_{ij}^s for all $i=1,\dots,k, j=1,\dots,c$ (here l_{ij}^s is the vector in V^s which corresponds to the j^{th} row of the matrix $[l_i]$). (Indeed, the coordinates in the basis $\mathfrak{B}_s$ of vectors l_{ij}^s are rows of the matrices $[l_i]$. Further, if $a_i \in D$ then the rows of the matrix $[a_i l_i] = i(a_i)[l_i]$ are linear combinations over K_s of the rows $[l_i]$. Then for any vector $x = \sum\limits_i a_i l_i$ the rows of $[x]$ are linear combinations over K_s of the rows of $[l_i], i=1,\dots,k$.) Note that vectors l_{ij}^s are linearly independent over K. (It is obvious if $\{l_1,\dots,l_k\}$ is a part of the basis $\mathfrak{B}$.) On the other hand, there exists a non-degenerate matrix $T \in \mathrm{GL}_k(D)$ such that the elements of the column

$$\begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_k \end{pmatrix} = T \begin{pmatrix} l_1 \\ l_2 \\ \dots \\ l_k \end{pmatrix} \quad \text{are a part of the basis} \quad \mathfrak{B}.$$

Then the definition of the operations $x \to [x], T \to i^*(T)$ implies

$$B = \begin{pmatrix} [b_1] \\ [b_2] \\ \dots \\ [b_k] \end{pmatrix} = i^*(T) \begin{pmatrix} [l_1] \\ [l_2] \\ \dots \\ [l_k] \end{pmatrix}.$$

Since $b_1, \dots, b_k \in \mathfrak{B}$, the rows of the matrix $B \in \mathrm{M}_{ck\times cd}(K)$ are linearly independent and hence the rows of the matrix

$$\begin{pmatrix} [l_1] \\ [l_2] \\ \dots \\ [l_k] \end{pmatrix} \in M_{ck\times cd}(K)$$

are also linearly independent. But the rows of this matrix are exactly the coordinates of the vectors l_{ij}^s.) Hence

$$\dim_{K_s} U_s = c \dim_D U.$$

The same arguments show

$$\dim_{\overline{K}} \overline{U} = c \dim_D U.$$

Proposition 5. *Let $U \leqslant V$ be a totally isotropic subspace with respect to the form $h = (\,,\,)$. Then U_s (respectively, $\overline{U}$) is a totally isotropic subspace of V_s (respectively, $\overline{V}$) with respect to the form $(\,,\,)_s$ (respectively, $(\,,\,)_{alg}$).*

Proof. This follows directly from Corollary 4. □

Recall that *a pseudo-hyperbolic subspace $U \leqslant V$* is a linear subspace such that

$$U = \operatorname{rad} U \perp H_{2s}$$

where $\operatorname{rad} U$ is the radical of U and $H_{2s} = H_2 \perp H_2 \perp \cdots \perp H_2$ is the orthogonal sum of s hyperbolic surfaces ([4]). Note that all pseudo-hyperbolic subspaces with given $\dim U$ and $I(U)$ are in the same $\mathrm{SU}(D,h)$-orbit except when $D = K$ and h is a totally split symmetric form, $\mathrm{SU}(D,h) = \mathrm{SO}_{2n}(K)$, and $\dim U = n, I(U) = 0$ ([4, Proposition 3.1]).

Further, if $U_1, U_2 \leqslant V$ are totally isotropic, then $U = U_1 + U_2$ is a pseudo-hyperbolic subspace of V ([4], Lemma 4.1)

Proposition 6. *Let $U_1, U_2 \leqslant V$ be totally isotropic spaces and $U = U_1 + U_2$. Then $\overline{U} = \overline{U}_1 + \overline{U}_2$ and*

$$I(\overline{U}) = cI(U).$$

Proof. If $D = K$, then $\overline{V} = V \otimes_K \overline{K}$ and the statement is obvious. In other cases, the $\mathrm{SU}(D,h)$-orbit of the pair (U_1, U_2) is uniquely defined by $\dim(U_1+U_2)$ and $I(U_1+U_2)$ (see [3, Lemma 1.2 and Theorem 1, 2 below]). Thus, we may assume that U_1, U_2 are generated by elements of the basis $\mathfrak{B}$ and therefore $\overline{U}_1, \overline{U}_2$ are generated over $\overline{K}$ by elements of the basis $\mathfrak{B}_{alg} = \mathfrak{B}_s$. Then the basis of $\overline{U}_1 + \overline{U}_2$ can be chosen among the elements of $\mathfrak{B}_{alg} = \mathfrak{B}_s$. The definition of $\mathfrak{B}_{alg} = \mathfrak{B}_s$ implies the statement. □

§2. Maximal parabolic subgroups $\widetilde{P}_u, \widetilde{P}_v$

Proposition 7. *There exists a maximal parabolic subgroup $\widetilde{P}_v$ of $\widetilde{\Gamma}$ which is defined over K and such that $P_v = \widetilde{P}_v(K)$. Moreover,*

$$\widetilde{P}_v(K_s) = \{g \in \widetilde{\Gamma}(K_s) \mid g(v_s) = v_s\}, \quad \widetilde{P}_v = \{g \in \widetilde{\Gamma} \mid g(\overline{v}) = \overline{v}\}.$$

Proof. The stabilizer of $\overline{v}$ in the group $\widetilde{\Gamma} = \widetilde{\Gamma}(\overline{K})$ is a maximal parabolic subgroup because $\overline{v}$ is a totally isotropic subspace $\overline{v} \leqslant \overline{V}$ (see Proposition 5). Let us denote this group by $\widetilde{P}_v$. Since $\widetilde{\Gamma}$ is defined and completely split over K_s, the group $\widetilde{P}_v$ is also defined over K_s. Then

$$\widetilde{P}_v(K_s) = \{g \in \widetilde{\Gamma}(K_s) \leqslant \mathrm{GL}(\overline{V}) \mid g(\overline{v}) = \overline{v}\}.$$

However, the space $\overline{v} = v_s \otimes_{K_s} \overline{K}$ is generated by the same basis as v_s, and for every $g \in \mathrm{GL}_{cd}(K_s) \leqslant \mathrm{GL}_{cd}(\overline{K})$ we have

$$g(\overline{v}) = \overline{v} \Leftrightarrow g(v_s) = v_s.$$

Hence

$$\widetilde{P}_v(K_s) = \{g \in \widetilde{\Gamma}(K_s) \leqslant \mathrm{GL}(V_s) \mid g(v_s) = v_s\}.$$

Now we assume that $v = \langle e_1, \ldots, e_k \rangle$ where $k \leqslant n$. The stabilizer $St_{\overline{v}}$ in the group SL_{cd} is the closed subgroup which is defined by zero equations $x_{ij} = 0$ for the appropriate entries of matrices. Obviously, this algebraic group is a K-defined subgroup of SL_{cd} and

$$St_{\overline{v}}(K_s) = St_{v_s} := \{g \in \mathrm{SL}_{cd}(K_s) \mid g(v_s) = v_s\}.$$

The equations $x_{ij} = 0$ defining the group $St_{\overline{v}}$ preserve the c-block structure of SL_{cd}, and therefore the group St_{v_s} is $\mathcal{G}$-stable. Now we have $\widetilde{P}_v = \widetilde{\Gamma} \cap St_{\overline{v}}$. Note that $\widetilde{P}_v$ is a K_s-defined group and therefore $\widetilde{P}_v(K_s) = \Gamma(K_s) \cap St_{\overline{v}}(K_s)$. Since $\widetilde{\Gamma}(K_s)$ and St_{v_s} are $\mathcal{G}$-stable groups the group $\widetilde{P}_v(K_s)$ is $\mathcal{G}$-stable and therefore $\widetilde{P}_v$ is a K-defined algebraic group ([7, 11.2.8]) such that

$$\widetilde{P}_v(K) = \widetilde{P}_v(K_s)^{\mathcal{G}} = \widetilde{\Gamma}(K) \cap St_{\overline{v}}(K). \qquad \square$$

The double cosets $\widetilde{P}_u\gamma\widetilde{P}_v$ *and* $P_u\gamma P_v$. Note that Theorems 1,2 hold for groups $\widetilde{\Gamma}$ and $\widetilde{\Gamma}(K_s)$ (here $D = \overline{K}$ or $D = K_s$ and $\star$ is a trivial involution). Recall that double cosets $\widetilde{P}_u\gamma_i\widetilde{P}_v$ are locally closed sets.

Proposition 8. *Suppose* $\widetilde{P}_u\gamma\widetilde{P}_v \cap \Gamma \neq \varnothing$. *Then the double coset* $\widetilde{P}_u\gamma\widetilde{P}_v$ *is* K*-defined.*

Proof. Suppose $\widetilde{P}_u\gamma\widetilde{P}_v = \widetilde{P}_u\gamma'\widetilde{P}_v$ for some $\gamma' \in \Gamma$. Consider the action of $\widetilde{\Gamma} \times \widetilde{\Gamma}$ on $\widetilde{\Gamma}$ by left-right multiplications: $(g_1, g_2)(x) = g_1 x g_2^{-1}$. Since $\widetilde{\Gamma}$ is a K-defined group this action endows $\widetilde{\Gamma}$ with the structure of $\widetilde{\Gamma} \times \widetilde{\Gamma}$ space, defined over K. Then $\widetilde{\Gamma}$ is also a $\widetilde{P}_u \times \widetilde{P}_v$-space which is also defined over K because $\widetilde{P}_u \times \widetilde{P}_v$ is a K-defined subgroup of $\widetilde{\Gamma}$. The set $\widetilde{P}_u\gamma'\widetilde{P}_v$ is an orbit in $\widetilde{\Gamma}$ of the element $\gamma' \in \widetilde{\Gamma}(K) = \Gamma$. Hence $\widetilde{P}_u\gamma'\widetilde{P}_v$ is K-defined ([7, Proposition 12.1.2, ii]). $\square$

Definition 9. *We say that the double coset* $\widetilde{P}_u\gamma\widetilde{P}_v$ *is properly* K*-defined if* $\widetilde{P}_u\gamma\widetilde{P}_v \cap \Gamma \neq \emptyset$.

Proposition 10. *If the double coset* $\widetilde{P}_u\gamma\widetilde{P}_v$ *is properly* K*-defined and* $\gamma \in \Gamma$ *then*

$$\widetilde{P}_u\gamma\widetilde{P}_v(K) = P_u\gamma P_v.$$

Proof. We have

$$\widetilde{P}_u\gamma\widetilde{P}_v(K) = \widetilde{P}_u\gamma\widetilde{P}_v \cap \Gamma$$

(because of Proposition 8). If we are in the Special Case the group Γ is a Chevalley group where each double coset $\widetilde{P}_u\gamma\widetilde{P}_v$ can be defined by the same element $\gamma \in \Gamma$ as a corresponding coset $P_u\gamma P_v$, and therefore

$$\widetilde{P}_u\gamma\widetilde{P}_v \cap \Gamma = P_u\gamma P_v. \tag{2.1}$$

Now we assume Γ being in the General Case. Then every double coset $P_u\gamma P_v = P_u\gamma_{pq}P_v$ is uniquely defined by $\dim(u+\gamma(v)) = l+p+q$ and $q = I(u+g(v))$. We have the inclusion $P_u\gamma P_v \subset \widetilde{P}_u\gamma\widetilde{P}_v \cap \Gamma$. Suppose $\gamma' \in \widetilde{P}_u\gamma\widetilde{P}_v \cap \Gamma, \gamma' \notin P_u\gamma P_v$. Then $\gamma' \in P_u\gamma_{p'q'}P_v$ where $(p',q') \neq (p,q)$. Thus, $\widetilde{P}_u\gamma_{pq}\widetilde{P}_v = \widetilde{P}_u\gamma_{p'q'}\widetilde{P}_v$. However, Propositions 7, 6 and Theorems 1, 2 imply that any double coset $\widetilde{P}_u\gamma\widetilde{P}_v$ uniquely defines the numbers $cq = I(\overline{u}+\gamma(\overline{v})), \dim(\overline{u}+\gamma(\overline{v}))-cl-cq$. This is a contradiction. Hence we have (2.1). □

If the group Γ is in the Special Case then we have a one-to-one correspondence between double cosets $P_u\gamma P_v$ of Γ and $\widetilde{P}_u\gamma\widetilde{P}_v$ of $\widetilde{\Gamma}$. Let Γ and $\widetilde{\Gamma}$ be in the General Case. Then Propositions 6, 10 and Theorem 1 show that the double coset $\widetilde{P}_u\gamma\widetilde{P}_v$ is properly K-defined if and only if

$$\widetilde{P}_u\gamma\widetilde{P}_v = \widetilde{P}_u\gamma_{p'q'}\widetilde{P}_v, \quad p' = cp, q' = cq, (p,q) \in X_{pq}. \tag{2.2}$$

Let Γ be in the General Case and $\widetilde{\Gamma}$ be in the Special Case. Then in $\widetilde{\Gamma}$ we may have negative cosets if and only if the space $(\bar{u}+g(\bar{v})^{\bar{u}})$ is a maximal totally isotropic subspace which belongs to a negative $\widetilde{\Gamma}$-orbit. However, all subspaces of the form $\bar{w}$, where $w \in \mathcal{I}_r$, are in the same Γ-orbit if the number r is fixed (because Γ is in the General Case). Thus, we always may assume that we choose the orientation in $\widetilde{\Gamma}$ such that all properly K-defined double cosets are positive. Also, in the Special Case i4. we have $\operatorname{sign}(\bar{u},\bar{v}) = 1$ because $c = 2^s \mid \mathrm{d_{in}}(\bar{u},\bar{v})$. Hence Theorems 1, 2 imply that we also have (2.2) when Γ is in the General Case and $\widetilde{\Gamma}$ is in the Special Case. We get

Proposition 11. *Suppose Γ is in the General Case. Then the double coset $\widetilde{P}_u\gamma\widetilde{P}_v$ is properly K-defined if and only if*

$$\widetilde{P}_u\gamma\widetilde{P}_v = \widetilde{P}_u\gamma_{p'q'}\widetilde{P}_v, \quad p' = cp, q' = cq, (p,q) \in X_{pq}.$$

§3. The proof of the Theorem

We have to prove the result on adherence of properly K-defined double cosets $\widetilde{P}_u\gamma\widetilde{P}_v$. Thus, if we prove the corresponding result on adherence of all double cosets of $\widetilde{\Gamma}$ we will have our statement. We may and we will assume that $D = K = \overline{K}$ and $\Gamma = \widetilde{\Gamma}$, $P_u = \widetilde{P}_u$, $P_v = \widetilde{P}_v$.

Recall that, for $g \in \Gamma$, we write $v_g = u + g(v)$.

Lemma 12. *Let r be a non-negative integer. Then*

$$S_r = \langle g \in \Gamma \mid \dim_F v_g \leqslant r\}, \quad T_r = \langle g \in \Gamma \mid I(v_g) \leqslant r\}$$

are closed subsets of Γ.

Proof. Let $r_1, \dots, r_l$ be any fixed basis of u and let $s_1, \dots, s_k$ be any fixed basis of v. For $g \in \Gamma$ put $t_1 = g(s_1), \dots, t_k = g(s_k)$. Further, let us fix a basis of V and let $X_g \in \mathrm{M}_{(k+l)\times d}(K)$ be the matrix which consists of the rows of coordinates of $t_1, \dots, t_k, r_1, \dots, r_l$ in our fixed basis of V. Then $\dim_K v_g = \mathrm{rank}_K X_g$. Hence

$$g \in S_r \Leftrightarrow \text{ every minor of the rank } \quad r+1 \quad \text{of the matrix } X_g$$
$$\text{is equal to zero.}$$

Thus, the set S_r is defined by the system of algebraic equations on the entries of $\mathrm{M}_{(k+l)\times d}(K)$. Since the values of entries of the matrix X_g can be expressed as polynomials on entries of $g \in \Gamma \leqslant \mathrm{GL}(V)$ the set S_r is defined by the system of algebraic equations on entries of $\mathrm{GL}(V)$. Thus, S_r is a closed subset of Γ.

Now let $M_g = (m_{ij})$ be the $l \times k$-matrix where $m_{ij} = (r_i, t_j)$. We have

$$\mathrm{rank}\, M_g = q = I(v_g). \tag{3.1}$$

Indeed, we may change the basis $r_1, \dots, r_l$ of u for a basis $r'_1, \dots, r'_l$ and the basis $t_1, \dots, t_k$ of $g(v)$ by a basis $t'_1, \dots, t'_k$ such that the matrix $M'_g = ((r'_i, t'_j))$ will have the block-form

$$M'_g = \left(\begin{array}{c|c} E_q & \mathbf{0}_{(k-q)\times q} \\ \hline \mathbf{0}_{(l-q)\times q} & \mathbf{0}_{(l-q)\times(k-q)} \end{array}\right)$$

where $q = I(v_g)$. Obviously,

$$\begin{aligned} \operatorname{rank} M'_g = q \quad &\text{and} \quad M'_g = AM_gB \\ &\text{for some} \quad A \in \mathrm{GL}_l(K),\ B \in \mathrm{GL}_k(K). \end{aligned} \tag{3.2}$$

Now (3.1) follows from (3.2).

The equality (3.1) implies

$$g \in T_r \Leftrightarrow \text{ every minor of the rank } r+1 \text{ of the matrix } M_g$$
$$\text{is equal to zero.}$$

The same arguments as above show that T_r is a closed subset of Γ. □

Now we prove the implication

$$P_u\gamma^{\pm}_{pq}P_v \subset \overline{P_u\gamma^{\pm}_{p'q'}P_v} \Rightarrow p+q \leqslant p'+q', q \leqslant q'. \tag{3.3}$$

We have

$$\begin{aligned} &g \in P_v\gamma^{\pm}_{p'q'}P_v \\ &\overset{\text{Theorems 1,2}}{\Rightarrow} \dim v_g = l + p' + q' \Rightarrow g \in S_{l+p'+q'} \\ &\overset{\text{Lemma 12}}{\Rightarrow} \overline{P_u\gamma^{\pm}_{p'q'}P_v} \subset S_{l+p'+q'} \Rightarrow P_u\gamma^{\pm}_{pq}P_v \subset S_{l+p'+q'} \\ &\overset{\text{Theorems 1,2}}{\Rightarrow} p+q \leqslant p'+q'. \end{aligned} \tag{3.4}$$

Further,

$$\begin{aligned} g \in P_v\gamma^{\pm}_{p'q'}P_v &\overset{\text{Theorems 1,2}}{\Rightarrow} g \in T_{q'} \overset{\text{Lemma 12}}{\Rightarrow} \overline{P_u\gamma^{\pm}_{p'q'}P_v} \subset T_{q'} \\ &\Rightarrow P_u\gamma^{\pm}_{pq}P_v \subset T_{q'} \Rightarrow q \leqslant q'. \end{aligned} \tag{3.5}$$

From (3.4) and (3.5) we get the implication (3.3).

Let us prove

$$P_u\gamma^{\pm}_{pq}P_v \subset \overline{P_u\gamma^{\pm}_{p'q'}P_v} \Leftarrow p+q \leqslant p'+q', q \leqslant q' \qquad (3.6)$$

for two different pairs $(p,q) \neq (p',q')$.

Recall that by the symbol X_{pq} we denote the set of all pairs (p,q) in the General Case such that there exists a corresponding double coset $P_u\gamma_{pq}P_v$ of Γ. By the same symbol we also denote below the set of all pairs (p,q) in the Special Case such that there exists a corresponding double coset $P_u\gamma_{pq}P_v$ of Γ. We may see from Theorem 2 that the description of X_{pq} in the Special Case is the same as in the General Case

$$(p,q) \in X_{pq} \Leftrightarrow 0 \leqslant p \leqslant \min\{k, n-l\}, \quad 0 \leqslant q \leqslant k-p \qquad (3.7)$$

except for the Special Case i4) when X_{pq} consists of the pairs $(0,q)$ where $q \leqslant n$ and all q are odd or all q are even.

Lemma 13. *Assume we are in the General Case or in the Special Case but not in* (*i*4)*. Further, let* $(p',q') \in X_{pq}$ *and let* (p'',q'') *be a pair of non-negative integers such that*

$$p''+q'' \leqslant p'+q',\ p'' \leqslant \min\{k, n-l\}.$$

Then

$$(p'',q'') \in X_{pq}.$$

Proof. The inequality $0 \leqslant p'' \leqslant n-l$ holds because of the conditions of the Lemma. Suppose, $q'' > k-p''$. Then $p'+q' \geqslant p''+q'' > k$ and therefore $q' > k-p'$ which is a contradiction to (3.7). Thus, we also have $q'' \leqslant k-p''$ and, by (3.7), $(p'',q'') \in X_{pq}$. □

Lemma 14. *Suppose the implication* (3.6) *holds for all such cases:*
case 1. $q' = q+1, p'+1 = p$,
case 2. $q' = q, p' = p+1$,
case 3. $p' = p, q' = q+1$,
case 4. (*only for the case* (i4) *of Theorem* 2) $p' = p = 0, q' = q+2$.
Then the implication (3.6) *holds for any possible pairs* $(p,q) \neq (p',q') \in X_{pq}$*, where* $p+q \leqslant p'+q', q \leqslant q'$.

Proof. Note that if

$$p+q \leqslant p''+q'' \leqslant p'+q' \quad \text{and} \quad q \leqslant q'' \leqslant q',$$

then we have the implication

$$\begin{cases} P_u\gamma^{\pm}_{pq}P_v \subset \overline{P_u\gamma^{\pm}_{p''q''}P_v} \\ \text{and} \\ P_u\gamma^{\pm}_{p''q''}P_v \subset \overline{P_u\gamma^{\pm}_{p'q'}P_v} \end{cases} \Rightarrow P_u\gamma^{\pm}_{pq}P_v \subset \overline{P_u\gamma^{\pm}_{p'q'}P_v}. \tag{3.8}$$

Consider the General Case or in the Special Case (i1)–(i3).

Let $q' = q$ (or $p' = p$). Then $p' > p$ (resp. $q' > q$). Put $p'' = p'-1, q'' = q$ (resp. $q'' = q'-1, p' = p$). Then $p'' \geqslant p$ (resp. $q'' \geqslant q$) and the pair (p'', q'') satisfies the conditions of Lemma (13) and therefore $(p'', q'') \in X_{pq}$.

Consider the Special Case (i4). Then $p' = p = 0$ and therefore $q' \geqslant q+2$. Put $p'' = 0, q'' = q'-2$. Then $(p'', q'') \in X_{pq}$ (Theorem 2).

Using the assertion for case 2 (resp. case 3) or for case 4 (if we are in the Special Case (i4), (3.8) and the induction on $p'-p$ (resp. $q'-q$) we can get the statement for p, q, p', q'.

If $q' > q$ and $p' > p$, we put $q'' = q', p'' = p'-1$. Lemma 13 implies $(p'', q'') \in X_{pq}$. Using the case 2, (3.8) and the induction we can get the previous case $q' > q, p' = p$.

If $q' > q$ and $p' < p$, we put $q'' = q'-1, p'' = p'+1$. Since $p'' = p'+1 \leqslant p \leqslant n-l$ and $p''+q'' = p'+q'$ Lemma (13) implies $(p'', q'') \in X_{pq}$. Using the case 1, (3.8) and the induction on $\min\{q'-q, p-p'\}$ we can get one of the previous cases. □

Lemma 15. *Suppose the numbers p, q, p', q' satisfy one of the conditions of cases of Lemma 14. Then there exists a morphism $\chi : G_a \to \Gamma$ such that*

$$\chi(\alpha) \in P_u\gamma^{\pm}_{p'q'}P_v \quad \textit{for every} \quad \alpha \neq 0$$

and $\chi(0) \in P_u\gamma^{\pm}_{pq}P_v$.

Proof. Recall

$$V = \langle e_1, \ldots, e_n, f_1, \ldots f_n\rangle$$

or

$$V = \langle e_1, \ldots, e_n, e_0, f_1, \ldots f_n \rangle,$$

where $(e_i, e_j) = (f_i, f_j) = 0, \quad (e_i, f_j) = \delta_{ij}, (e_0, e_0) = 1$. We may assume

$$u = \langle e_1, \ldots, e_l \rangle.$$

Further, we use the following scheme of the proof. We point out the subspace v_0 which is generated by some elements of the basis e_i, f_j and such that

$$\begin{aligned} \dim(u + v_0) &= l + p + q, \\ I(u + v_0) &= q, \quad v_0 = \gamma(v) \text{ for some } \gamma \in \Gamma. \end{aligned} \tag{3.9}$$

Then we define the morphism $\zeta : G_a \to \Gamma$ such that the elements $g_\alpha = \zeta(\alpha)$ satisfy the following conditions: $g_0 = 1$ and

$$\begin{aligned} \dim(u + g_\alpha(v_0)) &= l + p' + q', \\ I(u + g_\alpha(v_0)) &= q', \quad \text{for every} \quad 0 \neq \alpha \in K. \end{aligned} \tag{3.10}$$

Then (3.9), (3.10) imply that the map $\chi : G_a \to \Gamma$, which is defined by the formula $\chi(\alpha) = g_\alpha \gamma$, satisfies the condition of the statement of the Lemma.

The map $\zeta : G_a \to \Gamma$ will be constructed below by the following scheme. We point out the subspace $U \leqslant V$ which is either the hyperbolic space of dimension $2, 4$ or an orthogonal subspace of the dimension 3, and we construct the appropriate map $\zeta : G_a \to \Delta$ where Δ is the subgroup of Γ consisting of orthogonal or symplectic transformations of V which are trivial on $U^\perp$. Here we have to point out only the values $g_\alpha(x)$ for elements of a fixed basis of U.

Below the consideration of cases 1.–4. has a splitting into subcases which we have numerated using the lexicographical order.
Case 1. $q' = q + 1, p' + 1 = p$.

Subcase 1.1: $P_u\gamma^{\pm}_{pq}P_v = P_u\gamma_{pq}P_v$ *and* $P_u\gamma^{\pm}_{p'q'}P_v = P_u\gamma_{p'q'}P_v$ (*that is, both double cosets are positive*). Put

$$v_0 := \langle f_1,\dots,f_q\rangle + \langle e_{l+1},\dots,e_{l+p}\rangle + \begin{cases}\langle e_l, e_{l-1},\dots,e_{l-(k-p-q)+1}\rangle & \text{if } k-p-q>0,\\ \{0\} & \text{if } k-p-q=0.\end{cases}$$

Since $q < l-(k-p-q)+1 = l-k+p+q+1$ the space v_0 is totally isotropic and v_0 satisfies the conditions (3.9). Note that $l-k+p+q+1 > q+1$ (because $p = p'+1 > 0$) and $q+1 = q' \leqslant k \leqslant l$. Hence

$$e_{q+1} \notin v_0, \quad e_{q+1} \in u. \tag{3.11}$$

Put $U := \langle e_{q+1}, f_{q+1}\rangle + \langle e_{l+p}, f_{l+p}\rangle = H_4$, and for every $\alpha \in K$ we may define the element $g_\alpha \in \mathrm{SL}(V)$ by the formula

$$g_\alpha(e_{l+p}) = e_{l+p} + \alpha f_{q+1}, \quad g_\alpha(f_{q+1}) = f_{q+1},$$
$$g_\alpha(e_{q+1}) = e_{q+1} \pm \alpha f_{l+p}, \quad g_\alpha(f_{l+p}) = f_{l+p}$$

and $g_\alpha(x) = x$ for every $x \in U^\perp$. It is easy to see that $g_\alpha \in \Gamma$ (with an appropriate choice of signs in $g_\alpha(e_{q+1}) = e_{q+1} \pm \alpha f_{l+p}$).

Further, the construction of g_α implies

$$g_\alpha(v_0) = \langle e_{l-k+p+q+1},\dots,e_{l+p-1}, e_{l+p}+\alpha f_{q+1}, f_1 \dots, f_q\rangle.$$

Then, if $\alpha \neq 0$ we have (see (3.11))

$$u + g_\alpha(v_0) = \underbrace{\langle e_1,f_1\rangle \perp \cdots \perp \langle e_q,f_q\rangle \perp \langle e_{q+1}, e_{l+p}+\alpha f_{q+1}\rangle}_{=H_{2(q+1)}} \perp \langle e_{q+2},\dots,e_{l+p-1}\rangle$$

and therefore we have (3.10). For $\alpha = 0$ we have $g_\alpha = g_0 = 1$ and therefore we have (3.9) instead of (3.10). Since v_0 is a totally isotropic space of dimension $k \leqslant l < n$ (because $l+p \leqslant n$ and $p > 1$) there is an element $\gamma \in \Gamma$ such that $v_0 = \gamma(v)$. Hence the map $\chi : G_a \to \Gamma$, defined by the formula $\chi(\alpha) = \gamma_\alpha\gamma$, is a morphism and it satisfies the condition of the Lemma.

Subcase 1.2: $P_u\gamma^{\pm}_{pq}P_v = P_u\gamma^{-}_{pq}P_v$ and $P_u\gamma^{\pm}_{p'q'}P_v = P_u\gamma_{p'q'}P_v$ *where* $p = n - l > 0$ (*see Theorem* 2, i1.), $p' = n - l - 1$ (*that is, one double coset is negative and one is positive*). Note that the case $P_u\gamma^{\pm}_{pq}P_v = P_u\gamma_{pq}P_v$ and $P_u\gamma^{\pm}_{p'q'}P_v = P_u\gamma^{-}_{p'q'}P_v$ is impossible here because if $p' = n-l$ then $p = p'+1 > n-l$, yielding a contradiction to 3.7.

Recall that an element $g \in \Gamma$ belongs to a negative double coset only if $v_g^u \in \mathcal{I}_n^-$ (see Theorem 2, i1.).

Put

$$v_0 = \langle f_1, \dots, f_q\rangle + \langle e_{l+1}, \dots, e_{n-1}, f_n\rangle + \begin{cases} \langle e_l, e_{l-1}, \dots, e_{n+q-k+1}\rangle & \text{if} \quad k+l-n-q>0, \\ \{0\} & \text{if} \quad k+l-n-q=0. \end{cases}$$

Since $q < n+q-k+1$, the space v_0 is totally isotropic. Also, v_0 satisfies the condition (3.9), and

$$(u + g_0(v_0))^u = \langle e_1, \dots, e_{n-1}, f_n\rangle \in \mathcal{I}_n^-.$$

Further, $n-k+q+1 > q+1$ (because $k \leqslant l < n$) and $q+1 = q' \leqslant k \leqslant l$. Hence we have (3.11).

Put $U = \langle e_{q+1}, f_{q+1}\rangle + \langle e_n, f_n\rangle = H_4$, and for every $\alpha \in K$ define the element $g_\alpha \in \mathrm{SL}(V)$ by the formula

$$g_\alpha(f_n) = f_n + \alpha f_{q+1}, \quad g_\alpha(f_{q+1}) = f_{q+1},$$
$$g_\alpha(e_{q+1}) = e_{q+1} \pm \alpha e_n, \quad g_\alpha(e_n) = e_n$$

and $g_\alpha(x) = x$ for every $x \in U^\perp$. Then $g_\alpha \in \Gamma$ (with an appropriate choice of signs in $g_\alpha(e_{q+1}) = e_{q+1} \pm \alpha e_n$).

Further, the construction of g_α implies

$$g_\alpha(v_0) = \langle e_{n+q-k+1}, \dots, e_{n-1}, f_n + \alpha f_{q+1}, f_1 \dots, f_q\rangle.$$

Then, if $\alpha \neq 0$, we have (see (3.11))

$$u + g_\alpha(v_0) = \underbrace{\langle e_1, f_1\rangle \perp \cdots \perp \langle e_q, f_q\rangle \perp \langle e_{q+1}, f_n + \alpha f_{q+1}\rangle}_{=H_{2(q+1)}} \perp \langle e_{q+2}, \ldots, e_{n-1}\rangle$$

and therefore we have (3.10). For $\alpha = 0$ we have $g_\alpha = g_0 = 1$ and (3.9). Since $k \leqslant l < n$ there is an element $\gamma \in \Gamma$ such that $v_0 = \gamma(v)$. Also, $(u + g_\alpha(v_0))^u = \langle e_1, \ldots, e_{n-1}\rangle \notin \mathcal{I}_n^-$.

Case 2. $q' = q, p' = p + 1$.

Subcase 2.1*:* $P_u\gamma_{pq}^{\pm}P_v = P_u\gamma_{pq}P_v$ *and* $P_u\gamma_{p'q'}^{\pm}P_v = P_u\gamma_{p'q'}P_v$ (*that is, both double cosets are positive*)*.* Here we put

$$v_0 := \langle f_1, \ldots, f_q\rangle + \langle e_{l+1}, \ldots e_{l+p}\rangle + \langle e_l, \ldots, e_{l-k+p+q+1}\rangle,$$

and

$$U := \langle e_{l-k+p+q+1}, f_{l-k+p+q+1}\rangle + \langle e_{l+p+1}, f_{l+p+1}\rangle$$

(since $q' + p' \leqslant k$ then $q + (p+1) \leqslant k$ and therefore $k - p - q > 0$ and $l - k + p + q + 1 \leqslant l < l + p + 1$) and

$$\begin{aligned} g_\alpha(e_{l-k+p+q+1}) &:= e_{l-k+p+q+1} + \alpha e_{l+p+1}, \\ g_\alpha(e_{l+p+1}) &:= e_{l+p+1}, \\ g_\alpha(f_{l+p+1}) &:= f_{l+p+1} \pm \alpha f_{l-k+p+q+1}, \\ g_\alpha(f_{l-k+p+q+1}) &:= f_{l-k+p+q+1}. \end{aligned}$$

Then we have

$$u + g_\alpha(v_0) = \langle e_1, \ldots, e_{l+p+1}\rangle + \langle f_1, \ldots, f_q\rangle$$

and (3.10).

Subcase 2.2*:* $P_u\gamma_{pq}^{\pm}P_v = P_u\gamma_{pq}P_v$ *and* $P_u\gamma_{p'q'}^{\pm}P_v = P_u\gamma_{p'q'}^{-}P_v$*, where* $p' = n - l > 0, p = n - l - 1$ (*that is, one double coset is positive and one is negative*)*.* This is only one possibility in the case $p' = p+1$ for the pair of double cosets of the different signs (because $p < p' \leqslant n-l$, see 3.7). Here $k \leqslant l < n$ and $p' > 0$ (because the negative cosets do appear only in the Special Case i1)). Hence $q \leqslant n-k+q \leqslant l < n$. Put

$$\begin{aligned} v_0 &:= \langle f_1, \ldots, f_q\rangle + \langle e_{l+1}, \ldots e_{n-1}\rangle + \langle e_l, \ldots, e_{n-k+q}\rangle, \\ U &:= \langle e_{n-k+q}, f_{n-k+q}\rangle + \langle e_n, f_n\rangle, \\ g_\alpha(e_{n-k+q}) &= e_{n-k+q} + \alpha f_n, \quad g_\alpha(f_n) = f_n, \\ g_\alpha(e_n) &= e_n \pm \alpha f_{n-k+q}, \quad g_\alpha(f_{n-k+q}) = f_{n-k+q}. \end{aligned}$$

We have

$$u + g_\alpha(v_0) = \langle e_1, \ldots, e_{n-1}, f_n\rangle + \langle f_1, \ldots, f_q\rangle$$

and (3.10) and $(u + g_\alpha(v_0))^u = \langle e_1, \ldots, e_{n-1}, f_n\rangle \in \mathcal{I}_n^-$.

Case 3. $q' = q+1, p' = p$.

Subcase 3.1 : $P_u\gamma_{pq}^{\pm}P_v = P_u\gamma_{pq}P_v$ *and* $P_u\gamma_{p'q'}^{\pm}P_v = P_u\gamma_{p'q'}P_v$ (*that is, both double cosets are positive*). Here again $v_0 := \langle f_1, \ldots, f_q\rangle + \langle e_{l+1}, \ldots e_{l+p}\rangle + \langle e_l, \ldots, e_{l-k+p+q+1}\rangle$ and $k-p-q > k-p-q' \geqslant 0$.

Subcase 3.1.1. $q+1 < l$. Put $U := \langle e_l, f_l\rangle + \langle e_{q+1}, f_{q+1}\rangle$,

$$\begin{aligned} g_\alpha(e_l) &:= e_l + \alpha f_{q+1}, \quad g_\alpha(f_{q+1}) := f_{q+1}, \\ g_\alpha(e_{q+1}) &:= e_{q+1} \pm \alpha f_l, \quad g_\alpha(f_l) := f_l. \end{aligned}$$

Then for $\alpha \neq 0$

$$u + g_\alpha(v_0) = \langle e_1, \ldots, e_{l+p}\rangle + \langle f_1, \ldots, f_q, f_{q+1}\rangle$$

and (3.10).

Subcase 3.1.2. $q+1 = l$. Then $l = k = q+1$, $p = 0$, $v_0 = \langle f_1, \ldots, f_{l-1}, e_l\rangle$.

Subcase 3.1.2.1. $l = q+1 < n$. Put $U = \langle e_l, f_l\rangle \perp \langle e_{l+1}, f_{l+1}\rangle$ and

$$g_\alpha(e_l) = e_l \pm \alpha e_{l+1} \pm \alpha f_{l+1} \pm \alpha^2 f_l,$$
$$g_\alpha(e_{l+1}) = e_{l+1} \pm \alpha f_l \pm \alpha^2 f_{l+1},\ g_\alpha(f_l) = f_l,\ g_\alpha(f_{l+1}) = f_{l+1} \pm \alpha f_l.$$

Then

$$u + g_\alpha(v_0) = \langle e_1, \ldots, e_l\rangle + \langle f_1, \ldots, f_{l-1}, e_l \pm \alpha e_{l+1} \pm \alpha f_{l+1} \pm \alpha^2 f_l\rangle$$

and we have (3.10).

Subcase 3.1.2.2. $l = k = q + 1 = n$. Then we are in the General Case. (Indeed, in the Special Case the assumption $l = k = n$ implies that the values of the parameter q are all odd or all even.). Thus, $\Gamma = \mathrm{Sp}_{2n}(K)$ or $\Gamma = \mathrm{SO}_{2n+1}(K)$.

We have here $u = \langle e_1, \dots, e_n \rangle$ and $v_0 = \langle f_1, \dots, f_{n-1}, e_n \rangle$.

If $\Gamma = \mathrm{Sp}_{2n}(K)$ put $U := \langle e_n, f_n \rangle$ and

$$g_\alpha(e_n) = e_n + \alpha f_n, g_\alpha(f_n) = f_n.$$

Then

$$u + g_\alpha(v_0) = \langle e_1, \dots, e_n \rangle + \langle e_n + \alpha f_n, f_1, \dots, f_{n-1} \rangle = V$$

and we have (3.10).

If $\Gamma = \mathrm{SO}_{2n+1}(K)$, put $U := \langle e_n, e_0, f_n \rangle$ and

$$g_\alpha(e_n) := e_n + \alpha e_0 - \frac{\alpha^2}{2} f_n, \quad g_\alpha(e_0) := e_0 - \alpha f_n, \quad g_\alpha(f_n) := f_n.$$

Then

$$u + g_\alpha(v_0) = \langle e_1, \dots, e_n \rangle + \langle e_n + \alpha e_0 - \frac{\alpha^2}{2} f_n, f_1, \dots, f_{n-1} \rangle$$

and we have (3.10).

Subcase 3.2. $P_u \gamma^{\pm}_{pq} P_v = P_u \gamma^{-}_{pq} P_v$ and $P_u \gamma^{\pm}_{p'q'} P_v = P_u \gamma^{-}_{p'q'} P_v$ (*that is, both double cosets are negative*). Here $n - l = p = p' > 0$ and $k - p - q > 0$ (because $k - p - q' \geqslant 0$). Also,

$$l - k + p + q + 1 = l - k + (n - l) + q + 1 = n - k + q + 1 > q + 1$$

(because $k \leqslant l < n$). Put

$$\begin{aligned} v_0 &:= \langle f_1, \dots, f_q, e_{l+1}, \dots, e_{n-1}, f_n, e_l, \dots, e_{l-k+p+q+1} \rangle, \\ U &:= \langle e_l, f_l \rangle + \langle e_{q+1}, f_{q+1} \rangle \end{aligned}$$

and

$$\begin{aligned} g_\alpha(e_l) &= e_l + \alpha f_{q+1}, \quad g_\alpha(f_{q+1}) = f_{q+1}, \\ g_\alpha(e_{q+1}) &= e_{q+1} \pm \alpha f_l, \quad g_\alpha(f_l) = f_l. \end{aligned}$$

Then

$$u + g_\alpha(v_0) = \langle e_1, \dots, e_{n-1}, f_n\rangle + \langle e_l + \alpha f_{q+1}, f_1, \dots, f_q\rangle$$

and we have (3.10) and

$$(u + v_0)^u = (u + g_\alpha(v_0))^u = \langle e_1, \dots, e_{n-1}, f_n\rangle \in \mathcal{I}_n^-.$$

Subcase 3.3. $P_u\gamma^{\pm}_{pq}P_v = P_u\gamma_{pq}P_v$ and $P_u\gamma^{\pm}_{p'q'}P_v = P_u\gamma^{-}_{p'q'}P_v$ (*that is, the first coset is positive and the second is negative*). Here $n-l = p = p' > 0$ and $k-p-q > 0$ (because $k-p-q' \geqslant 0$). Also,

$$l-k+p+q+1 = l-k+(n-l)+q+1 = n-k+q+1 > q+1$$

(because $k \leqslant l < n$), and we put

$$v_0 := \langle f_1, \dots, f_q, e_{l+1}, \dots, e_n, e_l, \dots, e_{l-k+p+q+1}\rangle,$$
$$U := \langle e_l, f_l\rangle + \langle e_n, f_n\rangle,$$

$$g_\alpha(e_l) := e_l + \alpha f_n, \quad g_\alpha(f_n) := f_n, \quad g_\alpha(e_n) := e_n - \alpha f_l, \quad g_\alpha(f_l) := f_l.$$

Then

$$u + g_\alpha(v_0) = \langle e_1, \dots, e_{n-1}, f_n, f_1, \dots, f_q, e_n - \alpha f_l\rangle$$

and we have (3.10) and

$$(u + v_0)^u = \langle e_1, \dots, e_n\rangle \in \mathcal{I}_n^+,$$
$$(u + g_\alpha(v_0))^u = \langle e_1, \dots, e_{n-1}, f_n\rangle \in \mathcal{I}_n^-.$$

Subcase 3.4. $P_u\gamma^{\pm}_{pq}P_v = P_u\gamma^{-}_{pq}P_v$ and $P_u\gamma^{\pm}_{p'q'}P_v = P_u\gamma_{p'q'}P_v$ (*that is, the first coset is negative and the second is positive*). Here we put

$$v_0 := \langle f_1, \dots, f_q, e_{l+1}, \dots, e_{n-1}, f_n, e_l, \dots, e_{l-k+p+q+1}\rangle,$$
$$U := \langle e_l, f_l\rangle + \langle e_n, f_n\rangle,$$

$$g_\alpha(f_n) := f_n + \alpha f_l, \quad g_\alpha(e_n) := e_n, \quad g_\alpha(e_l) = e_l - \alpha e_n, \quad g_\alpha(f_l) = f_l.$$

Then

$$u + g_\alpha(v_0) := \langle e_1, \dots, e_{n-1}, e_n, f_1, \dots, f_q, f_n + \alpha f_l\rangle$$

and we have (3.10) and $(u + v_0)^u = \langle e_1, \ldots, e_{n-1}, f_n\rangle \in \mathcal{I}_n^-, (u + g_\alpha(v_0))^u = \langle e_1, \ldots, e_{n-1}, e_n\rangle \in \mathcal{I}_n^+$.

Case 4. $k = l = n, p = p' = 0$ and $q + 2 \leqslant n$. Here we put

$$v_0 = \langle f_1, \ldots, f_q, e_n, \ldots, e_{q+1}\rangle,$$
$$U := \langle e_{q+1}, f_{q+1}\rangle + \langle e_n, f_n\rangle,$$

$$g_\alpha(e_n) := e_n + \alpha f_{q+1}, \quad g_\alpha(f_{q+1}) := f_{q+1},$$
$$g_\alpha(e_{q+1}) := e_{q+1} - \alpha f_n, \quad g_\alpha(f_n) := f_n.$$

Then

$$u + g_\alpha(v_0) = \langle e_1, \ldots, e_{n-1}, e_n, f_1, \ldots, f_q, f_{q+1}, f_n\rangle$$

and we have (3.10). Note, that we take $u = \langle e_1, \ldots, e_n\rangle \in \mathcal{I}_n^+$ and we have sign $(u, v) = (-1)^{\mathrm{d_{in}}(u,v_0)} = (-1)^q$. Also, Theorem 2, (i4) implies that q is odd if and only if sign $(u, v) = -1$. Thus, the subspaces v, v_0 both are contained in $\mathcal{I}_n^-$ or in $\mathcal{I}_n^+$ and therefore $v_0 = \gamma(v)$ for some $\gamma \in \Gamma$. □

Now we can prove the implication (3.6). Lemma 14 reduces the general case to one of the cases 1.-4. of the Lemma 15. Lemma 15 shows that for cases 1.-4. we have the morphism $\chi : G_a \to \Gamma$ of the irreducible variety G_a such that $\chi(\alpha) \in P_u\gamma^{\pm}_{p'q'}P_v$ for $\alpha \neq 0$ and $\chi(0) \in P_u\gamma^{\pm}_{pq}P_v$. Then we have the inclusion

$$P_u\gamma^{\pm}_{pq}P_v \subset \overline{P_u\gamma^{\pm}_{p'q'}P_v}.$$

We may finish the proof of the theorem by a comparison of the double cosets $P_u\gamma^{\pm}_{pq}P_v$.

Lemma 16. $P_u\gamma_{pq}P_v \nsubseteq \overline{P_u\gamma^{-}_{pq}P_v}$ *and* $P_u\gamma^{-}_{pq}P_v \nsubseteq \overline{P_u\gamma_{pq}P_v}$.

Proof. Since we consider double cosets of different signs we are in the Special Case i1). Hence $p = n - l > 0$. We may assume $u = \langle e_1, \ldots, e_l\rangle, l < n$. Let

$$v_{\gamma_{pq}} = u + \gamma_{pq}(v), \quad v_{\gamma^{-}_{pq}} = u + \gamma^{-}_{pq}(v).$$

Then $v^u_{\gamma_{pq}} \in \mathcal{I}^+_n$ and $v^u_{\gamma^-_{pq}} \in \mathcal{I}^-_n$ (Theorem 2, (ii1)).

Put $v_0 = \langle e_1, \ldots, e_k \rangle$. Since $k \leqslant l < n$, there is an element $\sigma \in \Gamma$ such that $\sigma(v_0) = v$. Obviously, $P_v = \sigma P_{v_0} \sigma^{-1}$. Hence

$$P_u \gamma^{\pm}_{pq} P_v = P_u \gamma^{\pm}_{pq} \sigma P_{v_0} \sigma^{-1}. \tag{3.12}$$

Consider the pair of isotropic subspaces u, v_0. We have

$$\begin{aligned} u + \gamma_{pq}\sigma(v_0) &= u + \gamma_{pq}(v) = v_{\gamma_{pq}}, \\ u + \gamma^-_{pq}\sigma(v_0) &= u + \gamma^-_{pq}(v) = v_{\gamma^-_{pq}}. \end{aligned} \tag{3.13}$$

Put $\bar\gamma_{pq} = \gamma_{pq}\sigma$, $\bar\gamma^-_{pq} = \gamma^-_{pq}\sigma$. The formulas (3.13) show that $P_u \bar\gamma^{\pm}_{pq} P_{v_0}$ are positive and negative double cosets which correspond to the pair (u, v_0) and to the parameters (p, q) (that is, $\dim(u + g(v_0)) = l + p + q, I(u + g(v_0)) = q$ for every $g \in P_u \bar\gamma^{\pm}_{pq} P_{v_0}$). Now (3.12) shows that we may prove our statement for the case $v = v_0$.

Let $\tau \in \mathrm{O}_{2n}(K)$ be the orthogonal transformation such that

$$\tau(e_i) = e_i, \tau(f_i) = f_i \quad \text{for every } i < n$$
$$\text{and} \quad \tau(e_n) = f_n, \quad \tau(f_n) = e_n.$$

The definitions of $u, v = v_0$ and τ imply $\tau(u) = u, \tau(v) = v$ and therefore $\tau P_u \tau^{-1} = P_u$, $\tau P_v \tau^{-1} = P_v$. Let us show

$$\tau P_u \gamma_{pq} P_v \tau^{-1} = P_u \gamma^-_{pq} P_v. \tag{3.14}$$

We have

$$\tau P_u \gamma_{pq} P_v \tau^{-1} = \tau P_u \tau^{-1} (\tau \gamma_{pq} \tau^{-1}) \tau P_v \tau^{-1} = P_u (\tau \gamma_{pq} \tau^{-1}) P_v$$

and therefore to prove (3.14) we have to prove

$$\tau \gamma_{pq} \tau^{-1} \in P_u \gamma^-_{pq} P_v. \tag{3.15}$$

First of all, note that

$$\tau(\mathcal{I}^+_n) = \mathcal{I}^-_n. \tag{3.16}$$

(Indeed, the definition of τ implies $\tau(V^+_n) = V^-_n$. Since

$$\mathrm{O}_{2n}(K) = \langle \tau \rangle \cdot \mathrm{SO}_{2n}(K) \quad \text{and} \quad \Gamma = \mathrm{SO}_{2n}(K) \lhd \mathrm{O}_{2n}(K)$$

then for every $g \in \Gamma$ there is an element $h \in \Gamma$ such that $\tau g = h\tau$. Now, if $w = g(V^+_n) \in \mathcal{I}^+_n$ then $\tau(w) = \tau g(V^+_n) = h\tau(V^+_n) = h(V^-_n) \in \mathcal{I}^-_n$.)

Further,

$$\tau(v_{\gamma_{pq}}) = \tau(u + \gamma_{pq}(v)) = u + \tau\gamma_{pq}(v) = u + (\tau\gamma_{pq}\tau^{-1})(v).$$

Hence

$$\dim(u+(\tau\gamma_{pq}\tau^{-1})(v)) = l+p+q, \quad I(u+(\tau\gamma_{pq}\tau^{-1})(v)) = q. \quad (3.17)$$

Further, $v^u_{\gamma_{pq}} \in \mathcal{I}^+_n$ because γ_{pq} is an element of the positive double coset and

$$\tau(v^u_{\gamma_{pq}}) = v^u_{\tau\gamma_{pq}\tau^{-1}} \in \mathcal{I}^-_n,$$

because of (3.16). Thus,

$$\tau\gamma_{pq}\tau^{-1} \in P_u\gamma^-_{pq}P_v.$$

Hence we have (3.14) therefore the locally closed sets

$$P_u\gamma_{pq}P_v \quad \text{and} \quad P_u\gamma^-_{pq}P_v$$

have the same dimension. It implies

$$P_u\gamma_{pq}P_v \not\subseteq \overline{P_u\gamma^-_{pq}P_v}$$

and

$$P_u\gamma^-_{pq}P_v \not\subseteq \overline{P_u\gamma_{pq}P_v}.$$

Now the Theorem has been proved. □

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Department of Mathematics,
Russian State
Pedagogical University,
Moijka 48, St.Petersburg,
191186, Russia;
St.Petersburg State University,
Universitetsky prospekt, 28,
Peterhof, St. Petersburg, 198504, Russia
E-mail: nickgordeev@mail.ru

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Department of Mathematics,
Bielefeld University,
Universitätsstrasse 25,
D-33615 Bielefeld, Germany
E-mail: rehmann@math.uni-bielefeld.de