Записки научных
семинаров ПОМИ
Том 459, 2015 г.

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ON THE LOCAL SMOOTHNESS OF SOME CLASS OF AXI–SYMMETRIC SOLUTIONS TO THE MHD EQUATIONS

Abstract. In this paper we consider a special class of weak axi-symmetric solutions to the MHD equations for which the velocity field has only poloidal component and the magnetic field is toroidal. We prove local regularity for such solutions. The global strong solvability of the initial-boundary value problem for the corresponding system in a cylindrical domain with non-slip boundary conditions for the velocity on the cylindrical surface is established as well.

§1. Introduction and Main Results

Let $\Omega \subset \mathbb{R}^3$ be a cylindrical domain with the axis of symmetry x_3. In this paper we study regularity of some special class of weak axi-symmetric solutions to the equations of magnetohydrodynamics (MHD) in $Q_T := \Omega \times (0,T)$. The MHD system describes the dynamics of a conductive incompressible viscous fluid:

$$\left\{ \begin{array}{rcl} \partial_t u + \operatorname{rot} u \times u - \Delta u + \nabla\Big(p + \dfrac{|u|^2}{2}\Big) & = & \operatorname{rot} H \times H \\ \operatorname{div} u & = & 0 \\ \partial_t H - \Delta H & = & \operatorname{rot}(u \times H) \\ \operatorname{div} H & = & 0 \end{array} \right. \qquad \text{in} \quad Q_T. \tag{1.1}$$

Key words and phrases: magnetohydrodynamics, axially symmetric solutions, regularity.

The research leading to these results has received funding from the People Programme (Marie Curie Actions) of the European Union's Seventh Framework Programme FP7/2007-2013/ under REA grant agreement n- 319012 and from the Funds for International Co-operation under Polish Ministry of Science and Higher Education grant agreement n- 2853/7.PR/2013/2. The author is also supported by RFBR, grant 17-01-00099-a.

Here $u : Q_T \to \mathbb{R}^3$ is the velocity field, $p : Q_T \to \mathbb{R}$ is pressure and $H : Q_T \to \mathbb{R}^3$ is the magnetic field, and for any a, $b \in \mathbb{R}^3$ we denote by $a \times b$ its vector product in $\mathbb{R}^3$.

Let (x_1, x_2, x_3) be Cartesian and (r, φ, z) be cylindrical coordinates of the point $x \in \mathbb{R}^3$, i.e. $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$, $x_3 = z$. Denote the basis vectors of orthonormal cylindrical coordinate system by $\mathbf{e}_r$, $\mathbf{e}_\varphi$, $\mathbf{e}_z$. For every vector field $u : \Omega \to \mathbb{R}^3$, $u = u_r\mathbf{e}_r + u_\varphi\mathbf{e}_\varphi + u_z\mathbf{e}_z$ we denote by u^P and u^T its poloidal and toroidal components respectively:

$$u = u^P + u^T, \qquad u^P = u_r\mathbf{e}_r + u_z\mathbf{e}_z, \qquad u^T = u_\varphi\mathbf{e}_\varphi.$$

We say the scalar function is axi-symmetric if (been represented in cylindrical coordinates) it does not depend on φ. The vector field u is *axi-symmetric* if functions u_r, u_φ, u_z are axi-symmetric. We say the vector field u is poloidal if its toroidal component is identically zero and toroidal if it has no poloidal component.

In this paper we are interested in those axi-symmetric solutions to the system (1.1) which possess some additional symmetry. To analyze the possible symmetries of solutions to the MHD equations we project the equations onto the subspaces of toroidal and poloidal vector fields. Note that in the space of axi-symmetric vector fields the toroidal and poloidal subspaces are invariant under the Laplacian. Hence from (1.1) we obtain that toroidal and poloidal components of the velocity and magnetic fields in the axi-symmetric case satisfy the following equations:

$$\begin{aligned} \partial_t u^T - \Delta u^T + \operatorname{rot} u^T \times u^P &= \operatorname{rot} H^T \times H^P \qquad (1.2)\\ \partial_t u^P - \Delta u^P + \operatorname{rot} u^T \times u^T + \operatorname{rot} u^P \times u^P + \nabla\widetilde{p} & \\ = \operatorname{rot} H^P \times H^P + \operatorname{rot} H^T \times H^T & \\ \partial_t H^T - \Delta H^T &= \operatorname{rot}(u^T \times H^P) + \operatorname{rot}(u^P \times H^T) \\ \partial_t H^P - \Delta H^P &= \operatorname{rot}(u^P \times H^P) \qquad (1.3)\\ \operatorname{div} u^P = 0, \qquad \operatorname{div} H^P &= 0 \end{aligned}$$

As the equation (1.3) is linear with respect to H^P, we can expect that for the apropriate initial boundary-value problem the poloidal part of magnetic field is zero if it vanishes at the initial moment of time. On the other hand, from (1.2) we see that the toroidal component of the velocity is governed by the external force $\operatorname{rot} H^T \times H^P$ and hence v^T is identically zero if H^P is absent and v^T vanishes at the initial moment of time. So, we can expect that for the MHD system the following symmetry of solutions is preserved under the evolution:

$$u^T(\cdot,0)=0,\ H^P(\cdot,0)=0 \quad \Longrightarrow \quad u^T(\cdot,t)=0,\ H^P(\cdot,t)=0, \quad \forall\, t>0$$

In this paper we study the class of weak axi-symmetric solutions to the MHD equations for which the velocity field has only poloidal component and the magnetic field is toroidal:

$$u(x,t)=u^P(x,t), \quad H(x,t)=H^T(x,t), \quad \forall\, t>0, \quad x\in\Omega. \tag{1.4}$$

In this case we have

$$u(x,t)=u_r(r,z,t)\mathbf{e}_r+u_z(r,z,t)\mathbf{e}_z, \quad H(x,t)=H_\varphi(r,z,t)\mathbf{e}_\varphi \tag{1.5}$$

and the system (1.1) in cylindrical coordinates reduces to the equations for evolution of the poloidal part of the velocity field coupled by the equation which is linear with respect to the scalar function H_φ

$$\begin{gathered}
\partial_t u_r+(u_r u_{r,r}+u_z u_{r,z})-\Big(\Delta_{r,z}u_r-\frac{u_r}{r^2}\Big)+\Big(p+\frac{H_\varphi^2}{2}\Big)_{,r}=-\frac{H_\varphi^2}{r}\\
\partial_t u_z+(u_r u_{z,r}+u_z u_{z,z})-\Delta_{r,z}u_z+\Big(p+\frac{H_\varphi^2}{2}\Big)_{,z}=0\\
u_{r,r}+u_{z,z}+\frac{u_r}{r}=0\\
\partial_t H_\varphi+(u_r H_{\varphi,r}+u_z H_{\varphi,z})-\Big(\Delta_{r,z}H_\varphi-\frac{H_\varphi}{r^2}\Big)=\frac{u_r H_\varphi}{r}
\end{gathered} \tag{1.6}$$

Here by $\Delta_{r,z}$ we denote the Laplacian of an axi-symmetric scalar function with respect to cylindrical coordinates:

$$\Delta_{r,z}\psi=\psi_{,rr}+\psi_{,zz}+\frac{\psi_{,r}}{r}.$$

Our main interest to the system (1.6) is due to its formal similarity to the system describing the axi-symmetric solutions to the Navier-Stokes equations with swirl. The latter consists of the equations for evolution of the poloidal part of the velocity field coupled by the equation which is linear with respect to the angular component of the velocity field u_φ:

$$\begin{gathered}
\partial_t u_r \ + \ (u_r u_{r,r} + u_z u_{r,z}) - \Big(\Delta_{r,z} u_r - \frac{u_r}{r^2}\Big) + p_{,r} = \frac{u_\varphi^2}{r} \\
\partial_t u_z + (u_r u_{z,r} + u_z u_{z,z}) - \Delta_{r,z} u_z + p_{,z} = 0 \\
u_{r,r} + u_{z,z} + \frac{u_r}{r} = 0 \\
\partial_t u_\varphi + (u_r u_{\varphi,r} + u_z u_{\varphi,z}) - \Big(\Delta_{r,z} u_\varphi - \frac{u_\varphi}{r^2}\Big) = -\frac{u_r u_\varphi}{r}
\end{gathered} \tag{1.7}$$

As we see, after the replacement $H_\varphi \leftrightarrow u_\varphi$ the both systems are almost identical with the only difference in the signs of the right-hand sides in (1.6) and (1.7). Although the system (1.7) is widely studied, its global strong solvability remains open for now. To the contrast, for the system (1.6) we are able to prove the global existence of smooth solutions. Roughly speaking the reason for it is following: it is well-known that for the system (1.7) the quantity ru_φ is governed by the equation

$$\partial_t (ru_\varphi) + u_r (ru_\varphi)_{,r} + u_z (ru_\varphi)_{,z} - \Delta_{r,z}(ru_\varphi) + \frac{2}{r}(ru_\varphi)_{,r} = 0$$

This relation provides some extra control for the quantity ru_φ. The different sign in the equations (1.6) results in the fact that in the case of (1.6) the corresponding quantity that "moves with the flow" is not rH_φ but $r^{-1}H_\varphi$:

$$\partial_t \Big(\frac{H_\varphi}{r}\Big) + u_r \Big(\frac{H_\varphi}{r}\Big)_{,r} + u_z \Big(\frac{H_\varphi}{r}\Big)_{,z} - \Delta_{r,z}\Big(\frac{H_\varphi}{r}\Big) - \frac{2}{r}\Big(\frac{H_\varphi}{r}\Big)_{,r} = 0$$

This relation provides some extra control for the quantity $r^{-1}H_\varphi$ which implies that in the case (1.6) the function H_φ is much more regular near the axis of the symmetry than the function u_φ in the case of (1.7). Essentially this is the reason why the global strong solvability for the system (1.6) turns out to be much easier than the analogous result for the system (1.7).

The global smooth solvability of the Cauchy problem for the system (1.6) was proved in [9]. The method of [9] was based on the ideas described above and it allowed the author to obtain the result even in the case where the magnetic diffusion is ignored.

In this paper we are interested in the study of the initial boundary value problem for the MHD system. Hence we need to supply our system with some boundary conditions. One of the common technical tools in the study of axi-symmetric solutions to the equations of hydrodynamics is the transfer of the equations for the velocity field to the vorticity form. That is why many authors consider slip boundary conditions for the velocity, see, for example, [12, 20, 21]. Indeed, conditions of such type allow one to introduce the flux function for the velocity field or to use vorticity itself as a test function for the equations and hence to deal with the initial-boundary value problem roughly speaking in the same way as it was done for the Cauchy problem.

In this paper we consider non-slip boundary conditions for the velocity field. In this case the direct transfer to the vorticity equations fails as vorticity "looses" information about boundary data. So, we use instead the method based on the local regularity theory developed in [15], see also [5]. As in the case of non-slip boundary conditions the boundary regularity of axi-symmetric solutions without swirl near points of intersection of the axis of symmetry with the boundary of the domain is an open issue even for the Navier-Stokes equations (see, for example, [3]), to avoid this difficulty we impose conditions of periodicity in x_3-direction. We assume that $\Omega = S\times(-L,L)$, where $S = \{\ (x_1,x_2)\in\mathbb{R}^2 : x_1^2+x_2^2 < a^2\ \}$ and we impose the following boundary conditions for the velocity field for all $t\in(0,T)$:

$$u|_{\partial S\times(-L,L)} = 0, \qquad u|_{S\times\{x_3=-L\}} = u|_{S\times\{x_3=L\}} \tag{1.8}$$

For the magnetic field we also consider conditions of periodicity in x_3 direction and on the cylindrical surface of Ω we can take, for example, conditions of ideal conductor:

$$H_\nu|_{\partial S\times(-L,L)} = 0,\ (\operatorname{rot} H)_\tau|_{\partial S\times(-L,L)} = 0,\ H|_{S\times\{x_3=-L\}} = H|_{S\times\{x_3=L\}}. \tag{1.9}$$

Here we denote by ν the external normal to $\partial\Omega$ and $H_\nu := H\cdot\nu$, $H_\tau := H-\nu H_\nu$. Note that for a toroidal vector field $H = H_\varphi \mathbf{e}_\varphi$ these conditions reduce to

$$\Big(H_{\varphi,r}+\frac{H_\varphi}{a}\Big)\Big|_{\partial S\times(-L,L)} = 0, \qquad H_\varphi|_{S\times\{x_3=-L\}} = H_\varphi|_{S\times\{x_3=L\}}.$$

Finally, we supply the system (1.1) by the initial conditions

$$u|_{t=0} = u_0, \quad u|_{t=0} = H_0, \tag{1.10}$$

where u_0 and H_0 are the divergence-free vector fields satisfying (1.8), (1.9) in the appropriate sense. Namely, for u_0, $H_0\in W^1_2(\Omega)$ we assume conditions $u_0|_{\partial S\times(-L,L)} = 0$, $(H_0\cdot\nu)|_{\partial S\times(-L,L)} = 0$ on the cylindrical surface of Ω and periodicity conditions on the top and the bottom of Ω hold in the sense of traces while the condition on the tangential part of $\operatorname{rot} H_0$ on the cylindrical surface is omitted.

Our first main result is concerned with the existence of global strong solutions to the initial boundary value problem (1.1), (1.8)–(1.10) (see the definition of strong solutions in Section 2):

Theorem 1.1. *Let u_0, $H_0 \in W^1_2(\Omega)$ be axi-symmetric divergence-free initial data satisfying the structure conditions* (1.4) *and the boundary conditions* (1.8), (1.9) (*in the sense described above*). *Then for any $T > 0$ there is a strong solution u, H, p to the initial boundary value problem* (1.1), (1.8)–(1.10) *in Q_T such that for any moment of time* (1.4) *is satisfied. Moreover, this solution is unique in the class of all weak Leray–Hopf type solutions to the problem* (*see the definition of the Leray-Hopf-type solutions in Section* 2).

As we mentioned above, in the case of non-slip boundary conditions for the velocity we can not pass directly to the vorticity equations for u and use the method developed in [9] to get the result. So, we employ a different approach based on partial regularity of suitable weak solutions to the MHD system with further local analysis of the regularity of suitable weak solutions satisfying (1.4) near the axis of symmetry.

The theory of partial regularity for the MHD equations was developed in [2] in the internal case, in [18] near the plane part of the boundary (under boundary conditions (1.8), (1.9)) and in [19] in the case of a curved boundary (under the same boundary conditions). This theory guarantees that for a suitable weak solution to the system (1.1) both u and H are Hölder continuous (up to the boundary) everywhere except for a closed set $\Sigma \subset \bar{\Omega} \times (0, T]$ (called *a singular set*) whose one-dimensional parabolic Hausdorff measure is zero:

$$\mathcal{P}^1(\Sigma) = 0.$$

For an axi-symmetric suitable weak solutions this implies that singularity can occur only on the axis of symmetry.

The idea of the local analysis of regularity of axi-symmetric solutions near the axis of symmetry we employ is borrowed from [15], see also [3]. This idea includes the reduction of our problem to the problem of the first-time-singularity, i.e. to the problem in the canonical domain in which solution is smooth up to the last moment of time. Then we transfer our equation for the magnetic field H_φ to the auxiliary equation for $\frac{H_\varphi}{r}$ and following to [15] interpret the obtained relation as the heat equation with a drift term in 5-dimensional space. This allows us to apply the maximum principle and obtain the estimate of the maximum of $\frac{H_\varphi}{r}$. Then we go back to the equations for the velocity, transfer them to the scalar equation for the angular component of vorticity ω_φ and applying the same idea obtain the estimate of the maximum of $\frac{\omega_\varphi}{r}$. With these two estimates the regularity of u and u at the initial moment of time easily follows from the ε-regularity theory for the MHD equations, see, for example, [4].

So, our second main result concerning the local regularity of axi-symmetric suitable weak solutions (to be defined in Section 2) to the MHD equations under the symmetry conditions (1.4) can be formulated as follows:

Theorem 1.2. *Denote $\mathcal{C}_R := \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 < R^2, \ |x_3| < R\}$, $\mathcal{Q}_R := \mathcal{C}_R \times (-R^2, 0)$. Let u, H and p be a suitable weak solution to the MHD system in $\mathcal{Q}_1$. Assume additionally functions u, H and p are axially symmetric, the vector field u is poloidal and the vector field H is toroidal, i.e.* (1.5) *holds. Then u, $H \in C^{\alpha,\frac{\alpha}{2}}(\bar{\mathcal{Q}}_\rho)$ with any $\rho \in (0,1)$ and any $\alpha \in (0, \frac{2}{3})$. Here we denote by $C^{\alpha,\frac{\alpha}{2}}(\bar{\mathcal{Q}}_\rho)$ the set of functions which are Hölder continuous on $\bar{\mathcal{Q}}_\rho$ with the exponent α with respect to the parabolic metric.*

Finally we would like to remark that as our approach is based on the partial regularity theory for the MHD equations with our method we can not ignore the magnetic diffusivity (i.e. the Laplacian term) in the magnetic equation in (1.1) and hence to prove a complete analogue of the result obtained in [9] for the Cauchy problem.

Our paper is organized as follows: in Section 2 we present the definitions of strong, Leray-Hopf-type and suitabe weak solutions to the MHD system and recall some known properties of them. In Section 3 we consider the model local problem for the MHD system "until the first singularity". In Section 4 we present the proofs of our main results (Theorems 1.1 and 1.2).

We use the following notation. For any a, $b \in \mathbb{R}^n$ we denote by $a \cdot b$ its scalar product in $\mathbb{R}^n$. For any $q \in [1, +\infty)$ we denote by $L_q(\Omega)$ and $W^k_q(\Omega)$ the usual Lebesgue and Sobolev spaces. The space $L_q(\Omega; d\mu)$ is the Lebesgue space with respect to Borel measure μ on Ω. We do not distinguish between spaces of scalar functions and vector fields in the notation. The space of measurable functions whose values are essentially bounded in Ω is denoted by $L_\infty(\Omega)$. We denote by $C^\infty(\bar\Omega)$ the set of all infinitely smooth functions on $\bar\Omega$ and by $\mathcal{D}'(\Omega)$ the set of distributions on Ω.

In contrast to traditional setting, we replace the usual balls with cylinders (which is quite convenient in the case of axial symmetry) and denote

$\mathcal{C}_R(x^*) := \{\ x\in\mathbb{R}^3 : (x_1-x_1^*)^2+(x_1-x_1^*)^2<R^2,\ |x_3-x_3^*|<R\ \}$, $\mathcal{C}_R=\mathcal{C}_R(0)$

$\mathcal{Q}_R(z^*)=\mathcal{C}_R(x^*)\times(t^*-R^2,t^*),\quad z^*=(x^*,t^*),\quad \mathcal{Q}_R=\mathcal{Q}_R(0,0)$

We define some "parabolic" functional spaces as follows:

$$W^{1,0}_q(Q_T)\equiv L_q(0,T;W^1_q(\Omega))=\{u\in L_q(Q_T):\ \nabla u\in L_q(Q_T)\},$$
$$W^{2,1}_q(Q_T)=\{u\in W^{1,0}_q(Q_T):\ \nabla^2u,\ \partial_t u\in L_q(Q_T)\},$$
$$L_{q,\infty}(Q_T)=L_\infty(0,T;L_q(\Omega)).$$

§2. Preliminaries

We start with the definition of strong solutions:

Definition 2.1. *We call functions u, H, p a strong solution to the MHD system if*
$$u,H\in W^{2,1}_2(Q_T),\quad p\in L_2(Q_T),$$
and u, H, p satisfy (1.1) *a.e. in Q_T.*

Similar to the Navier-Stokes equations, it turns out that a strong solution to the MHD equations, if exists, is unique in the class of weak solutions which are analogues to the Leray-Hopf solutions in the Navier-Stokes theory. To be precise we need to define Leray-Hopf-type solutions to the MHD equations. Below we denote by $J_\nu(\Omega)$ the closure in $L_2(\Omega)$ of the set of all infinitely smooth divergence-free vector fields in Ω which vanish near the cylindrical surface of Ω and satisfy periodicity condition (with the period $2L$) in x_3-direction.

Definition 2.2. *Assume u_0, $H_0\in J_\nu(\Omega)$. We say the divergence-free vector fields u and H are a weak Leray–Hopf–type solution to the initial-boundary value problem* (1.1), (1.8)–(1.10), *if*
$$u,H\in L_{2,\infty}(Q_T)\cap W^{1,0}_2(Q_T),$$
u and H are weakly continuous in time as functions with values in $L_2(\Omega)$, satisfy the boundary conditions (1.8), (1.9)*, satisfy the initial conditions*

(1.10) *in sense of strong convergence in* $L_2(\Omega)$*, satisfy the equations* (1.1) *in the sense of distributions and also satisfy the global energy inequality*

$$\|u(t)\|^2_{L_2(\Omega)}+\|H(t)\|^2_{L_2(\Omega)}+2\int\limits_0^t\Big(\|\nabla u(\tau)\|^2_{L_2(\Omega)}+\|\operatorname{rot}H(\tau)\|^2_{L_2(\Omega)}\Big)\,d\tau \leqslant \|u_0\|^2_{L_2(\Omega)}+\|H_0\|^2_{L_2(\Omega)}$$

Note that while Dirichlet-type and periodical boundary conditions in Definition 2.2 are understood in the sense of traces, the Neuman-type boundary conditions can be taken into account by the proper choice of the class of test functions in the integral identities for u and H, see precise definition, for example, in [7] or [18]. So, we have the following theorem:

Theorem 2.1. *Assume*

$$u_0, H_0 \in W^1_2(\Omega) \text{ are divergence-free and satisfy } (1.8), (1.9). \tag{2.1}$$

If u*,* H *and* p *is a strong solution to the problem* (1.1)*,* (1.8)*–*(1.10) *in* Q_T *and* $\widetilde{u}$*,* $\widetilde{H}$ *is a Leray-Hopf-type solution in* Q_T *which corresponds to the same initial data* u_0 *and* H_0 *then* $u=\widetilde{u}$ *and* $H=\widetilde{H}$ *a.e. in* Q_T*.*

Though the global existence of strong solution for the MHD system is an open problem, for sufficiently smooth initial data we can always guarantee local existence of strong solutions:

Theorem 2.2. *Assume* (2.1) *holds. Then there exists* $T_0>0$ *depending on* $\|u_0\|_{W^1_2(\Omega)}$ *and* $\|H_0\|_{W^1_2(\Omega)}$ *such that there exists a strong solution* u*,* H *and* p *to the problem* (1.1)*,* (1.8)*–*(1.10) *in* $\Omega\times(0,T_0)$*.*

Strong solutions to the MHD equations are locally smooth. Namely, we need the following fact:

Theorem 2.3. *Assume* u*,* H *and* p *are a strong solution to* (1.1) *in* Q_T*. Then for any* $k\in\mathbb{N}$ *the inclusions* $\nabla^{k-1}u$*,* $\nabla^{k-1}H\in C^{\alpha,\frac{\alpha}{2}}_{\rm loc}(\Omega\times(0,T])$ *hold with any* $\alpha\in(0,1)$*.*

Theorems 2.1–2.3 are analogues to the known facts in the Navier-Stokes theory. Their proofs can be found, for example, in [7], or can be obtained by straightforward modifications of the corresponding results for the Navier-Stokes equations.

Now we turn to the investigation of the specific class of strong solutions to the MHD system. Namely, we consider axially symmetrical solutions to the initial-boundary value problem and show that for the axial symmetry is preserved during the evolution as long as the solution remains a strong one. Moreover, for axially symmetric strong solutions we also have preservation of the poloidal-toroidal structure of the MHD flow if the initial data possess the corresponding structure.

Theorem 2.4. *Let u_0, $H_0 \in W_2^1(\Omega)$ be divergence-free vector fields satisfying the boundary conditions* (1.8), (1.9). *Assume that for some $T > 0$ there exists a strong solution u, H, p to the problem* (1.1), (1.8)–(1.10) *in Q_T. Then*

1) *if the vector fields u_0 and H_0 are axially symmetric then for any $t \in (0,T)$ the functions $u(\cdot,t)$, $H(\cdot,t)$ and $p(\cdot,t)$ are also axially symmetric*;
2) *moreover, if u_0 and H_0 satisfy additionally the structure assumptions* (1.4) *then for any $t \in (0,T)$ the functions $u(\cdot,t)$ and $H(\cdot,t)$ also satisfy* (1.4).

The first part of Theorem 2.4 is a trivial consequence of the uniqueness theorem for the initial-boundary value problem, see Theorem 2.1. The second part follows from the linearity of the equations (1.2) and (1.3).

Now we turn to the discussion of so-called suitable weak solutions to the MHD system. We define them as follows:

Definition 2.3. *We say the functions u, H, p are a suitable weak solution to the system* (1.1) *if u, $H \in L_{2,\infty}(Q_T) \cap W_2^{1,0}(Q_T)$ are divergence free, $p \in L_{\frac{3}{2}}(Q_T)$, u, H, p satisfy equations* (1.1) *in the sense of distributions and the local energy inequality holds: for any $\zeta \in C_0^\infty(\mathbb{R}^3 \times (0,T])$ such that $\frac{\partial\zeta}{\partial\nu}\Big|_{\partial S\times(-L,L)} = 0$, $\zeta|_{S\times\{x_3=-L\}} = \zeta|_{S\times\{x_3=L\}}$ we have*

$$\sup_{t\in(0,T)} \int\limits_\Omega \zeta\Big(|u|^2 + |H|^2\Big)\, dx + 2\int\limits_0^T\int\limits_\Omega \zeta\Big(|\nabla u|^2 + |\operatorname{rot} H|^2\Big)\, dxdt$$
$$\leqslant \int\limits_0^T\int\limits_\Omega \Big(|u|^2+|H|^2\Big)(\partial_t\zeta + \Delta\zeta)\, dxdt + \int\limits_0^T\int\limits_\Omega \Big(|u|^2 + 2p\Big) u\cdot\nabla\zeta\, dxdt$$
$$+ 2\int\limits_0^T\int\limits_\Omega (H\otimes H) : \nabla^2\zeta\, dxdt + 2\int\limits_0^T\int\limits_\Omega (u\times H)(\nabla\zeta\times H)\, dxdt$$

For the initial-boundary value problem the existence of suitable weak solutions to the MHD system in Q_T corresponding to the initial data u_0, $H_0 \in J_\nu(\Omega)$ can be proved for arbitrary $T > 0$ by standard regularization method, see, for example, [18]. Namely, we have the following theorem:

Theorem 2.5. *Assume u_0, $H_0 \in J_\nu(\Omega)$ and $T > 0$ is arbitrary. Then there exist u, H and p such that functions u and H are a Leray-Hopf-type solution to the initial-boundary value problem* (1.1), (1.8)–(1.10) *in Q_T and simultaneously u, H, p are a suitable weak solution to the system* (1.1) *in Q_T.*

The important property of suitable weak solutions is so-called *partial regularity*. To describe it we define a *singular point* $z_0 = (x_0, t_0) \in \bar{\Omega} \times (0,T]$ of a suitable weak solution u, H and p of the system (1.1) in Q_T as a point such that the function $|u| + |H|$ is unbounded in any parabolic neighborhood of z_0 (i.e. in any set $\mathcal{Q}_R(z_0) \cap Q_T$ with arbitrary $R > 0$). Then we define a *singular set* $\Sigma \subset \bar{\Omega} \times (0,T]$ of the suitable weak solution u, H, p as a set of all its singular points. All other points of the set $(\bar{\Omega} \times (0,T]) \setminus \Sigma$ are called *regular points* of the suitable weak solution. Note that thanks to the assumption on pressure $p \in L_{\frac{3}{2}}(Q_T)$ every regular point z_0 of a suitable weak solution u, H, p has some parabolic neighborhood $\mathcal{Q}_R(z_0) \cap Q_T$ such that $u, H \in C^{\alpha, \frac{\alpha}{2}}(\overline{\mathcal{Q}_R(z_0) \cap Q_T})$ with any $\alpha \in (0, \frac{2}{3})$.

The main result on partial regularity of suitable weak solutions to the MHD system is the following theorem:

Theorem 2.6. *Assume u, H, p is a suitable weak solution to the system* (1.1) *in Q_T and denote by $\Sigma \subset \bar{\Omega} \times (0,T]$ the singular set of this solution. Then*

$$\mathcal{P}^1(\Sigma) \;=\; 0,$$

where $\mathcal{P}^1$ is the one-dimensional parabolic Hausdorff measure on $\mathbb{R}^3 \times \mathbb{R}$.

Internal partial regularity for the Navier-Stokes equations (which is a particular case of (1.1) if $H \equiv 0$) was established in the celebrated paper [1]. Boundary partial regularity for the Navier-Stokes equations with non-slip boundary conditions was established in [13, 14] in the case of flat boundary and in [16] in the case of curved boundary, see also [17]. The internal partial regularity for the MHD equations was proved in [2]. Boundary partial regularity for the MHD equations with boundary conditions (1.8), (1.9) was proved in [18] in the case of flat boundary and in [19] in the case of curved boundary. So, our Theorem 2.6 is a combination of results of [2, 18] and [19].

Note that the condition of axial symmetry of solutions impose significant restrictions on the structure of the singular set of a suitable weak solution to the MHD system. Namely, a suitable weak solution can not have singular points away from the axis of symmetry (because if a point which does not belong to the axis of symmetry is singular then due to axial symmetry it immediately generates a singular curve, but a singular curve is forbidden by Theorem 2.6). So, the following theorem holds:

Theorem 2.7. *Assume* u_0, $H_0 \in J_\nu(\Omega)$ *and let* u, H, p *be a suitable weak solution to the problem* (1.1), (1.8)–(1.10) *in* Q_T. *Denote by* $\Sigma \subset \bar\Omega \times (0,T]$ *the singular set of this solution. Then*

$$\Sigma \subset \left\{ \ (x,t) \in \Omega \times (0,T] : \ x_1 = x_2 = 0 \right\}.$$

§3. Model problem

In this section we investigate the model problem "until the first singularity". This means that we assume that a local solution can develop singularity only at the final moment of time. Remind we denote $\mathcal{C}_R = \{ \ x \in \mathbb{R}^3 : x_1^2 + x_2^2 < R^2, \ |x_3| < R \ \}$, $\mathcal{Q}_R = \mathcal{C}_R \times (-R^2, 0)$ and $\mathcal{C} = \mathcal{C}_1$, $\mathcal{Q} = \mathcal{Q}_1$. The following theorem is the main result of this section:

Theorem 3.1. *Assume* u, H, p *are an axially symmetric suitable weak solution to* (1.1) *in* $\mathcal{Q}$ *such that* u *is poloidal and* H *is toroidal in* $\mathcal{Q}$, *i.e.* (1.5) *holds. Assume that for some* $\alpha \in (0, \frac{2}{3})$ *the inclusions* u, $H \in C^{\alpha,\frac{\alpha}{2}}(\bar{\mathcal{C}} \times [-1, t'])$ *hold for any* $t' \in (-1, 0)$ *and besides* u, $H \in C^{\alpha,\frac{\alpha}{2}}(\bar{\mathcal{Q}} \setminus \mathcal{Q}_R)$ *for some* $R \in (0,1)$. *Then* u, $H \in C^{\alpha,\frac{\alpha}{2}}(\bar{\mathcal{Q}})$.

We split the proof of Theorem 3.1 into few steps. First we obtain some additional regularity of $\omega_\varphi := u_{r,z} - u_{z,r}$ and H_φ on the internal subsets of $\mathcal{Q}$.

Theorem 3.2. *Assume all conditions of Theorem 3.1 hold. Denote* $\omega := \operatorname{rot} u$, $\omega = \omega_\varphi \mathbf{e}_\varphi$. *Then* $\frac{\omega_\varphi}{r}$, $\frac{H_\varphi}{r} \in W^{1,0}_{\infty,\mathrm{loc}}(\mathcal{Q})$. *Moreover, for any* R_1, R_2, *such that* $R < R_1 < R_2 < 1$, *the functions* $\frac{\omega_\varphi}{r}$ *and* $\frac{H_\varphi}{r}$ *are continuous on the set* $\bar{\mathcal{Q}}_{R_2} \setminus \mathcal{Q}_{R_1}$ *and locally continuous in* $\mathcal{Q}$.

Proof of Theorem 3.2. Theorem 3.2 is the direct consequence of the usual regularity theory for parabolic and the Navier-Stokes equations. Indeed, as u and H are Hölder continuous on the sets $\bar{\mathcal{C}} \times [-1, t']$, $\forall\, t' \in (-1, 0)$ and on $\bar{\mathcal{Q}} \setminus \mathcal{Q}_R$ by Theorem 2.3 we obtain that for any $k \in \mathbb{N}$ the derivatives

$\nabla^k u$, $\nabla^k H$ are also locally Hölder continuous in $\mathcal{Q}$ and Hölder continuous on the set $\bar{\mathcal{Q}}_{R_2} \setminus \mathcal{Q}_{R_1}$ for any $R < R_1 < R_2 < 1$.

Denote $h := \frac{H_\varphi}{r}$, $\psi := \frac{\omega_\varphi}{r}$. Then in cylindrical coordinates we have

$$\begin{aligned}\nabla^2 H = {} & H_{\varphi,rr}\mathbf{e}_\varphi \otimes \mathbf{e}_r \otimes \mathbf{e}_r + H_{\varphi,rz}(\mathbf{e}_r \otimes \mathbf{e}_r \otimes \mathbf{e}_z + \mathbf{e}_\varphi \otimes \mathbf{e}_z \otimes \mathbf{e}_r) \\ & + H_{\varphi,zz}\mathbf{e}_\varphi \otimes \mathbf{e}_z \otimes \mathbf{e}_z + +h_{,r}(\mathbf{e}_\varphi \otimes \mathbf{e}_\varphi \otimes \mathbf{e}_\varphi - \mathbf{e}_r \otimes \mathbf{e}_r \otimes \mathbf{e}_\varphi \\ & - \mathbf{e}_r \otimes \mathbf{e}_\varphi \otimes \mathbf{e}_r) - h_{,z}(\mathbf{e}_r \otimes \mathbf{e}_\varphi \otimes \mathbf{e}_z + h_{,z}\mathbf{e}_r \otimes \mathbf{e}_z \otimes \mathbf{e}_\varphi)\end{aligned}$$

Hence we have pointwise double-sided estimate

$$|\nabla^2 H|^2 \asymp H^2_{\varphi,rr} + H^2_{\varphi,rz} + H^2_{\varphi,zz} + h^2_{,r} + h^2_{,z}$$

Similarly we have

$$|\nabla^2 \omega|^2 \ \asymp \ \omega^2_{\varphi,rr} + \omega^2_{\varphi,rz} + \omega^2_{\varphi,zz} + \psi^2_{,r} + \psi^2_{,z}$$

So, from $\nabla^2 H$, $\nabla^2\omega \in L_{\infty,\mathrm{loc}}(\mathcal{Q})$ we obtain ∇h, $\nabla\psi \in L_{\infty,\mathrm{loc}}(\mathcal{Q})$.

Next, as the functions $\omega_\varphi(0,z,t) = 0$ and $H_\varphi(0,z,t) = 0$ we can define $\psi(r,z,t)$ and $h(r,z,t)$ for $r = 0$ by the values $\omega_{\varphi,r}(0,z,t)$ and $H_{\varphi,r}(0,z,t)$ respectively. It is easy to see that the obtained functions are continuous on $\bar{\mathcal{Q}}_{R_2} \setminus \mathcal{Q}_{R_1}$ and locally continuous in $\mathcal{Q}$. Theorem 3.2 is proved. $\square$

Next get the estimate of the magnetic field:

Theorem 3.3. *Assume u, $H \in L_{2,\infty}(\mathcal{Q}) \cap W^{1,0}_2(\mathcal{Q})$ are axially symmetric divergence-free vector fields such that u is poloidal and H is toroidal in $\mathcal{Q}$, i.e.* (1.5) *holds. Let u and H satisfy the equations (in the sense of distributions)*

$$\partial_t H - \Delta H \ = \ \mathrm{rot}(u \times H) \quad in \quad \mathcal{Q}$$

Assume additionally that u is Hölder continuous in $\bar{\mathcal{C}} \times [-1, t']$ for any $t' < 0$ and besides $\frac{H_\varphi}{r} \in L_\infty(\partial'\mathcal{Q}_R)$ for some $R < 1$. Then $\frac{H_\varphi}{r} \in L_\infty(\mathcal{Q}_R)$ and the following estimate holds:

$$\operatorname*{esssup}_{\mathcal{Q}_R} \left|\frac{H_\varphi}{r}\right| \ \leqslant \ \operatorname*{esssup}_{\partial'\mathcal{Q}_R} \left|\frac{H_\varphi}{r}\right|. \tag{3.1}$$

Proof of Theorem 3.3. It turns out that $\frac{H_\varphi}{r}$ satisfies to the equation which can be reduced to the heat equation with a drift term in the 5-dimensional space. This idea is borrowed from [15], see also [3].

Denote $D := K\times(-1,0)$ where $K := \{\, (r,z)\in\mathbb{R}^2 : r\in(0,1),\ |z|<1 \,\}$. Then H_φ satisfies the identity

$$\int\limits_D \Big(- H_\varphi \partial_t \eta + H_{\varphi,r}\eta_{,r} + H_{\varphi,z}\eta_{,z} + \frac{H_\varphi \eta}{r^2} + u_r H_{\varphi,r}\eta$$

$$+ u_z H_{\varphi,z}\eta - \frac{u_r H_\varphi \eta}{r} \Big)\, r\,dr\,dz\,dt = 0$$

for any test function $\eta\in C_0^\infty(D)$. Define the function h by

$$h(r,z,t) := \frac{H_\varphi(r,z,t)}{r}.$$

Then h satisfies the integral identity

$$\int\limits_D \Big(-h\partial_t\eta + h_{,r}\eta_{,r} + h_{,z}\eta_{,z} - 2\,\frac{h_{,r}\eta}{r} + u_r h_{,r}\eta + u_z h_{,z}\eta \Big)\, r dr\,dz\,dt = 0 \quad (3.2)$$

for any $\eta\in C_0^\infty(D)$. In Cartesian coordinates this means that $h(x,t) = \frac{H_\varphi(x,t)}{|x'|}$ is a weak solution to the equation

$$\partial_t h - \Delta h + u\cdot\nabla h - 2\,\frac{x'}{|x'|^2}\cdot\nabla h \;=\; 0 \qquad \text{in} \quad \mathcal{D}'(\mathcal{Q}\setminus\Gamma),$$

where $x' := (x_1,x_2,0)^T$ and $\Gamma := \{(x,t)\in\mathcal{Q} :\ x' = 0\}$. Moreover, replacing in (3.2) the test function η by $r^2\eta$ we obtain that h and u also satisfy the integral identity

$$\int\limits_D \Big(- h\partial_t\eta + h_{,r}\eta_{,r} + h_{,z}\eta_{,z} + u_r h_{,r}\eta + u_z h_{,z}\eta \Big)\, r^3\,dr\,dz\,dt = \ 0,$$

$$\forall\, \eta\in C_0^\infty(D). \quad (3.3)$$

Denote

$$\mathcal{C}_5 := \big\{ y\in\mathbb{R}^5 :\ y_1^2+y_2^2+y_3^2+y_4^2<1,\ |y_5|<1 \big\}, \quad \mathcal{Q}_5 := \mathcal{C}_5\times(-1,0) \quad (3.4)$$

$$r = \sqrt{y_1^2+y_2^2+y_3^2+y_4^2}, \quad z = y_5, \quad (3.5)$$

$$b_\alpha(y,t) = u_r(r,z,t)\,\frac{y_\alpha}{r}, \quad \alpha = 1,2,3,4, \qquad b_5(y,t) = u_z(r,z,t). \quad (3.6)$$

Denote also

$$w(y,t) = h(r,z,t).$$

Then by Theorem 3.2 we have $w\in W^{1,0}_{\infty,\mathrm{loc}}(\mathcal{Q}_5)$, and $u\in C^{\alpha,\frac{\alpha}{2}}(\bar{\mathcal{C}}\times[-1,t'])$ implies $b\in L_\infty(\mathcal{C}_5\times(-1,t'))$ for any $t'\in(-1,0)$.

Making the change of variables in (3.3) and adding to the obtained identity the spherical part of the Laplacian in $(y_1, y_2, y_3, y_4) \in \mathbb{R}^4$ (which is zero due to the cylindrical symmetry of w) we arrive at the identity

$$\int\limits_{\mathcal{Q}_5} \Big(- w\partial_t \eta + \nabla w \cdot \nabla \eta + b \cdot \nabla w\ \eta \Big)\, dydt = 0, \quad \forall\ \eta \in C_0^\infty(\mathcal{Q}_5 \setminus \Gamma_5), \tag{3.7}$$

where $\Gamma_5 := \{\ (y,t) \in \mathcal{Q}_5 : y' = 0\ \}$ and $y' = (y_1, y_2, y_3, y_4, 0)^T$. As the set Γ_5 is removable in the 5-dimensional space we conclude that (3.7) remains true for any $\eta \in C_0^\infty(\mathcal{Q}_5)$. Hence $w \in W^{1,0}_{2,\mathrm{loc}}(\mathcal{Q}_5)$ is a weak solution to the equation

$$\partial_t w - \Delta w + b \cdot \nabla w \ = \ 0 \quad \text{in} \quad \mathcal{Q}_5.$$

Now we can apply the following version of the maximum principle (see, for example, [11], Corollary 3.5 and remarks at the end of Section 3).

Theorem 3.4. *Denote* $\mathcal{C}_n := \{y \in \mathbb{R}^n : \ |y'| < 1, \ |y_n| < 1\}$, $n \geqslant 3$, $\mathcal{Q}_n = \mathcal{C}_n \times (-1, 0)$ *and assume that* $w \in W_2^{1,0}(\mathcal{Q}_n)$, $b \in L_{s,l}(\mathcal{Q}_n)$, $\frac{n}{s} + \frac{2}{l} = 1$, $s > n$, *satisfy the integral identity*

$$\int\limits_{\mathcal{Q}_n} \Big(- w\partial_t \eta + \nabla w \cdot \nabla \eta + b \cdot \nabla w \eta \Big)\, dy\, dt = 0, \quad \forall\ \eta \in C_0^\infty(\mathcal{Q}_n).$$

Assume also

$$w|_{\partial' \mathcal{Q}_n} = \varphi, \qquad \textit{where} \qquad \varphi \in L_\infty(\partial' \mathcal{Q}_n)$$

(the boundary condition is understood in the sense of traces and the initial condition makes sense as under above assumptions on b *we have inclusions* $\partial_t w \in L_2(-1, 0; W_2^{-1}(\mathcal{C}_n))$ *and* $w \in C([-1, 0]; L_2(\mathcal{C}_n))$*). Then* $w \in L_\infty(\mathcal{Q}_n)$ *and* w *satisfies the estimate:*

$$\operatorname*{esssup}_{\mathcal{Q}_n} |w| \ \leqslant \ \operatorname*{esssup}_{\partial' \mathcal{Q}_n} |\varphi|.$$

Denote $\mathcal{C}'_R := \{\ y \in \mathbb{R}^5 : |y'| < R, \ |y_5| < R\ \}$ and $\mathcal{Q}'_R := \mathcal{C}'_R \times (-R^2, 0)$. Now we can apply Theorem 3.4 to our function w in the parabolic cylinder $\mathcal{C}'_R \times (-R^2, t')$ with some $t' \in (-1, 0)$. By assumption $w \in L_\infty(\partial' \mathcal{Q}'_R)$ and we obtain

$$\operatorname*{esssup}_{\mathcal{C}'_R \times (-1, t')} |w| \leqslant \operatorname*{esssup}_{\partial' \mathcal{Q}'_R} |w| \quad \Longrightarrow \quad \operatorname*{esssup}_{\mathcal{C}_R \times (-R^2, t')} \Big| \frac{H_\varphi}{r} \Big| \leqslant \operatorname*{esssup}_{\partial' \mathcal{Q}_R} \Big| \frac{H_\varphi}{r} \Big|.$$

As the right-hand side in this inequalities is independent on t' we conclude that $\frac{H_\varphi}{r}$ is bounded in $\mathcal{Q}_R$ and (3.1) holds. Theorem 3.3 is proved. □

Now we turn to the estimates for the vorticity of the velocity field.

Theorem 3.5. *Assume u, $H \in L_{2,\infty}(\mathcal{Q}) \cap W^{1,0}_2(\mathcal{Q})$ are axially symmetric divergence-free vector fields such that u is poloidal and H is toroidal in $\mathcal{Q}$, i.e.* (1.5) *holds. Denote $\omega = \operatorname{rot} u = \omega_\varphi \mathbf{e}_\varphi$, where $\omega_\varphi := u_{r,z} - u_{z,r}$. Assume u, ω and H satisfy the equation* (*in the sense of distributions*)

$$\partial_t \omega - \Delta \omega + \operatorname{rot}(\omega \times u) \;=\; \operatorname{rot}(\operatorname{rot} H \times H) \qquad in \quad Q. \tag{3.8}$$

Assume additionally that u is Hölder continuous in $\bar{\mathcal{C}} \times [-1, t']$ for any $t' \in (-1, 0)$ and besides $\frac{\omega_\varphi}{r} \in L_\infty(\partial' \mathcal{Q}_R)$ and $\frac{H_\varphi}{r} \in L_\infty(\mathcal{Q}_R)$ for some $R < 1$. Then $\frac{\omega_\varphi}{r} \in L_\infty(\mathcal{Q}_R)$ and the following estimate holds:

$$\operatorname*{esssup}_{\mathcal{Q}_R} \left| \frac{\omega_\varphi}{r} \right| \leqslant \operatorname*{esssup}_{\partial' \mathcal{Q}_R} \left| \frac{\omega_\varphi}{r} \right| + c \operatorname*{esssup}_{\mathcal{Q}_R} \left| \frac{H_\varphi}{r} \right|^2 \tag{3.9}$$

where c is some absolute constant.

Proof of Theorem 3.5. We explore the ideas from [11]. Define the function

$$\psi(r, z, t) \;:=\; \frac{\omega_\varphi(r, z, t)}{r}$$

and represent ψ in Cartesian coordinates. Then from Theorem 3.2 we obtain $\psi \in W^{1,0}_{\infty,\mathrm{loc}}(\mathcal{Q})$ and by assumption $\psi \in L_\infty(\partial' \mathcal{Q}_R)$. Moreover, the function ψ is smooth away from the set $\Gamma := \{\ (x, t) :\ |x'| = 0\ \}$ and satisfies the identity

$$\partial_t \psi - \Delta \psi + u \cdot \nabla \psi = 2\, \frac{x'}{|x'|^2} \cdot \nabla \psi - F \quad \text{a.e. in} \quad \mathcal{Q}, \tag{3.10}$$

where $x' := (x_1, x_2, 0)^T$ and F is the representation in Cartesian coordinates of the function

$$F := \Big(\frac{H_\varphi^2}{r^2} \Big)_{,z}.$$

Take arbitrary $\varepsilon \in (0, R)$, $t' \in (-R^2, 0)$ and denote

$$\widetilde{\mathcal{C}}_\varepsilon := \{\ x \in \mathcal{C}_R :\ \varepsilon < |x'| < R\ \}, \quad \widetilde{\mathcal{Q}}_\varepsilon := \widetilde{\mathcal{C}}_\varepsilon \times (-R^2, t'), \quad k_0 := \sup_{\partial' \mathcal{Q}_R} |\psi|$$

Taking $k > k_0$ we see that the function $(\psi - k)_+$ vanishes on the parabolic boundary of $\mathcal{Q}_R$. Here we denote $(\psi - k)_+ := \max\{\psi - k, 0\}$.

Multiplying (3.10) by $(\psi - k)_+$, integrating the result over $\widetilde{\mathcal{C}}_\varepsilon$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|(\psi-k)_+\|^2_{L_2(\widetilde{\mathcal{C}}_\varepsilon)} + \|\nabla(\psi-k)_+\|^2_{L_2(\widetilde{\mathcal{C}}_\varepsilon)}$$
$$= \int\limits_{\widetilde{\mathcal{C}}_\varepsilon} \Big(\frac{x'}{|x'|^2} - \frac{u}{2}\Big)\cdot\nabla|(\psi-k)_+|^2\,dx \;+\; \int\limits_{\widetilde{\mathcal{C}}_\varepsilon} F\frac{\partial}{\partial x_3}(\psi-k)_+\,dx.$$

Using the divergence-free condition $\operatorname{div} u = 0$ and the fact that the divergence of the vector field $\frac{x'}{|x'|^2}$ is sign-definite in $\mathcal{D}'(\mathcal{Q})$ (see [11, Appendix]), namely,

$$\operatorname{div}\frac{x'}{|x'|^2} \;=\; -2\pi\delta_\Gamma \;\leqslant\; 0 \quad \text{in} \quad \mathcal{D}'(\mathcal{Q}),$$

integrating by parts we obtain

$$\int\limits_{\widetilde{\mathcal{C}}_\varepsilon} \Big(\frac{x'}{|x'|^2} - \frac{u}{2}\Big)\cdot\nabla|(\psi-k)_+|^2\,dx \;=\; -\frac{1}{\varepsilon}\int\limits_{|x'|=\varepsilon} |(\psi-k)_+|^2\,ds_x.$$

Discarding the non-positive term in the right-hand side and passing to the limit as $\varepsilon \to 0$ we arrive at

$$\frac{1}{2}\frac{d}{dt}\|(\psi-k)_+\|^2_{L_2(\mathcal{C}_R)} + \|\nabla(\psi-k)_+\|^2_{L_2(\mathcal{C}_R)} \;\leqslant\; \int\limits_{\mathcal{C}_R} F\frac{\partial}{\partial x_3}(\psi-k)_+\,dx.$$

Applying the Hölder inequality and integrating over $t \in (-R^2, t')$ we obtain

$$\sup_{t\in(-R^2,t')}\|(\psi-k)_+\|^2_{L_2(\mathcal{C}_R)} + \|\nabla(\psi-k)_+\|^2_{L_2(\mathcal{C}_R\times(-R^2,t'))}$$
$$\leqslant C\int\limits_{A_k\cap\mathcal{Q}_R}|F|^2\,dx$$

where $A_k := \{\ (x,t)\in\mathcal{C}_R\times(-R^2,t')\ |\ \psi(x,t) > k\ \}$. From Theorem 3.3 we know that $F \in L_\infty(\mathcal{Q}_R)$ and hence

$$\sup_{t\in(-R^2,t')}\|(\psi-k)_+\|^2_{L_2(\mathcal{C}_R)} + \|\nabla(\psi-k)_+\|^2_{L_2(\mathcal{C}_R\times(-R^2,t'))}$$
$$\leqslant C\ \|F\|^2_{L_\infty(\mathcal{Q}_R)}\ |A_k|, \quad \forall\ k \geqslant k_0.$$

From [8, Theorem 6.1] we obtain

$$\operatorname*{esssup}_{\mathcal{C}_R\times(-R^2,t')}(\psi-k_0)_+ \leqslant C\ \|F\|_{L_\infty(\mathcal{Q}_R)}, \quad \forall\ t'\in(-R^2,0).$$

Replacing ψ by $-\psi$ and repeating the same arguments we obtain

$$\|\psi\|_{L_\infty(\mathcal{C}_R\times(-R^2,t'))} \leqslant k_0 + C\ \|F\|_{L_\infty(\mathcal{Q}_R)}, \quad \forall\ t' \in (-R^2,0).$$

As the right-hand side of the last inequality is independent on $t' \in (-R^2,0)$ we arrive at the desired result. Theorem 3.5 is proved. □

Estimates (3.3) and (3.9) imply that for any $x^* \in \mathcal{C}_R$ belonging to the axis of symmetry (i.e. such that $x_1^* = x_2^* = 0$) at the final moment of time $t = 0$ the following identities hold:

$$\limsup_{\rho\to 0} \frac{1}{\rho^2} \int\limits_{\mathcal{Q}_\rho(z^*)} |H|^3\ dxdt = 0, \quad \limsup_{\rho\to 0} \frac{1}{\rho} \int\limits_{\mathcal{Q}_\rho(z^*)} |\omega|^2\ dxdt = 0. \quad (3.11)$$

Here we denote $z^* := (x^*,0)$, $\mathcal{Q}_\rho(z^*) = \mathcal{C}_\rho(x^*)\times(-\rho^2,0)$. Now Theorem 3.1 is a direct consequence of the following ε–regularity condition for the MHD system which can be found, for example, in [4], see Theorem 1.2 (b) (we present even a weaker version of the result in [4] which is sufficient for our purpose):

Theorem 3.6. *Assume u, $H \in L_{2,\infty}(\mathcal{Q}) \cap W_2^{1,0}(\mathcal{Q})$ and $p \in L_{\frac{3}{2}}(\mathcal{Q})$ are a suitable weak solution to the system* (1.1) *in $\mathcal{Q}$ and denote $\omega = \operatorname{rot} u$. Assume that for some point $z^* \in \mathcal{C}\times(-1,0]$ the identities* (3.11) *hold. Then there exists a parabolic neighborhood $\mathcal{Q}_{\rho_*}(z^*)$ of the point z^* such that u and H are Hölder continuous in $\bar{\mathcal{Q}}_{\rho_*}(z^*)$ with any exponent $\alpha \in (0,\frac{2}{3})$.*

Theorem 3.1 is proved. □

§4. Proof of Theorems 1.2 and 1.1

We start this section with the proof of Theorem 1.2.

Proof of Theorem 1.2. We assume that u, H and p is a suitable weak solution satisfying all assumptions of Theorem 1.2. By contradiction, we assume that the singular set $\Sigma \subset \mathcal{C}\times(-1,0]$ of this solution is non-empty. Then we reduce our problem to the model problem "until the first singularity" investigated in the previous section and obtain a contradiction.

First we establish existence of a "cylindrical layer" in which our solution is smooth. Indeed, from Theorem 2.6 the existence of the radius $R \in (0,1)$ follows such that $\partial'\mathcal{Q}_R \cap \Sigma = \varnothing$ (otherwise we obtain $\mathcal{P}^1(\Sigma) > 0$ which contradicts to the partial regularity established in Theorem 2.6). As the set

Σ is relatively closed in $\bar{\Omega}\times(0,T]$ the existence of $0<R_1<R_2<1$ follows such that $\Sigma \cap \left(\bar{\mathcal{Q}}_{R_2}\setminus \mathcal{Q}_{R_1}\right)=\varnothing$. Moreover, without loss of generality we can assume that $\Sigma\cap\mathcal{Q}_{R_1}\neq\varnothing$.

Now we define *the first singular moment of time* t_0 of our solution in the cylinder $\mathcal{Q}_{R_1}$. For any $t\in(-R_1^2,0]$ we denote

$$\Sigma_t=\Big\{\ x\in\mathcal{C}_{R_1}:\ (x,t)\in\Sigma\ \Big\}$$

and define

$$t_0\ :=\ \inf\Big\{\ t\in(-R_1^2,0]:\ \Sigma_t\neq\varnothing\ \Big\}.$$

The set Σ_{t_0} is non-empty (it follows from closedness of the set Σ). So, we can take $x_0\in\mathcal{C}_{R_1}$ such that $(x_0,t_0)\in\Sigma$ (and hence the function $|u|+|H|$ is unbounded in any parabolic neighborhood of (x_0,t_0)). Then there exist r_1 and r_2, $0<r_1<r_2$, such that

$$\mathcal{C}_{r_2}(x_0)\Subset\mathcal{C}_{R_1}\qquad\text{and}\qquad\big(\bar{\mathcal{C}}_{r_2}(x_0)\setminus\mathcal{C}_{r_1}(x_0)\big)\cap\Sigma_{t_0}=\varnothing.$$

So, the functions u and H are Hölder continuous in $\bar{\mathcal{C}}_{r_2}(x_0)\times[t_0-r_2^2,t']$ with any $t'\in(t_0-r_2^2,t_0)$, as well as they are Hölder continuous in $\bar{\mathcal{Q}}_{r_2}(x_0,t_0)\setminus\mathcal{Q}_{r_1}(x_0,t_0)$, where we denote $\mathcal{Q}_\rho(x_0,t_0)=\mathcal{C}_\rho(x_0)\times(t_0-\rho^2,t_0)$.

Now for any $(x,t)\in\mathcal{Q}$ we define functions

$$\begin{aligned}\widetilde{u}(x,t)&:=r_2u(x_0+r_2x,t_0+r_2^2t),\\ \widetilde{H}(x,t)&:=r_2H(x_0+r_2x,t_0+r_2^2t),\\ \widetilde{p}(x,t)&:=r_2^2p(x_0+r_2x,t_0+r_2^2t).\end{aligned}$$

Then the functions $\widetilde{u}$, $\widetilde{H}$ and $\widetilde{p}$ satisfy all conditions of Theorem 3.1 (if we take $R:=\frac{r_1}{r_2}$). By Theorem 3.1 we obtain that $\widetilde{u}$ and $\widetilde{H}$ are Hölder continuous in $\bar{\mathcal{Q}}$ and hence the original functions u and H are Hölder continuous in the cylinder $\bar{\mathcal{Q}}_{r_2}(x_0,t_0)$. This contradicts to the assumption that the point (x_0,t_0) is singular. The obtained contradiction implies that $\Sigma=\varnothing$. Theorem 1.2 is proved. □

Now we turn to the proof of Theorem 1.1.

Proof of Theorem 1.1. From the local well-posedness of the MHD system (see [7]) we know that for any divergence-free initial data u_0 and H_0 satisfying (1.8), (1.9) there exists $T_0=T_0(u_0,H_0)$, $T_0>0$, such that the system (1.1), (1.8), (1.9), (1.10) has the unique strong solution u, H and p in $\Omega\times(0,T_0)$. Denote

$$T_* := \sup\Big\{\ T\geqslant T_0 :$$
$$\text{there exists a strong solution to(1.1), (1.8)–(1.10) in}\Omega\times(0,T)\Big\}.$$

Note that if the initial data u_0 and H_0 are axially symmetric and satisfy (1.4) then from Theorem 2.4 we obtain that for any $T<T_*$ the strong solution u, H, p in $\Omega\times(0,T)$ corresponding to u_0 and H_0 is also axially symmetric and satisfies (1.4).

Our goal is to show that $T_*=+\infty$. By contradiction, assume $T_*<+\infty$. Let $\widetilde u$, $\widetilde H$, $\widetilde p$ be any suitable weak solution to the problem (1.1), (1.8)–(1.10) in $\Omega\times(0,T_*)$. Then for any $T\in(0,T_*)$ the functions $\widetilde u$, $\widetilde H$ coincide with the strong solution u, H. Hence the functions $\widetilde u$, $\widetilde H$, $\widetilde p$ are axially symmetric and satisfy (1.4). Denote by Σ the singular set of the suitable weak solution $\widetilde u$, $\widetilde H$, $\widetilde p$ in $\bar\Omega\times[0,T_*]$. Then evidently $\Sigma\subset\bar\Omega\times\{t=T_*\}$. Moreover, Σ lays on the axis of symmetry $\{x\in\mathbb R^3 : x'=0\}$. Assume $z_0:=(x_0,T_*)\in\Sigma$. By definition of a singular point $|u|+|H|$ must be unbounded in any parabolic neighborhood of z_0. On the other hand, if we take $R>0$ such that $\mathcal Q_R(z_0)\subset\Omega\times(0,T_*)$ then all conditions of Theorem 1.2 are satisfied for a suitable weak solution $\widetilde u$, $\widetilde H$, $\widetilde p$ in the cylinder $\mathcal Q_R(z_0)$. Thus $|u|+|H|$ is bounded in some parabolic neighborhood of z_0. The obtained contradiction implies that $\Sigma=\varnothing$ and the strong solution u, H, p can be extended onto $\Omega\times(0,T_*+\varepsilon)$ for some $\varepsilon>0$, but this contradicts to the definition of T_*. Thus $T_*=+\infty$. Theorem 1.1 is proved. $\square$

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Поступило 13 августа 2017 г.