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ON STABLY BISERIAL ALGEBRAS AND THE AUSLANDER–REITEN CONJECTURE FOR SPECIAL BISERIAL ALGEBRAS

ABSTRACT. By a result claimed by Pogorzały selfinjective special biserial algebras can be stably equivalent only to stably biserial algebras and these two classes coincide. By an example of Ariki, Iijima and Park the classes of stably biserial and selfinjective special biserial algebras do not coincide. In these notes we provide a detailed proof of the fact that a selfinjective special biserial algebra can be stably equivalent only to a stably biserial algebra following some ideas from the paper by Pogorzały. We will analyse the structure of symmetric stably biserial algebras and show that in characteristic $\neq 2$ the classes of symmetric special biserial (Brauer graph) algebras and symmetric stably biserial algebras indeed coincide. Also, we provide a proof of the Auslander–Reiten conjecture for special biserial algebras.

§1. INTRODUCTION

Derived equivalences of symmetric special biserial or equivalently Brauer graph algebras [19] have been extensively studied over the past few years [1–4, 9, 10, 13, 16–18, 21, 23–25]. These studies concern mainly attempts to classify symmetric special biserial algebras up to derived equivalence, classification of special tilting complexes over such algebras or computation of the derived Picard groups. It is well known that the class of symmetric special biserial algebras of finite representation type is closed under derived equivalence. The fact that the class of symmetric special biserial algebras is closed under derived equivalence followed from the results of Pogorzały [14]. Unfortunately, in [5] counterexamples for some of the statements of [14] were given.

In this paper we reprove the fact that if a selfinjective algebra (not isomorphic to the Nakayama algebra with $\mathrm{rad}^2 = 0$) is stably equivalent to a selfinjective special biserial algebra, then it is stably biserial. We do not use the original approach of Pogorzały via Galois coverings, instead we perform all combinatorial computations directly. We give a proof of the Auslander–Reiten conjecture for special biserial algebras using the reduction to the selfinjective case obtained by Martínez-Villa. The conjecture states that the number of non-isomorphic non-projective simple modules is invariant under stable equivalence. The proof for selfinjective special biserial algebras is more involved, since we have to consider systems of orthogonal stable bricks over stably biserial algebras. After that we describe all symmetric stably biserial algebras, showing that in characteristic $\neq 2$ this class indeed coincides with the class of symmetric special biserial algebras. This is the first step towards the proof of the fact that the class of symmetric special biserial algebras is closed under derived equivalence.

§2. Preliminaries

Throughout this paper A is a basic, connected, finite dimensional algebra over an algebraically closed field $\mathbf{k}$, mod-A is the category of finite-dimensional right A-modules, $\underline{\mathrm{mod}}$-A is the stable category of mod-A, i.e. the category of modules modulo the maps factoring through projective modules. In the case where A is selfinjective the category $\underline{\mathrm{mod}}$-A is triangulated. The Auslander-Reiten translation DTr will be denoted by τ, the Hom-spaces in $\underline{\mathrm{mod}}$-A will be denoted by $\underline{\mathrm{Hom}}$, for $f \in \mathrm{mod}$-A its class in $\underline{\mathrm{mod}}$-A will be denoted by $\underline{f}$, the syzygy or the Heller's loop functor will be denoted by $\Omega : \underline{\mathrm{mod}}$-$A \to \underline{\mathrm{mod}}$-$A$. A module will be called local, if it is an epimorphic image of an indecomposable projective module.

Definition 1. *Let Q be a quiver, I an admissible ideal of $\mathbf{k}Q$. A selfinjective algebra A' is called stably biserial if it is isomorphic to $A = \mathbf{k}Q/I$, where Q and I satisfy the following conditions:*

(a) *For each vertex $i \in Q$, the number of outgoing arrows and the number of incoming arrows are less than or equal to 2;*

(b) *For each arrow $\alpha \in Q$, there is at most one arrow $\beta \in Q$ that satisfies $\alpha\beta \notin \alpha\mathrm{rad}(A)\beta + \mathrm{soc}(A)$;*

(c) *For each arrow $\alpha \in Q$, there is at most one arrow $\beta \in Q$ that satisfies $\beta\alpha \notin \beta\mathrm{rad}(A)\alpha + \mathrm{soc}(A)$.*

The following description of stably biserial algebras was provided in [5]:

Proposition 1 (Proposition 7.5 [5])**.** *If* A *is stably biserial then there exists a presentation of* $A \simeq \mathbf{k}Q/I$ *such that the following conditions hold:*

(1) *If* $\alpha\beta \neq 0$, $\alpha\gamma \neq 0$, $\beta \neq \gamma$, *for arrows* α, β, γ *then either* $\alpha\beta \in \mathrm{soc}(A)$ *or* $\alpha\gamma \in \mathrm{soc}(A)$;

(2) *If* $\beta\alpha \neq 0$, $\gamma\alpha \neq 0$, $\beta \neq \gamma$, *for arrows* α, β, γ *then either* $\beta\alpha \in \mathrm{soc}(A)$ *or* $\gamma\alpha \in \mathrm{soc}(A)$.

Definition 2. *Let* Q *be a quiver,* I *an admissible ideal of* $\mathbf{k}Q$. *An algebra* A' *is called special biserial if it is isomorphic to* $A = \mathbf{k}Q/I$, *where* Q *and* I *satisfy the following conditions:*

(a) *For each vertex* $i \in Q$, *the number of outgoing arrows and the number of incoming arrows are less than or equal to* 2;

(b) *For each arrow* $\alpha \in Q$, *there is at most one arrow* $\beta \in Q$ *that satisfies* $\alpha\beta \neq 0$;

(c) *For each arrow* $\alpha \in Q$, *there is at most one arrow* $\beta \in Q$ *that satisfies* $\beta\alpha \neq 0$.

If additionally A *is selfinjective, then it is called selfinjective special biserial.*

§3. Stable equivalences

In this section we are going to prove following the ideas from [14] that if an algebra is stably equivalent to a selfinjective special biserial algebra (not isomorphic to the Nakayama algebra with $\mathrm{rad}^2 = 0$), then it is stably biserial.

Proposition 2 (Proposition 7.11 [5], see also Lemmas 5.3, 5.4 [14])**.** *Let* B *be an indecomposable selfinjective algebra which is not a local Nakayama algebra. Then, we have the following:*

(1) *If* P *is indecomposable projective, then* $\tau(P/\mathrm{soc}(P)) \not\simeq P/\mathrm{soc}(P)$;

(2) *If* S *is simple, then* S *is non-projective and* $\tau(S) \not\simeq S$.

From now on we are not going to consider local Nakayama algebras. Thus, we can assume that A does not have any simple modules of τ-period 1.

Definition 3. *Let* A *be a selfinjective* $\mathbf{k}$-*algebra. An indecomposable* A-*module* M *is said to be a stable brick if* $\underline{\mathrm{End}}(M) \simeq \mathbf{k}$. *A family* $\{M_i\}_{i \in I}$ *of mutually non-isomorphic stable bricks is a system of orthogonal stable bricks if the following conditions hold:*

(1) M_i *is not of* τ-*period 1 for every* $i \in I$;

(2) $\underline{\mathrm{Hom}}(M_i, M_j) = 0$ *for any* $i, j \in I$ *with* $i \neq j$.

A system of orthogonal stable bricks $\{M_i\}_{i \in I}$ *is called maximal if for every indecomposable A-module N that is neither projective nor of τ-period 1 there exist* $i, j \in I$ *such that* $\underline{\mathrm{Hom}}(M_i, N) \neq 0$ *and* $\underline{\mathrm{Hom}}(N, M_j) \neq 0$.

Remark 1. If there is an equivalence of categories $\underline{\mathrm{mod}}\text{-}B \to \underline{\mathrm{mod}}\text{-}A$, where A and B are selfinjective, then the image of the set of representatives of the iso-classes of simple modules is a maximal system of orthogonal stable bricks.

Since we are interested in maximal systems of orthogonal stable bricks which are images of the sets of simple modules, for now we can assume, that the cardinality of $\mathcal{M}$ is finite.

Definition 4. *Let* $\mathcal{M} = \{M_1, \cdots, M_n\}$ *be a maximal system of orthogonal stable bricks. An indecomposable A-module N is called s-projective with respect to $\mathcal{M}$ if the following conditions are satisfied:*

(1) *N is not of τ-period 1;*

(2) $\underline{\mathrm{Hom}}(N, \oplus_{i=1}^{n} M_i) \simeq \mathbf{k}$;

(3) *If* $\underline{\mathrm{Hom}}(N, M_i) \neq 0$, *then for every non-zero* $\underline{f} : X \to M_i$ *and* $\underline{g} : N \to M_i$ *there exists* $\underline{h} : N \to X$ *such that* $\underline{fh} = \underline{g}$.

An A-module N is called s-projective with respect to $\mathcal{M}$ if it is a sum of indecomposable s-projective modules; s-injective modules are defined dually.

It is clear that for an indecomposable s-projective A-module N there exists only one $i \in I$ such that $\underline{\mathrm{Hom}}(N, M_i) \neq 0$. In [15] it is proved that an indecomposable A-module N is s-projective with respect to $\mathcal{M}$ if and only if $N \simeq \tau^{-1}\Omega(M)$ for some $M \in \mathcal{M}$. Let N be an indecomposable s-projective A-module with respect to $\mathcal{M}$. We say that s-top$(N) \simeq M$ if $M \in \mathcal{M}$ and $\underline{\mathrm{Hom}}(N, M) \neq 0$. In this case s-top$(\tau^{-1}\Omega(M)) \simeq M$ for $M \in \mathcal{M}$. See also [5, Proposition 7.13].

Remark 2. If there is an equivalence of categories $\underline{\mathrm{mod}}\text{-}B \to \underline{\mathrm{mod}}\text{-}A$, where A and B are selfinjective and $\mathcal{M} = \{M_1, \cdots, M_n\}$ is the image of the set of simple B-modules, then the image of the module of the form $P/\mathrm{soc}(P)$, where P is an indecomposable projective B-module, is indecomposable s-projective with respect to $\mathcal{M}$.

We will denote by Q_0 the set of vertices of Q, by Q_1 the set of arrows of Q and by $s(\alpha)$, $e(\alpha)$ the maps from Q_1 to Q_0, which map an arrow to its beginning and end respectively.

From now on, when considering a selfinjective special biserial algebra $A = \mathbf{k}Q/I$ we will fix a presentation satisfying the conditions from Definition 2. Note that the generating set of relations in I can be chosen to consist of relations of three kinds: zero relations $\alpha\beta = 0$ for some $\alpha, \beta \in Q_1$; relations of the form $\alpha_1 \cdots \alpha_m = c\beta_1 \cdots \beta_n$ $(c \in \mathbf{k}^*)$ for $\alpha_1 \neq \beta_1$ and $s(\alpha_1) = s(\beta_1)$; relations of the form $\alpha_1 \cdots \alpha_m = 0$ in the case when there is only one arrow leaving $s(\alpha_1)$ $(\alpha_i, \beta_j \in Q_1)$.

Recall that an indecomposable non-projective module over a selfinjective special biserial algebra $A = \mathbf{k}Q/I$ is either a string or a band module. Since all the band modules are of τ-period 1 we are not going to use them.

Given an arrow $\alpha \in Q_1$, we will denote by α^{-1} its formal inverse; thus $s(\alpha^{-1}) = e(\alpha)$, $e(\alpha^{-1}) = s(\alpha)$, $(\alpha^{-1})^{-1} = \alpha$. The set of formal inverse arrows $\{\alpha^{-1}\}_{\alpha \in Q_1}$ will be denoted by Q_1^{-1}. A string of length n is a sequence of the form $c = c_1 \cdots c_n$, where $c_i \in Q_1 \cup Q_1^{-1}$, $s(c_{i+1}) = e(c_i)$, $c_i \neq c_{i+1}^{-1}$ and neither $c_i \cdots c_{i+t}$ nor $c_{i+t}^{-1} \cdots c_i^{-1}$ belong to $\mathrm{soc}(A)$ for any i and t. Additionally, for every vertex $x \in Q_0$, there is a string of length zero denoted by 1_x with $s(1_x) = e(1_x) = x$. For a string $c = c_1 \cdots c_n$ of positive length, let $s(c) := s(c_1)$, $e(c) := e(c_n)$.

Let $c = c_1 \cdots c_n$ be a string of length $n \geq 1$. A string module M_c is defined as follows: fix a basis $\{z_0, \cdots, z_n\}$, given an idempotent e_x, corresponding to the vertex x, $z_i e_x = z_i$ if $x = e(c_i)$ or $x = s(c_{i+1})$ and zero otherwise. Given an arrow $\alpha \in Q_1$, $z_i \alpha = z_{i-1}$ if $c_i = \alpha^{-1}$, $z_i \alpha = z_{i+1}$ if $c_{i+1} = \alpha$ and zero otherwise. To the string of length zero 1_x we associate the simple module corresponding to the vertex x. Two string modules corresponding to different strings c and c' are isomorphic if and only if $c = c_1 \cdots c_n$ and $c' = c_n^{-1} \cdots c_1^{-1}$. Usually we will depict the string and the corresponding module by the diagram of that module, e.g., the string $\alpha^{-1}\beta\gamma\delta^{-1}$ will be depicted as


We will call z_i a peak if there is no $\alpha \in Q_1$ such that $z_{i-1}\alpha = z_i$ or $z_{i+1}\alpha = z_i$. We will call z_i a deep if for all $\alpha \in Q_1$ we have $z_i\alpha = 0$. In the example above z_1, z_4 are peaks and z_0, z_3 are deeps. Note that this is not the standard use of the terms peak and deep. In cases when it does not lead to confusion,

we will omit the names of the arrows in the diagrams and we will use diagrammatic notation for the elements of the algebra A.

We shall now describe the Auslander–Reiten sequences in mod-A, containing string modules. The Auslander–Reiten sequences, containing an indecomposable projective module P in the middle term are of the form

$$0 \to \operatorname{rad}(P) \to \operatorname{rad}(P)/\operatorname{soc}(P) \oplus P \to P/\operatorname{soc}(P) \to 0.$$

Assume now that M_c is a non-projective indecomposable module not isomorphic to $P/\operatorname{soc}(P)$ for any projective module P. The module M_c is of the form
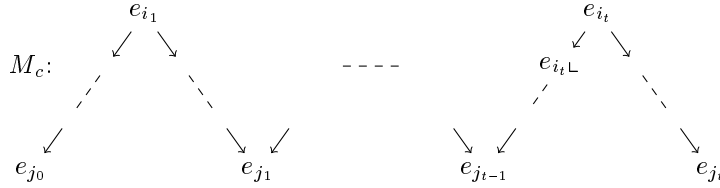
where the first or the last directed substring may be trivial. Let c^r be the maximal string extending c on the right by $e_{j_t} \to e_{j_t \lrcorner} \leftarrow \cdots \leftarrow e_{i_{t+1}}$ if such a string exists (adding a co-hook).
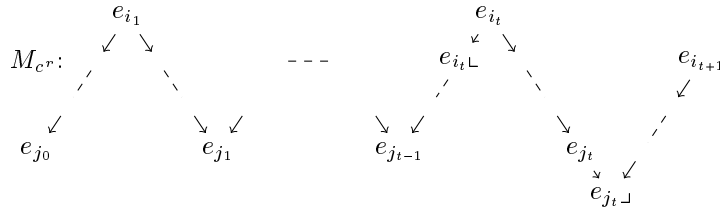
If not, let c^r be the string obtained from c by cancellation of the last directed substring including the vertex e_{i_t} (c^r may be empty),
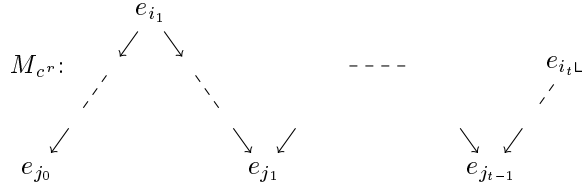
(deleting a hook). Similarly let $^l c$ be obtained from c by the corresponding operations on the left-hand side of c. Since M_c is not isomorphic to $P/\operatorname{soc}(P)$ for any projective module P, at least one of the strings $^l(c^r)$ or

$({}^l c)^r$ is non-empty, and if both are defined, then ${}^l(c^r) = ({}^l c)^r$, let ${}^l c^r$ be the non-trivial string ${}^l(c^r)$ or $({}^l c)^r$. Then the Auslander–Reiten sequence terminating at M_c is of the form

$$0 \to \tau(M_c) \simeq M_{{}^l c^r} \to M_{c^r} \oplus M_{{}^l c} \to M_c \to 0.$$

Similarly, τ^{-1} can be computed by adding hooks if possible and deleting co-hooks if not [8, 20, 22].

The following lemma follows immediately from the description of the Auslander–Reiten sequences.

Lemma 1 (see Lemma 6.4 [14])**.** *Let A be a selfinjective special biserial algebra and let $\mathcal{M}$ be a maximal system of orthogonal stable bricks in $\underline{\mathrm{mod}}$-A. Consider $M \in \mathcal{M}$ and let N be an indecomposable s-projective module with respect to $\mathcal{M}$ with s-top$(N) \simeq M$.*

case (1): If M is of the form

$t = 0, 1, \cdots$ then N is of the form
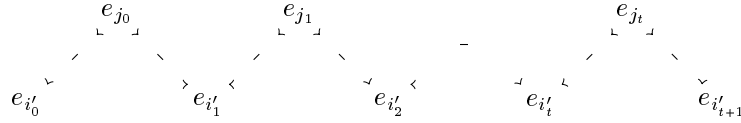
where $e_{j_0} \to \cdots \to e_{i'_0}$ and $e_{j_t} \to \cdots \to e_{i'_{t+1}}$ are maximal directed strings (may be trivial), $e_{i'_k} \leftarrow \cdots \leftarrow e_{j_{k-1}} \leftarrow \cdots \leftarrow e_{i_k} = c_k e_{i'_k} \leftarrow \cdots \leftarrow e_{j_k} \leftarrow \cdots \leftarrow e_{i_k}$ in A for $k = 1, 2, \cdots, t$ and some $c_k \in \mathbf{k}^*$.

case (2): If M is of the form

$t = 2, 3, \cdots$ then N is of the form

$$
\begin{array}{ccccc}
& e_{j_1} & & & e_{j_{t-1}} \\
& \swarrow \quad \searrow & & & \swarrow \quad \searrow \\
\swarrow \qquad\qquad \searrow & & \text{-- .} & & \swarrow \qquad\qquad \searrow \\
e_{i'_1\urcorner} \qquad\qquad e_{i'_2} & & & e_{i'_{t-1}} \qquad\qquad e_{i'_t\ulcorner}
\end{array}
$$

where $e_{i_1} \to \cdots \to e_{j_1} \to \cdots \to e_{i'_1\urcorner}$ and $e_{i_t} \to \cdots \to e_{j_{t-1}} \to \cdots \to e_{i'_t\ulcorner}$ (e_{j_1} may coincide with $e_{i'_1\urcorner}$, $e_{j_{t-1}}$ may coincide with $e_{i'_t\ulcorner}$) are maximal directed strings, $e_{i'_k} \leftarrow \cdots \leftarrow e_{j_{k-1}} \leftarrow \cdots \leftarrow e_{i_k} = c_k e_{i'_k} \leftarrow \cdots \leftarrow e_{j_k} \leftarrow \cdots \leftarrow e_{i_k}$ in A for $k = 2, 3, \cdots, t-1$ and $c_k \in \mathbf{k}^*$.
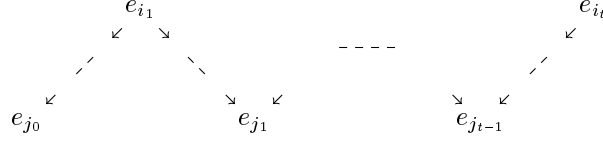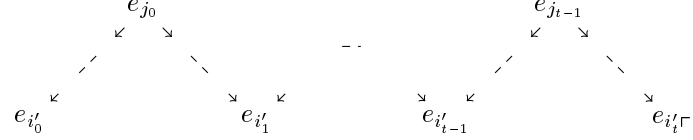
case (3): If M is of the form

$$
\begin{array}{ccccc}
e_{i_1} & & & & e_{i_t} \\
\swarrow \quad \searrow & & & & \swarrow \\
\swarrow \qquad\qquad \searrow & & \text{- - - -} & & \swarrow \\
e_{j_0} & & e_{j_1} & & e_{j_{t-1}}
\end{array}
$$

$t = 1, 2, \cdots$ then N is of the form

$$
\begin{array}{ccccc}
& e_{j_0} & & & e_{j_{t-1}} \\
& \swarrow \quad \searrow & & & \swarrow \quad \searrow \\
\swarrow \qquad\qquad \searrow & & \text{-- .} & & \swarrow \qquad\qquad \searrow \\
e_{i'_0} \qquad\qquad e_{i'_1} & & & e_{i'_{t-1}} \qquad\qquad e_{i'_t\ulcorner}
\end{array}
$$

$e_{j_0} \to \cdots \to e_{i'_0}$ and $e_{i_t} \to \cdots \to e_{j_{t-1}} \to \cdots \to e_{i'_t\ulcorner}$ are maximal directed strings, $e_{i'_k} \leftarrow \cdots \leftarrow e_{j_{k-1}} \leftarrow \cdots \leftarrow e_{i_k} = c_k e_{i'_k} \leftarrow \cdots \leftarrow e_{j_k} \leftarrow \cdots \leftarrow e_{i_k}$ in A for $k = 1, 2, \cdots, t-1$ and $c_k \in \mathbf{k}^*$.

The canonical map from N to M sends e_{j_k} from the top of N to $d_k e_{j_k}$ in the socle of M ($d_k \in \mathbf{k}$) with all d_k but one equal to 0. In the stable category all these maps belong to the same one-dimensional subspace of $\underline{\mathrm{Hom}}(N, M)$.

Lemma 2. *Let Q be a quiver of a selfinjective special biserial algebra, and let x be a vertex of Q. There is only one arrow entering x if and only if there is only one arrow leaving x.*

Proof. If there are no arrows entering vertex x, then the simple module corresponding to x is injective, and hence, it is projective and there are no arrows leaving x, the case with no arrows leaving x is similar. Assume there is one arrow α entering some vertex and two arrows β, γ leaving it. Then either $\alpha\beta = 0$ or $\alpha\gamma = 0$, say $\alpha\beta = 0$. Then $\beta \in \mathrm{soc}(A)$, hence β is

equal to some path starting from γ, which can not happen, since the ideal of relations is admissible. The case of one arrow leaving the vertex and two arrow entering is similar. $\qquad\square$

Lemma 3. *Let A be a selfinjective special biserial algebra and let $\mathcal{M}$ be a maximal system of orthogonal stable bricks in $\underline{\mathrm{mod}}$-A. For $M \in \mathcal{M}$, $\dim \underline{\mathrm{Hom}}(\tau^{-1}M, \oplus_{M_i \in \mathcal{M}} M_i) \le 2$ and $\dim \underline{\mathrm{Hom}}(\oplus_{M_i \in \mathcal{M}} \tau^{-1}M_i, M) \le 2$.*

Proof. We will prove only $\dim \underline{\mathrm{Hom}}(\tau^{-1}M, \oplus_{M_i \in \mathcal{M}} M_i) \le 2$, the other statement follows from the duality. Indeed, A is selfinjective special biserial if and only if A^{op} is selfinjective special biserial. There is a duality $D : \underline{\mathrm{mod}}\text{-}A \to \underline{\mathrm{mod}}\text{-}A^{\mathrm{op}}$, which sends τ_A to $\tau_{A^{\mathrm{op}}}^{-1}$ and maximal systems of orthogonal stable bricks in $\underline{\mathrm{mod}}$-A to maximal systems of orthogonal stable bricks in $\underline{\mathrm{mod}}$-A^{op}. Hence, if we prove $\dim \underline{\mathrm{Hom}}(\tau^{-1}M, \oplus_{M_i \in \mathcal{M}} M_i) \le 2$ for any maximal system of orthogonal stable bricks in $\underline{\mathrm{mod}}$-A^{op}, then $\dim \underline{\mathrm{Hom}}(\oplus_{M_i \in \mathcal{M}} \tau^{-1}M_i, M) \le 2$ holds for any maximal system of orthogonal stable bricks in $\underline{\mathrm{mod}}$-A.

Let $M \in \mathcal{M}$ be a module of the form
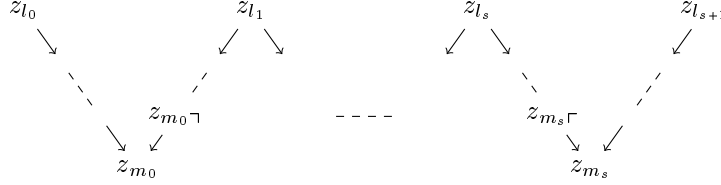
where the first or the last directed substring may be trivial. The diagram of $\tau^{-1}M$ is formed by adding hooks $z_{m_{-1}} \leftarrow \cdots \leftarrow z_{l_0 \ulcorner} \to z_{l_0}$ and $z_{l_{s+1}} \leftarrow z_{l_{s+1} \urcorner} \to \cdots \to z_{m_{s+1}}$ (case i) or by deleting co-hooks $z_{l_0} \to \cdots \to z_{m_0} \leftarrow z_{m_0 \urcorner}$ and $z_{m_s \ulcorner} \to z_{m_s} \leftarrow \cdots \leftarrow z_{l_{s+1}}$ (case ii) or by adding a hook $z_{m_{-1}} \leftarrow \cdots \leftarrow z_{l_0 \ulcorner} \to z_{l_0}$ and deleting a co-hook $z_{m_s \ulcorner} \to z_{m_s} \leftarrow \cdots \leftarrow z_{l_{s+1}}$ (case iii). Note that after deleting a co-hook of the form $z_{l_0} \to \cdots \to z_{m_0} \leftarrow z_{m_0 \urcorner}$ the vertex $z_{m_0 \urcorner}$ stays intact. If $M \simeq \mathrm{rad}\, P$ for a projective module P then $\tau^{-1}M \simeq P/\mathrm{soc}\, P$ (case iv).

We are going to use the same notation for morphisms in mod-A and the corresponding morphisms in $\underline{\mathrm{mod}}$-A. There are canonical diagram morphisms $M \to \tau^{-1}M$ induced by the intersection of diagrams. In case (i) there is a monomorphism $f : M \to \tau^{-1}M$, in case (ii) there is an epimorphism $f : M \to \tau^{-1}M$, in case (iii) there is a composition $f : M \to \tau^{-1}M$ of a monomorphism and an epimorphism. The map f is equal to zero in

the stable category iff in case (iii) module M is a maximal directed string $z_{l_0} \to \cdots \to z_{m_0}$ (case iii'). Note that in this last case M can be a simple module corresponding to a vertex with one incoming and one outgoing arrow. In case (iv) there are two morphisms f and f', with images equal to two indecomposable summands of $\mathrm{rad}P/\mathrm{soc}P$ (if P is not uniserial), $f = cf' \neq 0$ $(c \in \mathbf{k}^*)$ in $\underline{\mathrm{mod}}$-A. If P is uniserial, then $f = 0$.

If there is a morphism $g : \tau^{-1}M \to M_i$, then it factors through $\mathrm{Cone}(f)$, since $gf = 0$ in $\underline{\mathrm{mod}}$-A by the definition of the maximal system of orthogonal stable bricks, even if $M_i \simeq M$. Here $\mathrm{Cone}(f)$ denotes the cone of a morphism f in the triangulated structure on $\underline{\mathrm{mod}}$-A. Let us compute $\mathrm{Cone}(f)$.

Case (i): Since f is a monomorphism, $\mathrm{Cone}(f) \simeq \mathrm{Coker}(f) \simeq z_{m_{-1}} \leftarrow \cdots \leftarrow z_{l_0 \ulcorner} \oplus z_{l_{s+1} \urcorner} \to \cdots \to z_{m_{s+1}}$ is a sum of two maximal directed strings. (If the hook was trivial, then this is just a simple module.)

Case (ii): Since f is an epimorphism, $\mathrm{Cone}(f) \simeq \Omega^{-1}\mathrm{Ker}(f) \simeq \Omega^{-1}(z_{l_0} \to \cdots \to z_{m_0} \oplus z_{m_s} \leftarrow \cdots \leftarrow z_{l_{s+1}}) \simeq z_{m_0 \urcorner} \leftarrow \cdots \leftarrow z_{l_1} \leftarrow \cdots \leftarrow z_{l_0 \ulcorner} \oplus z_{l_{s+1} \urcorner} \to \cdots \to z_{l_s} \to \cdots \to z_{m_s \ulcorner}$ is a sum of two maximal directed strings (in the case, where z_{m_0} corresponds to a vertex with one incoming and one outgoing arrow and the co-hook is trivial there still is a maximal directed string ending at $z_{m_0 \urcorner}$ and we are going to use the notation $z_{m_0 \urcorner} \leftarrow \cdots \leftarrow z_{l_1} \leftarrow \cdots \leftarrow z_{l_0 \ulcorner}$ for it).

Case (iii): The morphism f is a composition of a monomorphism and an epimorphism, $\mathrm{Cone}(f)$ can be easily computed by the octahedron axiom or by the definition of triangles in $\underline{\mathrm{mod}}$-A. As before, $\mathrm{Cone}(f) \simeq z_{m_{-1}} \leftarrow \cdots \leftarrow z_{l_0 \ulcorner} \oplus z_{l_{s+1} \urcorner} \to \cdots \to z_{l_s} \to \cdots \to z_{m_s \ulcorner}$ is a sum of two maximal directed strings. In case $(iii$') let M be of the form $z_{m_s} = z_{l_0} \leftarrow \cdots \leftarrow z_{l_{s+1}}$, then $\mathrm{Cone}(f) = \tau^{-1}M \oplus \Omega^{-1}(M) \simeq z_{m_{-1}} \leftarrow \cdots \leftarrow z_{l_0 \ulcorner} \oplus z_{l_{s+1} \urcorner} \to \cdots \to z_{m_s \ulcorner} = z_{l_0 \ulcorner}$.

Case (iv): In this case $s = 0$. Assume that the projective module P is given by the relation $z_{l_0 \ulcorner} \to z_{l_0} \to \cdots \to z_{m_0 \ulcorner} \to z_{m_0} = cz_{l_{s+1} \urcorner} \to z_{l_{s+1}} \to \cdots \to z_{m_s \urcorner} \to z_{m_s}$ $(c \in \mathbf{k}^*)$, where $z_{l_0 \ulcorner} = z_{l_{s+1} \urcorner}$ and $z_{m_0} = z_{m_s}$. By the definition of triangles in $\underline{\mathrm{mod}}$-A we get $\mathrm{Cone}(f) \simeq z_{l_0 \ulcorner} \to z_{l_0} \to \cdots \to z_{m_0 \ulcorner} \oplus z_{l_{s+1} \urcorner} \to z_{l_{s+1}} \to \cdots \to z_{m_s \urcorner}$ is again a sum of two maximal directed strings. If P is uniserial, $f = 0$, then $\mathrm{Cone}(f) = P/\mathrm{soc}P \oplus \Omega^{-1}(\mathrm{rad}P) = P/\mathrm{soc}P \oplus \mathrm{top}P$ is a sum of two maximal directed strings, one of which is trivial.

Let M_i be a module of the form (1), (2) or (3) from Lemma 1, assume there is a non-zero morphism $\tilde{g} : \mathrm{Cone}(f) \to M_i$ in $\underline{\mathrm{mod}}$-A. Without loss of generality assume there is a morphism $\tilde{g} : (z_{m_{-1}} \leftarrow \cdots \leftarrow z_{l_0 \ulcorner}) \to M_i$. This morphism is non-zero only in the following cases:

- case (1) $e_{j_0} = z_{l_0\Gamma}$ and the composition of the last arrow in $e_{j_0} \leftarrow \cdots \leftarrow e_{i_1}$ and the first arrow in $z_{m-1} \leftarrow \cdots \leftarrow z_{l_0\Gamma}$ is zero;
- case (1) $e_{j_t} = z_{l_0\Gamma}$ and the composition of the last arrow in $e_{j_t} \leftarrow \cdots \leftarrow e_{i_t}$ and the first arrow in $z_{m-1} \leftarrow \cdots \leftarrow z_{l_0\Gamma}$ is zero;
- case (2) $e_{i_1} = z_{l_0\Gamma}$ and $e_{i_1} \to \cdots \to e_{j_1}$ is a substring of $z_{m-1} \leftarrow \cdots \leftarrow z_{l_0\Gamma}$;
- case (2) $e_{i_t} = z_{l_0\Gamma}$ and $e_{i_t} \to \cdots \to e_{j_{t-1}}$ is a substring of $z_{m-1} \leftarrow \cdots \leftarrow z_{l_0\Gamma}$;
- case (3) $e_{j_0} = z_{l_0\Gamma}$ and the composition of the last arrow in $e_{j_0} \leftarrow \cdots \leftarrow e_{i_1}$ and the first arrow in $z_{m-1} \leftarrow \cdots \leftarrow z_{l_0\Gamma}$ is zero;
- case (3) $e_{i_t} = z_{l_0\Gamma}$ and $e_{i_t} \to \cdots \to e_{j_{t-1}}$ is a substring of $z_{m-1} \leftarrow \cdots \leftarrow z_{l_0\Gamma}$.

Only one of all these cases can occur, and for only one $M_i \in \mathcal{M}$, otherwise, there would be a non-zero morphism between two objects from $\mathcal{M}$, which is not identity in the case they coincide. With the same cases for the other maximal directed string we get $\dim \underline{\mathrm{Hom}}(\tau^{-1}M, \oplus_{M_i \in \mathcal{M}} M_i) \le 2$. $\qquad\square$

Remark 3. We have seen that $\dim \underline{\mathrm{Hom}}(\tau^{-1}M, \oplus_{M_i \in \mathcal{M}} M_i) \le 2$. Now we are going to list all the cases, where $g : \tau^{-1}M \to M_i \ne 0$ in $\underline{\mathrm{mod}}$-A, i.e. $\tilde{g}h \ne 0$, where $h : \tau^{-1}M \to \mathrm{Cone}(f)$. In the above notation:

- For M_i of the form (1) from Lemma 1 the map $g \ne 0$, if and only if one of the following holds (we will write out the condition only for one end of the diagram):

  ○ M is of the form (i), $e_{j_0} = z_{l_0\Gamma}$ and the composition of the last arrow in $e_{j_0} \leftarrow \cdots \leftarrow e_{i_1}$ and the first arrow in $z_{m-1} \leftarrow \cdots \leftarrow z_{l_0\Gamma}$ is zero, additionally, the subdiagram of $\tau^{-1}M$ starting from z_{l_0} and coinciding with the subdiagram of $\Omega(M_i)$ starting from $e_{j_0\llcorner}$ ends in a deep of $\tau^{-1}M$ which is not a deep of $\Omega(M_i)$ or it ends on a peak of $\Omega(M_i)$ which is not a peak of $\tau^{-1}M$.

  ○ M is of the form (iii), the condition is the same as in the previous case.

  ○ M is of the form (iii'), $e_{j_0} = z_{l_0\Gamma}$ and the composition of the last arrow in $e_{j_0} \leftarrow \cdots \leftarrow e_{i_1}$ and the first arrow in $z_{m-1} \leftarrow \cdots \leftarrow z_{l_0\Gamma}$ is zero.

- For M_i of the form (2) from Lemma 1 the map $g \ne 0$, if and only if one on the following holds (we will write out the condition only for one end of the diagram):

○ M is of the form (i), $e_{i_1} = z_{l_0\ulcorner}$ and $e_{i_1} \to \cdots \to e_{j_1}$ is a subdiagram of $z_{m_{-1}} \leftarrow \cdots \leftarrow z_{l_0\ulcorner}$.

○ M is of the form (ii) $e_{i_1} = z_{l_0\ulcorner}$ and $e_{i_1} \to \cdots \to e_{j_1}$ is a subdiagram of $z_{m_0\urcorner} = z_{m_{-1}} \leftarrow \cdots z_{l_1} \cdots \leftarrow z_{l_0\ulcorner}$, z_{l_1} belongs to $e_{i_1} \to \cdots \to e_{j_1}$, if $z_{l_1} = e_{j_1}$ the subdiagram of $\tau^{-1}M$ starting from z_{l_1} and coinciding with the subdiagram of $\Omega(M_i)$ starting from e_{j_1} (going in the direction of e_{j_2}) ends in a deep of $\tau^{-1}M$ which is not a deep of $\Omega(M_i)$ or it ends on a peak of $\Omega(M_i)$ which is not a peak of $\tau^{-1}M$.

○ M is of the form (iii), condition here coincides with the previous case.

○ M is of the form (iii') $e_{i_1} = z_{l_0\ulcorner}$ and $e_{i_1} \to \cdots \to e_{j_1}$ is a subdiagram of $z_{m_{-1}} \leftarrow \cdots z_{l_1} \cdots \leftarrow z_{l_0\ulcorner}$.

○ M is of the form (iv) $e_{i_1} = z_{l_0\ulcorner}$.

• For M_i of the form (3) from Lemma 1 condition for g to be nonzero can be easily obtained as a combination of previous cases.

In all other cases the composition is either zero or factors through a projective module.

Corollary 1. *Let A be a selfinjective special biserial algebra, let B be a selfinjective algebra and let $F : \underline{\mathrm{mod}}\text{-}B \to \underline{\mathrm{mod}}\text{-}A$ be an equivalence of categories. Then in the quiver of B there are at most two incoming and at most two outgoing arrows at each vertex.*
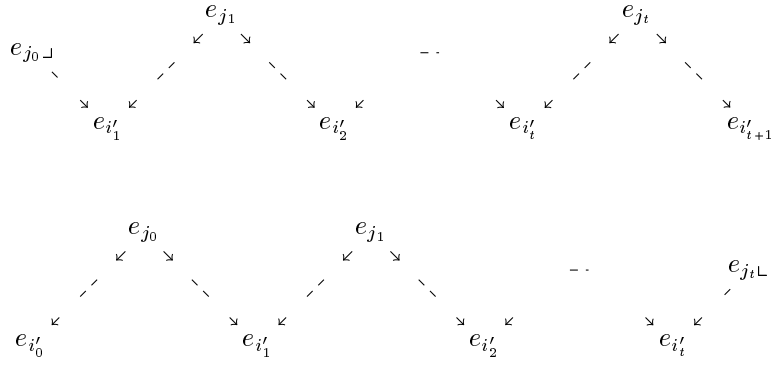
Proof. Let $\mathcal{M} = \{M_1, \cdots, M_n\}$ be the image of the set of simple B-modules under F. Let S, S_i be simple B-modules sent to $M, M_i \in \mathcal{M}$. By Auslander formula $\mathrm{Ext}^1(S, S_i) \simeq D\underline{\mathrm{Hom}}(\tau^{-1}S_i, S)$ and $\mathrm{Ext}^1(S_i, S) \simeq D\underline{\mathrm{Hom}}(\tau^{-1}S, S_i)$, but $\underline{\mathrm{Hom}}(\tau^{-1}S_i, S) \simeq \underline{\mathrm{Hom}}(\tau^{-1}M_i, M)$ and $\underline{\mathrm{Hom}}(\tau^{-1}S, S_i) \simeq \underline{\mathrm{Hom}}(\tau^{-1}M, M_i)$. The number of arrows from the vertex corresponding to S to the vertex corresponding to S_i coincides with $\dim \mathrm{Ext}^1(S, S_i)$, thus by the previous lemma there are at most tow incoming and at most two outgoing arrows at the vertex corresponding to S. $\qquad\square$

Definition 5. *Let N be an indecomposable s-projective module with respect to a maximal system of orthogonal stable bricks $\mathcal{M}$. An A-module R is said to be the s-radical of N (we denote R by $\mathrm{s\text{-}rad}N$) if the following conditions are satisfied:*
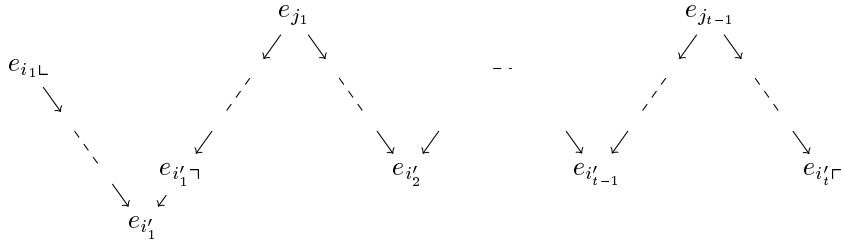
(1) R does not contain any projective direct summands.

(2) There is a projective A-module P and a right minimal almost split morphism $R \oplus P \to N$ in $\mathrm{mod}\text{-}A$, here P may be zero.

Lemma 4 (see Lemma 6.6 [14]). *Let A be a selfinjective special biserial algebra and let $\mathcal{M}$ be a maximal system of orthogonal stable bricks in $\underline{\mathrm{mod}}\text{-}A$. Let $M \in \mathcal{M}$ and let N be an indecomposable s-projective A-module such that $\mathrm{s\text{-}top}(N) \simeq M$. Then $\mathrm{s\text{-}rad}(N) = R_1 \oplus R_2$, where R_1, R_2 are indecomposable, in the notation of Lemma 1, R_1 and R_2 can be computed applying operations $^l(-)$ and $(-)^r$ to the string corresponding to N:*

case (1): R_1 and R_2 are of the form

$$
\begin{array}{ccccccccc}
 & & e_{j_1} & & & & & e_{j_t} & \\
e_{j_0\lrcorner} & \swarrow & & \searrow & \cdots & & \swarrow & & \searrow \\
 & e_{i'_1} & & & e_{i'_2} & & e_{i'_t} & & e_{i'_{t+1}} \\
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & e_{j_0} & & & e_{j_1} & & & & e_{j_t\llcorner} \\
\swarrow & & \searrow & \swarrow & & \searrow & \cdots & & \\
e_{i'_0} & & & e_{i'_1} & & & e_{i'_2} & & e_{i'_t} \\
\end{array}
$$

case (2): R_1 and R_2 are of the form

$$
\begin{array}{ccccccccc}
 & & e_{j_1} & & & & & e_{j_{t-1}} & \\
e_{i_1\llcorner} & & & & \cdots & & & & \\
 & e_{i'_1\urcorner} & & e_{i'_2} & & e_{i'_{t-1}} & & e_{i'_t\ulcorner} \\
 & e_{i'_1} & & & & & & \\
\end{array}
$$

where $e_{i_1} \to e_{i_1\llcorner} \to \cdots \to e_{i'_1} = c e_{i_1} \to \cdots \to e_{j_1} \to \cdots \to e_{i'_1\urcorner} \to e_{i'_1}$ in A ($c \in \mathbf{k}^*$);

$$
\begin{array}{c}
e_{j_1} \qquad\qquad\qquad\qquad e_{j_{t-1}} \\
\swarrow\quad\searrow \qquad\qquad \swarrow\quad\searrow \qquad\qquad e_{i_t \lrcorner} \\
\cdots \qquad\qquad\qquad\qquad\qquad \swarrow \\
e_{i'_1 \urcorner} \qquad\qquad e_{i'_2} \qquad\qquad e_{i'_{t-1}} \qquad\qquad e_{i'_t \ulcorner} \\
e_{i'_t}
\end{array}
$$
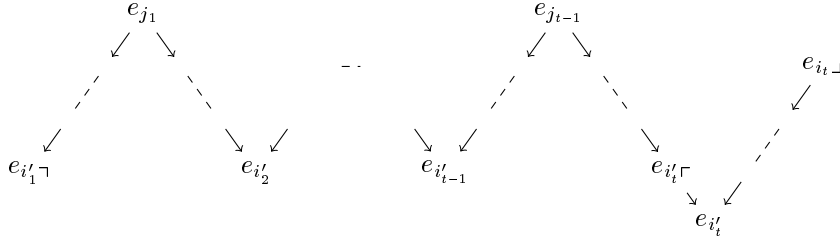
where $e_{i_t} \to \cdots \to e_{j_{t-1}} \to \cdots \to e_{i'_t \ulcorner} \to e_{i'_t} = c e_{i_t} \to e_{i_t \lrcorner} \to \cdots \to e_{i'_t}$ in A $(c \in \mathbf{k}^*)$;

    case (3): R_1 *and* R_2 *are of the form*

$$
\begin{array}{c}
e_{j_{t-1}} \\
e_{j_0 \lrcorner} \qquad\qquad \cdots \qquad\qquad \swarrow\quad\searrow \\
\searrow \qquad\qquad \swarrow\quad\searrow \qquad\qquad\qquad\searrow \\
e_{i'_1} \qquad\qquad e_{i'_{t-1}} \qquad\qquad e_{i'_t \ulcorner}
\end{array}
$$

$$
\begin{array}{c}
e_{j_0} \qquad\qquad\qquad\qquad e_{j_{t-1}} \\
\swarrow\quad\searrow \qquad\qquad \swarrow\quad\searrow \qquad\qquad e_{i_t \lrcorner} \\
\cdots \qquad\qquad\qquad\qquad\qquad \swarrow \\
e_{i'_0} \qquad\qquad e_{i'_1} \qquad\qquad e_{i'_{t-1}} \qquad\qquad e_{i'_t \ulcorner} \\
e_{i'_t}
\end{array}
$$
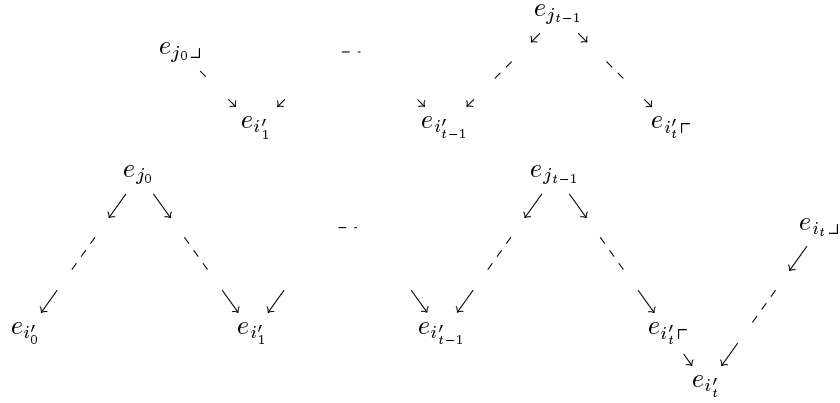
where $e_{i_t} \to \cdots \to e_{j_{t-1}} \to \cdots \to e_{i'_t \ulcorner} \to e_{i'_t} = c e_{i_t} \to e_{i_t \lrcorner} \to \cdots \to e_{i'_t}$ in A $(c \in \mathbf{k}^*)$. *Note that* R_1 *may be zero.*

Corollary 2. *Let* $\underline{\mathrm{mod}}\text{-}B \to \underline{\mathrm{mod}}\text{-}A$ *be an equivalence of categories, where* B *is selfinjective and* A *is selfinjective special biserial. Let* $\mathcal{M} = \{M_1, \ldots, M_n\}$ *be the image of the set of simple* B*-modules and let* $\{N_1, \cdots, N_n\}$ *be the image of the corresponding modules of the form* $P/\mathrm{soc}P$*, where* P *is indecomposable projective. Then* s-rad(N_i) *is the image of the module of the form* $\mathrm{rad}P/\mathrm{soc}P$*. Moreover, indecomposable summands of* s-rad(N_i) *have simple* s-top*, that is, if* s-rad$(N_i) = R_1 \oplus R_2$*, where* R_1, R_2 *are indecomposable, then* $\dim \underline{\mathrm{Hom}}(R_j, \oplus_{M_i \in \mathcal{M}} M_i) = 1$ *for non-zero* R_j*.*

Proof. Note that for an indecomposable projective module P, $\dim \mathrm{top}(\mathrm{rad}P/\mathrm{soc}P)$ corresponds to the number of arrows going out of the

vertex corresponding to P. By Corollary 1 there are at most two arrows going out of the vertex corresponding to P; thus, if $\operatorname{rad}P/\operatorname{soc}P$ has two non-zero non-projective summands $\tilde{R}_1$ and $\tilde{R}_2$, then both R_1 and R_2 are non-zero, hence $\dim\underline{\operatorname{Hom}}(\tilde{R}_j, \oplus_{S_i\in\mathcal{S}}S_i) = 1$ and $\dim\underline{\operatorname{Hom}}(R_j, \oplus_{M_i\in\mathcal{M}}M_i) = 1$.

Assume now that for some N the module s-$\operatorname{rad}(N) = R$ is indecomposable. By the description of the Auslander–Reiten triangles in $\underline{\operatorname{mod}}$-$A$, the diagram of N is a maximal directed string (which may or may not coincide with $P/\operatorname{soc}P$ for a uniserial projective module P). Then, $M =$ s-$\operatorname{top}(N)$ is a maximal directed string or a simple module (in case $P/\operatorname{soc}P$). The A^{op}-module DM is also a maximal directed string or a simple module. By arguments analogous to the proof of Lemma 3 and Remark 3, $\dim\underline{\operatorname{Hom}}(\tau^{-1}DM, \oplus_{M_i\in\mathcal{M}}DM_i) = 1$. Note that, if DM is simple corresponding to a vertex with one incoming and one outgoing arrow the result follows from Lemma 2. As

$$\dim\underline{\operatorname{Hom}}(\tau^{-1}DM, \bigoplus_{M_i\in\mathcal{M}} DM_i) = 1 = \dim\underline{\operatorname{Hom}}(\tau^{-1}\bigoplus_{M_i\in\mathcal{M}} M_i, M),$$

there is one arrow going out of the vertex corresponding to P and

$$\dim\underline{\operatorname{Hom}}(R, \bigoplus_{M_i\in\mathcal{M}} M_i) = 1. \qquad \square$$

In the notation of Corollary 2, for an indecomposable projective B-module P with $\operatorname{top}(P) = S$, the dimension of $\operatorname{top}(\operatorname{rad}(P))$ corresponds to the dimension of $\operatorname{Ext}^1(S, \oplus S_i)$, where $\oplus S_i$ is the sum of representatives of iso-classes of simple B-modules. If $\operatorname{Ext}^1(S, S_i) \neq 0$, then S_i is a summand of $\operatorname{top}(\operatorname{rad}(P))$. Since $\operatorname{Ext}^1(S, S_i) \simeq \underline{\operatorname{Hom}}(\tau^{-1}S_i, S)$, using the equivalence of stable categories, s-$\operatorname{top}(R)$ corresponds to $\underline{\operatorname{Hom}}(\tau^{-1}M_i, M)$. The following lemma follows easily from Remark 3. (The roles of M and M_i are switched.)

Lemma 5 (see Lemma 6.9 [14])**.** *Let A be a selfinjective special biserial algebra and let $\mathcal{M}$ be a maximal system of orthogonal stable bricks in $\underline{\operatorname{mod}}$-$A$, assume additionally, that $\mathcal{M}$ is an image of the set of simple B-modules under some stable equivalence, where B is selfijective. Let $M \in \mathcal{M}$ be as in Lemma 1. Moreover, let N be an indecomposable s-projective A-module such that s-$\operatorname{top}(N) \simeq M$, in the notation of Lemma 4, s-$\operatorname{rad}(N) = R_1 \oplus R_2$, then s-$\operatorname{top}(R_1)$ and s-$\operatorname{top}(R_2)$ are of the following form:*
case (1): s-$\operatorname{top}(R_1)$ is

$$
\begin{array}{ccccc}
e_{j_0 \lrcorner} & z_{l_1} & z_{l_s} & z_{l_{s+1}} \\
& & & \\
& z_{m_0} & & z_{m_s}
\end{array}
$$

where either the diagrams of s-top(R_1) and R_1 coincide or the subdiagram of s-top(R_1) starting from $e_{j_0 \lrcorner}$ and coinciding with the subdiagram of R_1 starting from $e_{j_0 \lrcorner}$ ends on a deep of s-top(R_1) which is not a deep of R_1 or it ends on a peak of R_1 which is not a peak of s-top(R_1) (note that this guarantees the existence of a non-zero morphism from R_1 to s-top(R_1) which sends $e_{j_0 \lrcorner}$ to $e_{j_0 \lrcorner}$ and is non-zero in the stable category; note also that this intersection can consist of one vertex);

s-top(R_2) is

$$
\begin{array}{ccccc}
e_{j_t \llcorner} & z_{l_1} & z_{l_s} & z_{l_{s+1}} \\
& & & \\
& z_{m_0} & & z_{m_s}
\end{array}
$$

where either the diagrams of s-top(R_2) and R_2 coincide or the subdiagram of s-top(R_2) starting from $e_{j_t \llcorner}$ and coinciding with the subdiagram of R_2 starting from $e_{j_t \llcorner}$ ends on a deep of s-top(R_2) which is not a deep of R_2 or it ends on a peak of R_2 which is not a peak of s-top(R_2);

case (2): s-top(R_1) is

$$
\begin{array}{ccccc}
e_{i_1 \llcorner} & z_{l_1} & z_{l_s} & z_{l_{s+1}} \\
& & & \\
& z_{m_0} & & z_{m_s}
\end{array}
$$

where either the diagrams of s-top(R_1) and R_1 coincide or the subdiagram of s-top(R_1) starting from $e_{i_1 \llcorner}$ and coinciding with the subdiagram of R_1 starting from $e_{i_1 \llcorner}$ ends on a deep of s-top(R_1), which is not a deep of R_1 or it ends on a peak of R_1 which is not a peak of s-top(R_1);

s-top(R_2) is

$$e_{i_t\lrcorner} \qquad\qquad z_{l_1} \qquad\qquad z_{l_s} \qquad\qquad z_{l_{s+1}}$$

$$z_{m_0} \qquad\qquad\qquad\qquad z_{m_s}$$

where either the diagrams of s-top(R_2) *and* R_2 *coincide or the subdiagram of* s-top(R_2) *starting from* $e_{i_t\lrcorner}$ *and coinciding with the subdiagram of* R_2 *starting from* $e_{i_t\lrcorner}$ *ends on a deep of* s-top(R_2) *which is not a deep of* R_2 *or it ends on a peak of* R_2 *which is not a peak of* s-top(R_2);

case (3): R_1 is analogous to R_1 from case (1); R_2 is analogous to R_2 from case (2).

We are going to use the following criterion to prove that a selfinjective algebra stably equivalent to a special biserial algebra is stably biserial. Here we cite only the part of the result that we need. Note that this proposition was reproved in [5]:

Proposition 3 (Proposition 2.7 [14], Proposition 7.8 [5]). *If a selfinjective algebra B satisfies the following conditions, then B is Morita equivalent to an algebra, that satisfies conditions (a) and (c) from Definition 1.*
   (a) *For each indecomposable projective module P, we have*

$$\mathrm{rad}(P)/\mathrm{soc}(P) = X' \oplus X'',$$

(where $X' \neq 0$) such that $\mathrm{top}(X')$, $\mathrm{top}(X'')$, $\mathrm{soc}(X')$, $\mathrm{soc}(X'')$ are simple modules (or zero, in case X'' is zero).
   (b) *Let $X = X'$ or X'', and let Q be the projective cover of X. Then X is non-projective and we denote by p the epimorphism $Q/\mathrm{soc}(Q) \to X$. Suppose that $\mathrm{rad}(Q)/\mathrm{soc}(Q) = Y_1 \oplus Y_2$, where Y_1 and Y_2 are indecomposable modules. Then, for irreducible morphisms $w_1 : Y_1 \to Q/\mathrm{soc}(Q)$, $w_2 : Y_2 \to Q/\mathrm{soc}(Q)$, pw_1 or pw_2 factors through a projective module.*

To use the criterion above we need the following lemma.

Lemma 6 (see Proposition 7.1 [14]). *Let A be selfinjective special biserial, let $\mathcal{M}$ be a maximal system of orthogonal stable bricks which is an image of the set of simple B-modules under some stable equivalence, where B is selfinjective. Let N be s-projective and $M \in \mathcal{M}$ be s-top(N). For s-rad$(N) = R_1 \oplus R_2$, where R_1, R_2 are indecomposable, let s-top$(R_i) = Y \in \mathcal{M}$ and let Q be an indecomposable s-projective such that s-top$(Q) = Y$, let $L_1 \oplus L_2$ be the s-radical of Q, where L_1, L_2 are indecomposable. There exist $\underline{f} : Q \to R_i$ and*

$\underline{h} : R_i \to Y$ *with* $\underline{hf} \neq 0$ *such that for irreducible morphisms* $g_1 : L_1 \to Q$, $g_2 : L_2 \to Q$, *we have* $\underline{fg_1} = 0$ *or* $\underline{fg_2} = 0$.

Proof. By Lemma 5 in all the cases R_i and s-top(R_i) start from the same vertex and their intersection ends on a deep of s-top(R_i) which is not a deep of R_i, or it ends on a peak of R_i which is not a peak of s-top(R_i) or R_i and s-top(R_i) coincide. That guarantees the existence of a morphism from R_i to s-top(R_i), which sends this intersection to itself and this morphism is non-zero in $\underline{\text{mod}}$-A, let us denote this morphism by h.

Without loss of generality we can consider the case (1)-R_1. In each case s-top(R_1) is itself a module of the form (1)-(3) from Lemma 1.

If s-top(R_1) has the form (1), then there is a non-zero morphism f from Q to R_1, whose image consists only of $e_{j_0 \lrcorner}$. There is a summand L_1 of s-rad(Q) which is formed by deleting the hook starting with $e_{j_0 \lrcorner}$, clearly $\underline{fg_1} = 0$ and $\underline{hf} \neq 0$. If s-top(R_1) has the form (2), then there is a non-zero morphism f from Q to R_1, induced by $z_{m_0} \to e_{i'_1}$. There is a summand L_1 of s-rad(Q) which is formed by adding a co-hook starting from $e_{j'_0 \lrcorner}$, the composition fg_1 factors through the projective module with the top corresponding to $e_{j_0 \lrcorner}$, clearly $\underline{hf} \neq 0$. The case when s-top(R_1) has the form (3) is similar. $\qquad\square$

Theorem 1 (see Theorem 7.3 [14]). *Let A be a selfinjective special biserial* **k**-*algebra not isomorphic to the Nakayama algebra with* $\text{rad}^2 = 0$. *If B is a basic algebra stably equivalent to A, then B is stably biserial.*

Proof. Let $\Phi : \underline{\text{mod}}$-$B \to \underline{\text{mod}}$-$A$ be an equivalence of categories. Since A is a selfinjective special biserial **k**-algebra not isomorphic to the Nakayama algebra with $\text{rad}^2 = 0$ we can assume that B is selfinjective. Indeed, since over A for any Auslander–Reiten sequence $0 \to M \xrightarrow{f} N \oplus P \to L \to 0$, where P is projective and N is not projective, we have $\underline{f} \neq 0$, then by [7, Proposition 2.3] $0 \to \Phi^{-1}(M) \to \Phi^{-1}(N) \oplus Q \to \Phi^{-1}(L) \to 0$ is the Auslander–Reiten sequence for some projective Q. Hence, τ and τ^{-1} are defined for all not projective modules, so B is selfinjective.

Let B be a selfinjective algebra which is not a local Nakayama algebra. Then by Proposition 2 none of the simple B-modules $\{S_i\}_{i=1,\ldots n}$ and none of the modules of the form $P/\text{soc}P$ for an indecomposable projective B-module P are of τ-period 1. Thus $\{\Phi(S_i)\}_{i=1,\ldots,n} = \mathcal{M}$ is a maximal system of orthogonal stable bricks over A. As $\{\Phi(P_i/\text{soc}P_i)\}_{i=1,\ldots,n}$ is the set of s-projective modules with respect to $\mathcal{M}$, Φ sends $\text{rad}(P_i/\text{soc}P_i)$ to

s-rad$(\Phi(P_i/\mathrm{soc}P_i))$. Corollary 2 implies that $\mathrm{rad}(P_i/\mathrm{soc}P_i)$ is a sum of at most two modules with simple top.

The duality $D_B : \mathrm{mod}\text{-}B \to \mathrm{mod}\text{-}B^{\mathrm{op}}$ sends simple B-modules to simple B^{op}-modules, modules of the form $\mathrm{rad}(P_i/\mathrm{soc}P_i)$ to modules of the same form, top to socle and socle to top. The equivalence Φ induces an equivalence $\underline{\mathrm{mod}}\text{-}B^{\mathrm{op}} \to \underline{\mathrm{mod}}\text{-}A^{\mathrm{op}}$. Since A^{op} is also selfinjective special biserial, $\mathrm{rad}(P_i/\mathrm{soc}P_i)$ is a sum of at most two modules with simple top. Hence, the B-module $\mathrm{rad}(P_i)/\mathrm{soc}(P_i)$ is a sum of at most two modules with simple socle. Thus, the condition (a) of Proposition 3 holds. These conditions correspond to the fact that there are at most two incoming and at most two outgoing arrows in the quiver of B.

In the notations of Proposition 3, by Lemma 6 there exists $p : Q/\mathrm{soc}Q \to X$ such that condition (b) holds. Let us prove that condition (b) holds for any $p' : Q/\mathrm{soc}Q \to X$. Let us denote by $\pi_X : Q \to X$ the projective cover of X and by $\pi : Q \to Q/\mathrm{soc}Q$ the projective cover of $Q/\mathrm{soc}Q$. By assumption X has a simple top, thus without loss of generality we can assume that the image of $p'' = p - p'$ belongs to $\mathrm{rad}(X)$. The morphism p'' can be lifted to a morphism $\tilde{p} : Q \to Q$ between the projective covers ($\pi_X\tilde{p} = p''\pi$). The image of $\tilde{p}$ belongs to $\mathrm{rad}(Q)$; hence $\tilde{p}$ factors through $Q/\mathrm{soc}(Q)$ and $\tilde{p} = h\pi$ for some h. Thus, $\pi_X h\pi = p''\pi$ and since π is an epimorphism $\pi_X h = p''$. We get that p'' factors through a projective, and hence is zero in the stable category, $\underline{p} = \underline{p}'$ and condition (b) of Proposition 3 holds. This condition correspond to condition (c) in Definition 1.

It is clear that the conditions (b) and (c) of Definition 1 are dual to each other. By the previous paragraph condition (b) of Proposition 3 holds for B^{op}, and thus condition (c) in Definition 1 holds for B^{op}; thus condition (b) in Definition 1 holds for B and B is stably biserial. $\square$

§4. AUSLANDER–REITEN CONJECTURE

In this section we are going to prove the Auslander–Reiten conjecture for special biserial algebras.

Let B be a stably biserial algebra. It is clear that $B/\mathrm{soc}(B)$ is a string algebra, and hence the classification of indecomposable non-projective modules over B coincides with the usual classification using string and band modules. Then by [6, Proposition 4.5] all Auslander–Reiten sequences over B and $B/\mathrm{soc}(B)$ not ending with a B-module of the form $P/\mathrm{soc}(P)$ coincide. Hence, if there is a system of orthogonal stable bricks $\mathcal{M}$ over B, then all the modules in $\mathcal{M}$ are string modules.

Lemma 7 (compare to Lemma 4.1 [14]). *Let $A = \mathbf{k}Q/I$ be a stably biserial algebra and let $\mathcal{M} = \{M_1, \ldots, M_k\}$ be a system of orthogonal stable bricks. Then every simple A-module can appear in the multiset of endpoints of diagrams corresponding to $M_i \in \mathcal{M}$ at most twice.*

Proof. Let us fix some $v \in Q_0$. We will consider the simple module corresponding to v and diagrams of $M_i \in \mathcal{M}$ ending at v, that is $M_i = c_1 \cdots c_l$, $s(M_i) = v$ or $e(M_i) = v$. Suppose that some arrow α incident to v occurs twice at the endpoint v of some diagrams $M_{i_1} = c_1 \cdots c_l, M_{i_2} = d_1 \cdots d_t$ for some $1 \le i_1, i_2 \le k$ in the same manner. Taking the opposite strings $M_{i_j}^{-1}$ if necessary, we can assume that either $s(M_{i_j}) = v$, $\alpha = c_1 = d_1$ or $s(M_{i_j}) = v$, $\alpha = c_1^{-1} = d_1^{-1}$. In both cases, there is a non-zero morphism $f : M_{i_1} \to M_{i_2}$ or $f : M_{i_2} \to M_{i_1}$, corresponding to the common part of the diagrams M_{i_1}, M_{i_2}. The morphism f is non-zero in $\underline{\mathrm{mod}}$-B, this is a contradiction to the definition of a system of orthogonal bricks.

Now we are to show that at most two different arrows, incident to v can occur at the endpoint v of the diagrams of $M_i \in \mathcal{M}$. If there is only one incoming or outgoing arrow at v (and, consequently, only one outgoing or incoming arrow at v, see Lemma 2), there is nothing to prove. So suppose that there are $\alpha_1, \alpha_2, \beta_1, \beta_2$ with $s(\alpha_1) = s(\alpha_2) = e(\beta_1) = e(\beta_2) = v$ and consider two cases (if there are loops at the vertex v, some arrows may coincide): $\{\beta_i \alpha_j\}_{i,j=1,2} \not\subseteq \mathrm{soc}(A)$ and $\{\beta_i \alpha_j\}_{i,j=1,2} \subseteq \mathrm{soc}(A)$.
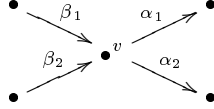
Case 1. Without loss of generality we can assume $\beta_1 \alpha_2 \notin \mathrm{soc}(A)$. In this case, by stably biserial condition, we have $\beta_1 \alpha_1 \in \mathrm{soc}(A), \beta_2 \alpha_2 \in \mathrm{soc}(A)$. Also in this case we have $\beta_2 \alpha_1 \ne 0$. Indeed, if $\beta_2 \alpha_1 = 0$, then $0 \ne \beta_2 \alpha_2 \in \mathrm{soc}(A)$ or $\beta_2 \in \mathrm{soc}(A)$, which is impossible, hence if we consider a maximal path q with $q\beta_1 \alpha_2 \ne 0$ (q is of positive length, since $\beta_1 \alpha_2 \notin \mathrm{soc}(A)$) we have $\beta_2 \alpha_2 = lq\beta_1 \alpha_2$ for some $l \in \mathbf{k}^*$. As $q\beta_1 \alpha_1 \in q \cdot \mathrm{soc}(A) = 0$, we have $\beta_2 - lq\beta_1 \in \mathrm{soc}(A)$, a contradiction, and thus $\beta_2 \alpha_1 \ne 0$.

Let us prove that at least one of β_1^{-1}, α_1 does not occur at the endpoint of some $M_i \in \mathcal{M}$, and at least one of β_2^{-1}, α_2 does not occur at the endpoint of some $M_i \in \mathcal{M}$ — that is all we need. Take $j \in \{1, 2\}$ and assume that both β_j^{-1}, α_j occur at the endpoint of some $M, N \in \mathcal{M}$.

Let M be a module with the diagram starting from α_i ($c_1 = \alpha_i$), $x \in M$ be an element corresponding to v, that is $xe_v = x, x\alpha_i \neq 0, x\alpha_{3-i} = 0$, note that x is non-zero in the top of M. Let N be a module with diagram starting with β_i^{-1} ($d_1 = \beta_i^{-1}$), $y \in N$ is an element corresponding to v. Note that y belongs to the socle of N. Let $f : M \to N$ be the morphism with $f(x) = y$, which is zero in $\underline{\mathrm{mod}}\text{-}(A)$ by the definition of a system of orthogonal stable bricks. We claim that in this case $\tau N = N$ − this also contradicts the definition of orthogonal stable bricks.
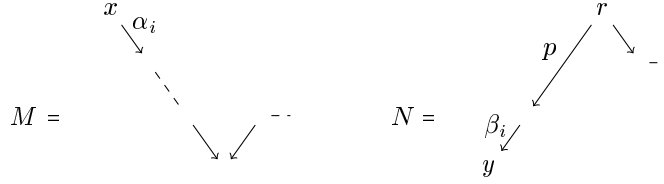
We prove the latter claim by induction on the number of maximal directed substrings of N. Let $p\beta_i$, where p is a path, correspond to the first maximal directed substring of N. Clearly $p\beta_i \notin \mathrm{soc}(A)$, as N does not contain projective summands, and therefore $p\beta_i\alpha_s \neq 0$ for some s. We can assume that $s \neq i$. Indeed, if $s = i$, then $\beta_i\alpha_i \in \mathrm{soc}(A)$ implies $p = e_{s(\beta_i)}$ and in this case $p\beta_i\alpha_{3-i} \neq 0$ as well.

Let $t = s(p)$. The projective cover of N is of the form $(g_1, g_2) : P = P_t \oplus P' \to N$ where $g_1(e_t) = r$ is the element of the basis corresponding to the first peak of N (so we have $rp\beta_i = y$) and $y \notin \mathrm{Im}(g_2)$. If $f = 0 \in \underline{\mathrm{mod}}\text{-}A$, we have $f = gh = g_1h_1 + g_2h_2$ for some $h = \binom{h_1}{h_2} : M \to P_t \oplus P'$. As $g_1(p\beta_i) = y$ we can set $h(x) = (p\beta_i + z_1, z_2)$, where $(z_1, z_2) \in \mathrm{Ker}(g)$. By construction of the projective cover, z_1 is a linear combination of paths not equal to $p\beta_i$ or subpaths of $p\beta_i$. Now $(0,0) = h(x\alpha_{3-i}) = (p\beta_i\alpha_{3-i} + z_1\alpha_{3-i}, z_2\alpha_{3-i})$, and therefore $0 \neq p\beta_i\alpha_{3-i} = kp_1\alpha_{3-i}$ for some path $p_1 \neq p\beta_i$ ($k \in \mathbf{k}^*$). The case $p_1 = p_1'\beta_i$ is impossible (in this case either both paths $p\beta_i\alpha_{3-i}, p_1\alpha_{3-i}$ have lengths at least 3 and contain subpaths of the form $\delta\gamma, \eta\gamma$ − a contradiction, or $\beta_i\alpha_{3-i}$ is equal to a longer path ending with $\beta_i\alpha_{3-i}$, which is also impossible), therefore, as $\beta_{3-i}\alpha_{3-i} \in \mathrm{soc}(A)$, we have $p_1 = \beta_{3-i}$. Note, that we get $p\beta_i\alpha_{3-i} \in \mathrm{soc}(A)$. Note that $p \neq \beta_{3-i}p_2$ for any path p_2 (else $p_1 = \beta_{3-i}$ is a subpath of $p\beta_i$).

Now we can prove the base of our induction. The previous paragraph shows that $s(p) = s(\beta_{3-i})$. If N is a directed string, corresponding to a maximal path $p\beta_i$ then $\tau^{-1}(N)$ is formed by adding a hook and deleting

a co-hook, as $e(\beta_i) = e(\beta_{3-i})$, this hook is a maximal directed string, corresponding to $p\beta_i$. We see that $\tau^{-1}(N) = N$, as desired.

Note that we can compute $\tau^{-1}(N)$ in the usual way, since N is not isomorphic to rad$P$ for some projective module P.

Now suppose that the diagram of N contains more than one maximal directed substrings. As $0 = f(x\alpha_i) = g(p\beta_i\alpha_i + z_1\alpha_i, z_2\alpha_i) = g(z_1\alpha_i, z_2\alpha_i)$ we have $g_1(z_1\alpha_i) = 0$ (since $\mathrm{Im}(g_1)\alpha_i \cap \mathrm{Im}(g_2)\alpha_i = 0$, as $\mathrm{Im}(g_1) \cap \mathrm{Im}(g_2) \in$ soc(N)), and, as $p\beta_i\alpha_{3-i} \in$ soc(P_t), we have $g_1(p\beta_i\alpha_{3-i}) = 0$. This implies that $g_1(\beta_{3-i}\alpha_i) = 0$, $g_1(\beta_{3-i}\alpha_{3-i}) = 0$, since $\beta_{3-i}\alpha_{3-i} \in$ soc(A), and hence the second maximal directed substring of the diagram of N is an arrow β_{3-i} ($g_1(\beta_{3-i}) = g_1(z_1) \in$ soc(N)). Consider a module $N' \leq N$, corresponding to the subdiagram, containing all but first two directed substrings of N (deleting a hook of N). Then we have $\mathrm{Im}(g_2) \subseteq N'$ and $g_2h(x) = g_2(z_2) = -g_1(z_1) = lr\beta_{3-i}$ for some $l \in \mathbf{k}^*$ (since $0 \neq g_1(\beta_{3-i}) = g_1(z_1)$). This means that the module N' and the morphism $f' = g_2h$ is of the same form as N and f (in particular, N' begins with β_i^{-1} as well). By induction, the string corresponding to N is of the form $\beta_i^{-1}p^{-1}\beta_{3-i}\beta_i^{-1}p^{-1}\beta_{3-i}\cdots\beta_i^{-1}p^{-1}$, and hence N has τ-period 1.

Case 2. $\{\beta_i\alpha_j\} \subseteq$ soc(A). For each i, $\beta_i \notin$ soc(A), so suppose that $\beta_i\alpha_{3-i} \neq 0$ (note that we can choose different j_1, j_2 for β_1, β_2 with $\beta_1\alpha_{j_1} \neq 0$, $\beta_2\alpha_{j_2} \neq 0$, since in the other case we have $\beta_1\alpha_j = \beta_2\alpha_j = 0$ for some j and $\alpha_j \in$ soc(A), which is impossible). Let us prove, as above (and with above notation) that α_j and β_i^{-1} cannot occur as first arrows for some M, N by checking that the corresponding morphism f is non-zero in $\underline{\mathrm{mod}}$-A. As above, $f(x\alpha_{3-i}) = 0$ implies that there is a path $p \neq \beta_i$ and $l \in \mathbf{k}^*$ such that $\beta_i\alpha_{3-i} = lp\alpha_{3-i}$. As $\beta_i\alpha_{3-i}, \beta_{3-i}\alpha_{3-i} \in$ soc(A), we obtain that $p = \beta_{3-i}$ (otherwise a socle path would be a subpath of a longer path). This implies that $s(\beta_1) = s(\beta_2)$.

Now we have that all directed strings containing β_i has length 1 and are maximal directed strings, and therefore N is of the form $\beta_i^{-1}\beta_{3-i}\beta_i^{-1}\beta_{3-i}\ldots$. If the length of this word is odd, then $\tau(N) = N$ (deleting a co-hook and adding a hook doesn't change N), contradiction. In the case of even length (i.e. if $\dim(N) = 2n + 1$ is odd) let $y_1, \ldots, y_n \in N$ be the elements of the diagram of N corresponding to peaks. Then projective cover of N is of the form $g : (e_{s(\beta_i)}A)^n \to N$, $g(z_k) = y_k$ for $k = 1, \ldots, n$, where z_k is the generator of the corresponding copy of $e_{s(\beta_i)}A$ and Ker$(g) = \langle\{z_k\beta_{3-i} - z_{k+1}\beta_i\}\rangle$. Now suppose that $f = gh$ for some h. Then $h(x) = z_1\beta_i + \sum_{k=1}^{n-1} l_k(z_k\beta_{3-i} -$

$z_{k+1}\beta_i$). Multiplying this by α_{3-i}, we obtain

$$0 = h(x\alpha_{3-i}) = \sum_{k=1}^{n-1} z_k (l_k \beta_{3-i}\alpha_{3-i} - l_{k-1}\beta_i\alpha_{3-i}) - l_{n-1}z_n\beta_i\alpha_{3-i},$$

where $l_0 = -1$. As all coefficients in the sum are to be zero, we obtain consequently that $l_i \neq 0$ for all $i = 0, \ldots n - 1$, therefore the last summand is non-zero, contradiction. $\qquad\square$

Recall that a simple non-projective, non-injective module S is called a node if the Auslander–Reiten sequence starting at S has the form

$$0 \to S \to P \to \tau^{-1}S \to 0,$$

where P is projective. By the results of [11], any algebra with nodes is stably equivalent to an algebra without nodes. Let A be an algebra with nodes $S_1, \cdots, S_k$, $S = \oplus_{i=1}^{k} S_i$. Let a be the trace of S in A, i.e. $\Sigma_{h\in\mathrm{Hom}(S,A)}\mathrm{Im}(h)$. Note that a is a two-sided ideal of A. Let b be a right annihilator of a, note that A/b is semisimple and a is an A/a-A/b bimodule. Then the matrix algebra $\hat{T}_A = \begin{pmatrix} A/a & a \\ 0 & A/b \end{pmatrix}$ has no nodes and it is stably equivalent to A. The construction of $\hat{T}_A$ replaces every node in the quiver of A by two simple modules: a sink and a source. It is clear, that the number of non-projective simple modules is preserved under this stable equivalence.

Theorem 2 (compare to Theorem 0.1 [14]). *Let A, B be two finite dimensional algebras such that $\underline{\mathrm{mod}}$-$A \cong \underline{\mathrm{mod}}$-$B$ and A is special biserial. Then the number of isomorphism classes of non-projective simple modules over A and B coincides.*

Proof. Without loss of generality we can assume that A, B have no semisimple summands. First, let us prove the statement for selfinjective A and B. If one of the algebras (and hence the other as well) has isolated vertices in the Auslander–Reiten quiver of the stable category, then they correspond to $P/\mathrm{soc}P$ or to $\mathrm{rad}P$ for some projective module P of length 2. Hence A and B have as summands Nakayama algebras with $\mathrm{rad}^2 = 0$, the number of simple modules over these algebras is the number of isolated vertices in the Auslander–Reiten quiver of the stable category, hence it is the same for A, B. From now on we can assume, that A, B do not have a Nakayama algebra with $\mathrm{rad}^2 = 0$ as a summand. By Theorem 1, B is stably biserial. Let $\mathcal{M} = \{M_1, \ldots, M_k\}$ be the images of simple A-modules under equivalence $F : \underline{\mathrm{mod}}$-$A \to \underline{\mathrm{mod}}$-$B$. Then $\mathcal{M}$ is a maximal system

of orthogonal stable bricks. If some M_i is a simple module, then it can not occur as an endpoint of any other diagram in $\mathcal{M}$. The diagram of each non-simple M_i has two endpoints, labelled by simple B-modules S_i^1 and S_i^2. Suppose that the number of simple B-modules is less than k, then $S_{i_1}^{j_1} = S_{i_2}^{j_2} = S_{i_3}^{j_3}$ for some i_l, j_l. This contradicts the previous lemma. The same argument for the quasi-inverse $\tilde{F} : \underline{\text{mod}}\text{-}B \to \underline{\text{mod}}\text{-}A$ shows that the number of simple B-modules is less or equal to the number of simple A-modules and we are done.

Let us now consider arbitrary A, B, where A is special biserial. If A or B has nodes, we can replace it by the matrix algebra $\hat{T}_A$ or $\hat{T}_B$, respectively. If A is special biserial, then so is $\hat{T}_A$, so we can assume that A, B have no nodes. To algebras A, B one can associate selfinjective algebras Δ_A, Δ_B in the following way: let P_A be the set of isoclasses of projective-injective A-modules that remain projective-injective under the action of any power of the Nakayama functor ν^k. Define $\Delta_A := \text{End}(\oplus_{P \in P_A} P)$. If A is special biserial, then Δ_A is selfinjective special biserial. By [12] (since A, B have no nodes) the algebras Δ_A, Δ_B are stably equivalent, and hence by the previous paragraph they have the same number of simple modules. By [12] A, B have the same number of isomorphism classes of non-projective simple modules. $\qquad\square$

§5. SYMMETRIC STABLY BISERIAL ALGEBRAS

Recall the standard description of a symmetric special biserial algebra [19]. We will assume that all quivers are connected. Consider the following data:

(1) A quiver Q such that every vertex has two incoming and two outgoing arrows or one incoming and one outgoing arrow.
(2) A permutation π on Q_1 with $e(\alpha) = s(\pi(\alpha))$ for all $\alpha \in Q_1$
(3) A function $m : C(\pi) \to \mathbb{N}$, where $C(\pi)$ is the set of cycles of π.

We are going to denote the cycle containing α by $\langle \pi \rangle \alpha$ and its order by $|\langle \pi \rangle \alpha|$. Now consider the ideal $I \subseteq \mathbf{k}Q$ generated by the following elements:

(1) $\alpha\beta$ for all $\alpha, \beta \in Q_1$, $\beta \neq \pi(\alpha)$;
(2) $\left(\alpha\pi(\alpha)\pi^2(\alpha) \ldots \pi^{|\langle \pi \rangle \alpha|-1}(\alpha) \right)^{m(\langle \pi \rangle \alpha)}$
    $- \left(\beta\pi(\beta)\pi^2(\beta) \ldots \pi^{|\langle \pi \rangle \beta|-1}(\beta) \right)^{m(\langle \pi \rangle \beta)}$ for all $\alpha, \beta \in Q_1$ with $s(\alpha) = s(\beta)$;

(3) $\left(\alpha\pi(\alpha)\pi^2(\alpha)\ldots\pi^{|\langle\pi\rangle\alpha|-1}(\alpha)\right)^{m(\langle\pi\rangle\alpha)}\alpha$ and

$\pi^{-1}(\alpha)\left(\alpha\pi(\alpha)\pi^2(\alpha)\ldots\pi^{|\langle\pi\rangle\alpha|-1}(\alpha)\right)^{m(\langle\pi\rangle\alpha)}$ for all $\alpha\in Q_1$ such

that $s(\alpha)$ has only one incoming and one outgoing arrow.

Then $\mathbf{k}Q/I$ is a symmetric special biserial algebra (SSB-algebra), and each SSB-algebra can be described uniquely in this way, up to obvious isomorphisms. Note that one of the relations from (3) is redundant.

The main aim of this section is to show that any symmetric stably biserial algebra is in a sense a deformation of some SSB-algebra. To obtain this, we are going to define the permutation π and the multiplicities of π-cycles for the algebras from this class.

From now on let $A = \mathbf{k}Q/I$ be an arbitrary stably biserial algebra, with I admissible. Let $\mathrm{sc}(A) = \mathrm{soc}(A) \smallsetminus \{0\}$.

Case I. For $\alpha\in Q_1$ we put $\pi(\alpha) = \beta$ if $\alpha\beta\notin\mathrm{soc}(A)$, $\beta\in Q_1$. The definition of a stably biserial algebra implies that we have at most one such arrow.

If $\alpha\mathrm{rad}(A) \subseteq \mathrm{soc}(A)$ we are to define $\pi(\alpha)$ a bit more carefully.

Note that $\alpha\mathrm{rad}(A) = 0$ only for the case $A = \mathbf{k}[\alpha]/\alpha^2$ of the algebra with one vertex and one loop α, for that case $\pi(\alpha) = \alpha$, we are not going to consider this case from here on. We can assume $\alpha\mathrm{rad}(A) \neq 0$ for any $\alpha\in Q_1$. Then (if $\alpha\mathrm{rad}(A) \subseteq \mathrm{soc}(A)$) we have the following cases:

Case II. There exist $\beta_1, \beta_2 \in Q_1$ ($\beta_1 \neq \beta_2$) with $\alpha\beta_i \in \mathrm{sc}(A)$ ($i = 1, 2$).

If $|Q_0| = 1$ and Q_1 consists of two loops α, β, then $\alpha^2, \alpha\beta \in \mathrm{sc}(A)$ implies $\beta\alpha \in \mathrm{sc}(A)$. If $\beta^2 = 0$ set $\pi(\alpha) = \beta, \pi(\beta) = \alpha$, if $\beta^2 \in \mathrm{sc}(A)$ we can chose $\pi(\alpha) = \alpha, \pi(\beta) = \beta$. If $\beta^2 \notin \mathrm{soc}(A)$, set $\pi(\alpha) = \alpha, \pi(\beta) = \beta$. From now on $|Q_0| > 1$.

The arrow α isn't a loop – otherwise β_1, β_2 are loops in the same vertex and we have $|Q_0| = 1$. Due to the symmetry, we have $e(\beta_i) = s(\alpha), i = 1, 2$.

If $|Q_0| > 2$ there exists a unique $\gamma \in Q_1$ with $s(\gamma) = s(\alpha), e(\gamma) \neq e(\alpha)$ and there exists a unique $\delta \in Q_1$ with $e(\delta) = e(\alpha), s(\delta) \neq s(\alpha)$. Then we have $\delta\beta_i \notin \mathrm{soc}(A)$ and $\beta_i\gamma \notin \mathrm{soc}(A)$ for some i and $\delta\beta_{3-i} = 0$ and $\beta_{3-i}\gamma = 0$ (as $\delta\beta_{3-i}$ and $\beta_{3-i}\gamma$ belong to $\mathrm{soc}(A)$ by stably biserial condition and are not cycles). Then $\pi(\delta) = \beta_i, \pi(\beta_i) = \gamma$ as defined in Case I, and we can put $\pi(\alpha) = \beta_{3-i}, \pi(\beta_{3-i}) = \alpha$.

Now consider the case $|Q_0| = 2$. Due to the symmetry $\beta_1\alpha, \beta_2\alpha \neq 0$ and clearly $\beta_1\alpha, \beta_2\alpha \in \mathrm{sc}(A)$, $\beta_1\alpha = c\beta_2\alpha$, $c \in \mathbf{k}^*$. By symmetry $\alpha\beta_1 = c\alpha\beta_2$ as well. As $\beta_1 - c\beta_2 \notin \mathrm{sc}(A)$ (as a combination of non-closed paths), there

exists $\alpha_2 \in Q_1$ with $\beta_1\alpha_2 - c\beta_2\alpha_2 \neq 0$. Then by stably biserial condition $\beta_i\alpha_2 \in \operatorname{soc}(A)$ for some i, and hence $\alpha_2\beta_i \in \operatorname{soc}(A)$ for the same i. If $\beta_i\alpha_2 = 0$, then $\alpha_2\beta_i = 0$ and we can set $\pi(\beta_i) = \alpha$, $\pi(\alpha) = \beta_i$, $\pi(\beta_{3-i}) = \alpha_2$, $\pi(\alpha_2) = \beta_{3-i}$ and $\beta_{3-i}\alpha_2 \neq 0, \alpha_2\beta_{3-i} \neq 0$. If $\beta_i\alpha_2 \neq 0$ but $\beta_{3-i}\alpha_2 \notin \operatorname{soc}(A)$, then $\alpha_2\beta_{3-i} \notin \operatorname{soc}(A)$ and we can set $\pi(\beta_i) = \alpha$, $\pi(\alpha) = \beta_i$, $\pi(\beta_{3-i}) = \alpha_2$, $\pi(\alpha) = \beta_{3-i}$ and $\beta_{3-i}\alpha_2 \neq 0, \alpha_2\beta_{3-i} \neq 0$. If $\beta_{3-i}\alpha_2 \in \operatorname{sc}(A)$, $\beta_i\alpha_2 \in \operatorname{sc}(A)$, then we can choose π arbitrary, e.g. $\pi(\beta_i) = \alpha$, $\pi(\alpha) = \beta_i$, $\pi(\beta_{3-i}) = \alpha_2$, $\pi(\alpha_2) = \beta_{3-i}$. The remaining case is when $\beta_{3-i}\alpha_2 = 0$, then $\alpha_2\beta_{3-i} = 0$ and we set $\pi(\beta_{3-i}) = \alpha$, $\pi(\alpha) = \beta_{3-i}$, $\pi(\beta_i) = \alpha_2$, $\pi(\alpha_2) = \beta_i$.

Case III: Let $\alpha \in Q_1$ be such that $\alpha\beta \neq 0$ for a unique arrow β and $\alpha\beta \in \operatorname{soc}(A)$. Consider $\gamma\beta$ for $\gamma \neq \alpha$, if $\gamma\beta = 0$, we can set $\pi(\alpha) = \beta$. If $\gamma\beta \neq 0$, then there exist a path p and $c \in \mathbf{k}^*$ such that $p\gamma\beta - c\alpha\beta = 0$, so there is β_2 such that $(p\gamma - c\alpha)\beta_2 \neq 0$. Since $\alpha\beta_2 = 0$ by assumption $p\gamma\beta_2 \neq 0$, so p is a path of length 0 and we can set $\pi(\alpha) = \beta$, $\pi(\gamma) = \beta_2$.

Now π is defined on all Q_1 and clearly it is injective ($\pi(x) \neq \pi(y)$ for $x \neq y$ by stably biserial condition if both x, y belong to case I, otherwise $\pi(x) \neq \pi(y)$ by construction). Then, indeed, π is a permutation and it has the following properties:

(1) $\alpha\pi(\alpha) \neq 0$;

(2) If $\beta \neq \pi(\alpha)$, then $\alpha\beta \in \operatorname{soc}(A)$.

For any $\alpha \in Q_1$ let $\langle\pi\rangle\alpha = (\alpha = \alpha_1, \alpha_2, \ldots, \alpha_{n_\alpha})$. We define α_i for all natural i by the condition $\alpha_{i+n_\alpha} = \alpha_i$ and find maximal integer k_α with $\alpha_1\alpha_2 \ldots \alpha_{k_\alpha} \neq 0$. Note that $k_\alpha > 1$ by (1), and therefore $\alpha_1\alpha_2 \ldots \alpha_{k_\alpha}\beta = 0$ for $\beta \neq \alpha_{k_\alpha+1}$ as well (by (2)), i.e. $p_\alpha = \alpha_1\alpha_2 \ldots \alpha_{k_\alpha} \in \operatorname{sc}(A)$. Actually $p_\alpha \in e_{s(\alpha)}Ae_{s(\alpha)}$ by symmetry. Let us define $sc(\alpha) = \alpha_1 \ldots \alpha_{k_\alpha}$.

Lemma 8.

1. *For each $\alpha \in Q_1$ we have $k_\alpha = n_\alpha m_\alpha$ for some integer m_α.*

2. *If $\alpha, \beta \in Q_1$ lie on the common cycle of π, then $k_\alpha = k_\beta$ (and $m_\alpha = m_\beta$).*

3. *If alpha, $\beta \in Q_1$ with $s(\alpha) = s(\beta)$, then $sc(\alpha) = c_{\alpha,\beta} \cdot sc(\beta)$ for some $c_{\alpha,\beta} \in \mathbf{k}^*$.*

We say that m_α is the *multiplicity of the cycle* $\langle\pi\rangle\alpha$.

Proof. Put $k = k_\alpha$.

1. Since $\alpha_1\alpha_2\alpha_3 \ldots \alpha_k \in \operatorname{sc}(A)$, we have $\alpha_2\alpha_3 \ldots \alpha_k\alpha_1 \neq 0$. If $k > 2$ then $\alpha_k\alpha_1 \notin \operatorname{soc}(A)$. Therefore, by (2), $\alpha_{k+1} = \alpha_1$ as required. If $k = 2$, i.e. $\alpha_1\alpha_2 \in \operatorname{sc}(A)$, then α belongs to Case II or to Case III and we have $n_\alpha = 2 = k_\alpha$.

2. This follows from 1 and from the fact that a socle path cannot be a subpath of another socle path.

3. It follows from the fact that $\mathrm{soc}(e_{s(\alpha)}A)$ is one-dimensional. $\qquad \square$

Let us call a non-zero path $\beta_1 \dots \beta_k$ *admissible* if $\pi(\beta_i) = \beta_{i+1}$ for all i. In particular, for any $v \in Q_0$ we have an admissible path $sc(\alpha) \in \mathrm{sc}(e_v A)$ with $s(\alpha) = e_v$. So it follows from (2) that any non-zero non-admissible path is of length 2 and is equal (in A) to an admissible socle path: $\beta\gamma = k \cdot sc(\alpha)$ for some $\alpha \in Q_1, k \in \mathbf{k}^*$. Such an equality we call a *socle relation*. Note that replacing in any socle relation right-hand side by 0 we obtain a standard description of SSB-algebra (up to coefficients in the relations of the form $sc(\alpha) = k \cdot sc(\beta)$, $k \in \mathbf{k}^*$ but these coefficients can be eliminated for symmetric algebras).

Lemma 9. *In the notations of the previous lemma, we can assume that $c_{\alpha,\beta} = 1$ for all $\alpha, \beta \in Q_1$ with $s(\alpha) = s(\beta)$ (i.e. $sc(\alpha) = sc(\beta)$).*

Proof. Let $\varphi_A(x) = \langle x, 1 \rangle$ be induced by the symmetric form $\langle -, - \rangle$ on A, put $c_\alpha = \varphi_A(sc(\alpha))$. As the form is symmetric, for α, β belonging to the same π-orbit $c_\alpha = c_\beta$, it follows that $c_{\alpha,\beta} = 1$ for such α, β. Now let $\{\alpha_1, \dots, \alpha_k\}$ be a set of representatives of π-orbits. Put $\alpha_i' = \frac{\alpha_i}{c_i}$, where $c_i^{m_{\alpha_i}} = c_{\alpha_i}$. Then, replacing α_i by α_i', $1 \le i \le k$, for any new socle path $sc(\alpha)'$ we obtain $\varphi_A(sc(\alpha)') = \varphi_A(sc(\alpha))/c_i^{m_{\alpha_i}} = 1$, where i is defined by $\alpha_i \in \langle \pi \rangle \alpha$. Therefore, we obtain that if $p_1 = kp_2$ for socle paths and $k \ne 0$, then $k = 1$ as required. Clearly we have not changed any relations except for, possibly, changing non-zero coefficients in socle relations. $\qquad \square$

Lemma 10. *Let $A = \mathbf{k}Q/I$ be a stably biserial algebra with permutation π, multiplicities m and ideal I generated by the following relations:*

(1) $sc(\alpha) - sc(\beta)$ *for each* (α, β) *with* $s(\alpha) = s(\beta)$.

(2) $sc(\alpha)\alpha$, $\pi^{-1}(\alpha)sc(\alpha)$ *for each vertex* $s(\alpha)$ *with one incoming and one outgoing arrow.*

(3) $\beta\gamma - l_{\beta,\gamma}sc(\beta)$ *for all* $\beta\gamma \in Q_1$, $\gamma \ne \pi(\beta)$ $(l_{\beta,\gamma} \in \mathbf{k})$.

Consider the ideal I_1 obtained from I by replacing generators of the form $\beta\gamma - l_{\beta,\gamma}sc(\beta)$ by $\beta\gamma$ for char $\mathbf{k} \ne 2$. *If* char $\mathbf{k} = 2$ *we make this replacement only in the cases with* $\beta \ne \gamma$. *Then* $\mathbf{k}Q/I_1 \simeq A$.

Proof. We are going to prove this lemma by induction on the number of non-zero $l_{\beta,\gamma}$. Suppose that $l_{\beta_0,\gamma_0} \ne 0$. Put $sc(\beta_0) = \beta_0 p$. Then we have $\beta_0(\gamma_0 - l_{\beta_0,\gamma_0}p) = 0$. Let us consider two cases:

1. Suppose that $\beta_0 \neq \gamma_0$. Let us show that the substitution $\gamma_0 \to \gamma_1$, $\gamma_1 = \gamma_0 - l_{\beta_0,\gamma_0} p$ decreases the number of non-zero $l_{\beta,\gamma}$ (preserving all other relations).

Looking at the values of φ_A we get

$$\varphi_A(\gamma_0 \beta_0) = \varphi_A(\beta_0 \gamma_0) = \varphi_A(l_{\beta_0,\gamma_0}\beta_0 p) = \varphi_A(l_{\beta_0,\gamma_0}p\beta_0) \neq 0.$$

Let us consider two cases.

Case I. $\pi(\gamma_0) \neq \beta_0$. Then $\gamma_0 \beta_0 \in sc(A)$, this implies that $\gamma_0 \beta_0 = l_{\beta_0,\gamma_0}p\beta_0$. So in this case we have $\beta_0 \gamma_1 = 0$ and also $\gamma_1 \beta_0 = 0$.

If $\pi^{-1}(\gamma_0)p = p\pi(\gamma_0) = 0$, then the substitution $\gamma_0 \to \gamma_1$ clearly does not change any other relations and we are done.

If $\pi^{-1}(\gamma_0)p \neq 0$ or $p\pi(\gamma_0) \neq 0$ then p is an arrow with $s(p) = s(\gamma_0)$, $e(p) = e(\gamma_0)$ and $\pi^{-1}(\gamma_0)$ is an arrow with $s(\pi^{-1}(\gamma_0)) = s(\beta_0)$, $e(\pi^{-1}(\gamma_0)) = e(\beta_0)$ (as $\pi^{-1}(\gamma_0)p \in soc(A)$) and we have $|Q_0| = 2$ or $|Q_0| = 1$. If $|Q_0| = 2$, then clearly, $\pi^{-1}(\gamma_0)p \neq 0$ implies $p\pi(\gamma_0) \neq 0$ and visa versa. Then the substitution of γ_0 for γ_1 does not create any new non-zero $l_{\beta,\gamma}$. If $|Q_0| = 1$ and Q has two loops α, β, with $\pi(\alpha) = \alpha, \pi(\beta) = \beta$, and say α plays the role of γ_0, then $\alpha' = \alpha - l_{\alpha,\beta}p$ satisfies the desired relations. A coefficient can appear in the relation $sc(\alpha) = c \cdot sc(\beta)$, but we can make it equal to 1 as before. Thus, in this case we have changed exactly two relations, obtaining $l_{\beta_0,\gamma_1} = l_{\gamma_1,\beta_0} = 0$.

Case II. $\pi(\gamma_0) = \beta_0$. Then we have $\gamma_0 \beta_0 \notin sc(A)$ (else we have $\pi(\beta_0) = \gamma_0$ as well). Then $\gamma_1 \beta_0 = \gamma_0 \beta_0 - l_{\beta_0,\gamma_0}p\beta_0$, with $l_{\beta_0,\gamma_0}p\beta_0 \in soc(A)$, and therefore any other path, containing $\gamma_1 \beta_0$ is equal to the corresponding path after the substitution $\gamma_1 \to \gamma_0$. Also we have $\pi^{-1}(\gamma_0)\gamma_1 = \pi^{-1}(\gamma_0)\gamma_0 - l_{\beta_0,\gamma_0}\pi^{-1}(\gamma_0)p = \pi^{-1}(\gamma_0)\gamma_0$, as $\pi^{-1}(\gamma_0)p$ is of length at least 3 and $p \neq \gamma_0 p'$ for any path p'. By the same reasons $\gamma_1 \delta = \gamma_0 \delta$ where $\delta \neq \beta, s(\delta) = s(\beta)$. Thus, in this case we have changed exactly one relation, obtaining $l_{\beta_0,\gamma_1} = 0$.

2. Suppose char$\mathbf{k} \neq 2$ and $\beta_0 = \gamma_0$, $|Q_0| \neq 1$. In this case $s(\beta_0) = e(\beta_0)$, p is a path of length more than 1 (else we have two loops at one vertex), $\beta_0 p = p\beta_0 \in sc(A)$. Put $\beta_0' = \beta_0 - l_{\beta_0,\gamma_0}p/2$. Then $(\beta_0')^2 = (\beta_0 - p/2)^2 = \beta_0^2 - l_{\beta_0,\gamma_0}\beta_0 p - l_{\beta_0,\gamma_0}p\beta_0 + 0 = 0$. As $\alpha p = p\alpha = 0$ for all arrows $\alpha \neq \beta_0$ (p is not an arrow), all other relations are preserved.

If $|Q_0| = 1$ and p is a path of length more than 1, the proof goes similar. If p is a path of length 1, by construction of π we have $p^2 = 0$ and lemma also holds. □

By Lemma 10 and induction on the number of non-zero $l_{\beta,\gamma}$ we get the following theorem:

Theorem 3. 1. *Any symmetric stably biserial algebra over an algebraically closed field* $\mathbf{k}$ *with* char$\mathbf{k} \neq 2$ *is isomorphic to a special biserial algebra.*

2. *Consider a standard description of a symmetric special biserial algebra* $A = \mathbf{k}Q/I$ *and any set of loops* $\{\alpha_1, \ldots, \alpha_k\}$ *in* Q_1, *where* $\pi(\alpha_i) \neq \alpha_i$ *for all* i *(so that* $\alpha_i^2 = 0$ *in* A*), consider a set* $\{c_{\alpha_1}, \ldots, c_{\alpha_k}\}$, $c_{\alpha_i} \in \mathbf{k}^*$. *Replacing in the standard set of relations* α_i^2 *by* $\alpha_i^2 - c_{\alpha_i} sc(\alpha_i)$ *we obtain a new algebra* A' *and all stably biserial algebras can be obtained in this way.*

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