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# ON PROJECTORS TO SUBSPACES OF VECTOR VALUED FUNCTIONS SUBJECT TO CONDITIONS OF THE DIVERGENCE FREE TYPE

**ABSTRACT.** We study operators projecting a vector valued function  $v \in W^{1,2}(\Omega, \mathbb{R}^d)$  to subspaces formed by the condition that the divergence is orthogonal to a certain amount (finite or infinite) of test functions. The condition that divergence is equal to zero almost everywhere presents the first (narrowest) limit case while the integral condition of zero mean divergence generates the other (widest) case. Estimates of the distance between  $v$  and the respective projection  $P_S v$  on such a subspace are important for analysis of various mathematical models related to incompressible media problems (especially in the context of a posteriori error estimates, see [15–17]). We establish different forms of such estimates, which contain only local constants associated with the stability (LBB) inequalities for subdomains. The approach developed in the paper also yields two sided bounds of the inf–sup (LBB) constant.

## §1. INTRODUCTION

We study operators  $P_S v$  that project a function  $v \in V := W^{1,2}(\Omega, \mathbb{R}^d)$  to a subspace  $S$  containing solenoidal (divergence free) functions or functions subject to weaker (integral) conditions of the divergence free type. The function  $v$  additionally satisfies zero boundary conditions on the whole boundary  $\Gamma$  (in this case  $v \in W_0^{1,2}(\Omega, \mathbb{R}^d)$ ) or on a measurable part  $\Gamma_0 \subset \Gamma$  (then  $v \in W_{0,\Gamma_0}^{1,2}(\Omega, \mathbb{R}^d)$ ).

It is assumed that  $\Omega$  is a bounded and connected domain in  $\mathbb{R}^d$  ( $d \geq 2$ ) with Lipschitz continuous boundary and  $\|\cdot\|$  denotes the  $L^2$  – norm over  $\Omega$ . The projection is considered with respect to the norm equivalent to the norm of  $V$  and is defined by the relation

$$\|\nabla(v - P_S v)\| = \inf_{w \in S} \|\nabla(v - w)\|, \quad (1)$$

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where, in particular, we can set

$$\mathbb{S} = \mathbb{S}_0(\Omega, \mathbb{R}^d) := \{v \in W^{1,2}(\Omega, \mathbb{R}^d), \operatorname{div} v = 0, v = 0 \text{ on } \Gamma\}.$$

Henceforth, the quantity

$$d(v, \mathbb{S}) := \|\nabla(v - P_{\mathbb{S}}v)\|$$

is called the distance between  $v$  and the set  $\mathbb{S}$ .

In our analysis, the following result plays a principal role.

**Theorem 1** ([1, 9]). *For any  $f \in \widetilde{L}^2(\Omega) := L^2(\Omega) \perp \mathbb{R}$  there exists a vector valued function  $w_f \in V_0 := W_0^{1,2}(\Omega, \mathbb{R}^d)$  such that*

$$\operatorname{div} w_f = f \quad \text{and} \quad \|\nabla w_f\| \leq \mathbb{C}_\Omega \|f\|, \quad (2)$$

where  $\mathbb{C}_\Omega$  is a positive constant depending on  $\Omega$ .

This theorem is very important for the mathematical theory of viscous incompressible fluids (see, e.g., [8, 23]). In fact, it is equivalent to the following statement: for any  $v \in W_0^{1,2}(\Omega, \mathbb{R}^d)$

$$d(v, \mathbb{S}_0(\Omega, \mathbb{R}^d)) \leq \mathbb{C}_\Omega \|\operatorname{div} v\|. \quad (3)$$

Theorem 1 can be extended to  $L^q$  spaces for  $1 < q < +\infty$  (see [2, 13, 14]) and yields estimates of the distance analogous to (3) (see [19, 21]).

The estimate (3) also holds for vector fields vanishing only on  $\Gamma_0$  (in this case we assume that  $\operatorname{meas}_{\mathbb{R}^{d-1}} \Gamma_0 > 0$ ). Let

$$v \in W_{0,\Gamma_0}^{1,2}(\Omega, \mathbb{R}^d) := \{w \in W^{1,2}(\Omega, \mathbb{R}^d) \mid w = 0 \text{ on } \Gamma_0 \subset \Gamma\}$$

and additionally satisfy the condition  $\int_{\Gamma} v \cdot n \, ds = 0$ . Then (see [17])

$$d(v, \mathbb{S}_{0,\Gamma_0}(\Omega, \mathbb{R}^d)) \leq \mathbb{C}_\Omega \|\operatorname{div} v\|, \quad (4)$$

where  $\mathbb{S}_{0,\Gamma_0}(\Omega, \mathbb{R}^d)$  denotes the subset of  $W_{0,\Gamma_0}^{1,2}(\Omega, \mathbb{R}^d)$  containing the divergence free fields.

Estimates (1) and (4) show that the distance to the set of divergence free fields is easy to estimate from above provided that the constant  $\mathbb{C}_\Omega$  (or a suitable upper bound of it) is known. Getting quantitative estimates of  $\mathbb{C}_\Omega$  is a practically important and mathematically interesting problem. It has attracted a serious attention (see [4–7, 11, 22] and other publications cited therein). For plane domains, which are star shaped with respect to a ball, computable and rather efficient majorants of  $\mathbb{C}_\Omega$  has been recently obtained in [3]. However, in general, finding guaranteed majorants of  $\mathbb{C}_\Omega$  in the dimensions larger than  $d = 2$  is a very difficult problem. To the best

of the authors knowledge, the only one known result is presented in the paper [12], which is related to a special class of three dimensional domains.

Theorem 1 is often used in the form of the so-called the inf-sup (or LBB) condition

$$\inf_{\substack{\phi \in L^2(\Omega) \\ \{\phi\}_{\Omega=0, \phi \neq 0}}} \sup_{\substack{w \in V_0 \\ w \neq 0}} \frac{\int_{\Omega} \phi \operatorname{div} w \, dx}{\|\phi\| \|\nabla w\|} \geq c_{\Omega} > 0. \quad (5)$$

Here and later on,  $\{\phi\}_{\Omega} := \frac{1}{|\Omega|} \int_{\Omega} \phi \, dx$ . In fact, (5) can be represented in the form  $\|\phi\| \leq \mathbb{C}_{\Omega} \|\phi\|$  (see [10]), where

$$\|\phi\| := \sup_{w \in V_0} \frac{\int_{\Omega} \phi \operatorname{div} w \, dx}{\|\nabla w\|}. \quad (6)$$

It is not difficult to show that  $c_{\Omega} = (\mathbb{C}_{\Omega})^{-1}$ , so that getting majorants of  $\mathbb{C}_{\Omega}$  is equivalent to getting minorants of the LBB constant  $c_{\Omega}$  and vice versa.

A variational principle that could be useful in numerical evaluation of  $\mathbb{C}_{\Omega}$  has been recently derived in [20]. In computations, it will generate lower bounds for the constant  $\mathbb{C}_{\Omega}$  (or upper bounds for  $c_{\Omega}$ ).

This paper suggests a way to deduce computable majorants of the constant  $\mathbb{C}_{\Omega}$  (which are valid for  $d \geq 2$ ) and respective estimates for projectors. The estimates are derived in two steps. First, we consider projectors on a specially constructed “intermediate space”  $\mathbb{S}_{0,\Gamma_0}^{\Phi}$  (which is wider than  $\mathbb{S}_{0,\Gamma_0}$ ) and then find estimates of the distance between any  $v \in \mathbb{S}_{0,\Gamma_0}^{\Phi}$  and  $\mathbb{S}_{0,\Gamma_0}(\Omega, \mathbb{R}^d)$ . In short, the main idea can be described as follows.

Note that formally  $\mathbb{S}_{0,\Gamma_0}(\Omega, \mathbb{R}^d)$  can be defined as a subset of  $W_{0,\Gamma_0}^{1,2}(\Omega, \mathbb{R}^d)$  that contains the functions  $v$  satisfying the condition

$$\int_{\Omega} \phi \operatorname{div} v \, dx = 0 \quad (7)$$

for any  $\phi \in L^2(\Omega)$  (or any  $\phi$  in a set dense in  $L^2(\Omega)$ ). We introduce a collection of subspaces defined by weaker conditions that require (7) to hold for a certain amount of test functions  $\phi_i$  only. Let the set  $\Phi = \{\phi_i(x)\}_{i=1}^N$  contain bounded and linearly independent functions such that  $\|\phi_i\| = 1$ .

$\text{Lin}\Phi$  denotes the linear envelope based on these functions, and

$$\mathbb{S}_{0,\Gamma_0}^\Phi(\Omega, \mathbb{R}^d) := \left\{ w \in W_{0,\Gamma_0}^{1,2}(\Omega, \mathbb{R}^d) \mid \int_{\Omega} \phi_i \operatorname{div} w \, dx = 0 \, \forall i = 1, 2, \dots, N \right\}.$$

If  $\Gamma_0 = \Gamma$ , then there exists one function ( $\phi_0 = 1$ ) such that (7) holds for any  $w$ . Therefore, in this case, without a loss of generality we assume that the function in  $\Phi_0$  are orthogonal to  $\phi_0$ , (in other words  $\{\phi_i\}_{i=1}^N = 0$ , for any  $i = 1, 2, \dots, N$ ). The respective subspace is denoted by  $\mathbb{S}_{0,\Gamma}^\Phi(\Omega, \mathbb{R}^d)$ .

It is clear that

$$\mathbb{S}_{0,\Gamma_0}(\Omega, \mathbb{R}^d) \subset \mathbb{S}_{0,\Gamma_0}^\Phi(\Omega, \mathbb{R}^d) \subset W_{0,\Gamma_0}^{1,2}(\Omega, \mathbb{R}^d). \quad (8)$$

This fact is important for our analysis because estimates of the projection

$$\mathbb{S}_{0,\Gamma_0}^\Phi(\Omega, \mathbb{R}^d) \rightarrow \mathbb{S}_{0,\Gamma_0}(\Omega, \mathbb{R}^d)$$

are deduced relatively easy (see [19, 21] and Sect. 4.1 of this paper).

In Section 2, we study the projection

$$W_{0,\Gamma_0}^{1,2}(\Omega, \mathbb{R}^d) \rightarrow \mathbb{S}_{0,\Gamma_0}^\Phi(\Omega, \mathbb{R}^d)$$

and find  $d(v, \mathbb{S})$  for  $\mathbb{S} = \mathbb{S}_{0,\Gamma_0}^\Phi$ . Lemma 1 presents the respective result, which yields estimates of  $d(v, \mathbb{S}_{0,\Gamma_0}^\Phi)$  that does not contain "difficult" constants  $c_\Omega$  or  $C_\Omega$ . The main identity (9) contains a matrix  $\mathbb{A}$  formed by products of certain functions defined as exact solutions of auxiliary boundary value problems. In general, they can be found only approximately by suitable numerical procedures. This brings certain difficulties, which could be avoided if we use known a posteriori error estimation methods and explicitly estimate the respective errors. However, in Section 3 we suggest a simpler solution and show that  $d(v, \mathbb{S})$  for  $\mathbb{S} = \mathbb{S}_{0,\Gamma_0}^\Phi$  can be estimated from above by a modification of the method based on solutions of finite dimensional problems (which are indeed available). The respective estimate (30) contains the matrix  $\mathbb{A}_n$  formed by these solutions and an additional term that can be viewed as an interpolation error.

Section 4 contains the main result. It presents an upper bound of the quantity  $\|\nabla(v - P_s v)\|$  for the case  $\mathbb{S} = \mathbb{S}_{0,\Gamma_0}$ , which is derived by combining the results of Sections 2 and 4.1. In this way, the projection estimate is derived in accordance with (8) using the intermediate space  $\mathbb{S}_{0,\Gamma_0}^\Phi$ .

Finally in Section 5, we discuss relations between the above derived estimates (of the distance to sets of vector valued functions satisfying the

divergence free condition in a weak form) and inf-sup constants. Also, we show that they yield two sided estimates of the constant  $\mathbb{C}_\Omega$ .

## §2. DISTANCE TO THE SET $\mathbb{S}_{0,\Gamma_0}^\Phi(\Omega, \mathbb{R}^d)$

Our first goal is to find the distance between  $v \in W_{0,\Gamma_0}^{1,2}(\Omega, \mathbb{R}^d)$  and  $\mathbb{S}_{0,\Gamma_0}^\Phi(\Omega, \mathbb{R}^d)$ .

**Lemma 1.** *For any  $v \in W_{0,\Gamma_0}^{1,2}(\Omega, \mathbb{R}^d)$ ,*

$$\mathbf{d}^2(v, \mathbb{S}_{0,\Gamma_0}^\Phi) = \mathbb{A}^{-1} \mathbf{b}(\operatorname{div} v) \cdot \mathbf{b}(\operatorname{div} v), \quad (9)$$

where

$$\begin{aligned} \mathbb{A} &= \{a_{ij}\}_{i,j=1}^N, & a_{ij} &= \int_{\Omega} \nabla u^{(i)} : \nabla u^{(j)} dx, \\ \mathbf{b}(\operatorname{div} v) &= \{b_i\}_{i=1}^N, & b_i &= \int_{\Omega} \phi_i \operatorname{div} v dx, \end{aligned}$$

and  $u^{(i)}$  minimizes the functional

$$J_i(w) := \frac{1}{2} \|\nabla w\|^2 + \int_{\Omega} \phi_i \operatorname{div} w dx.$$

on the set  $W_{0,\Gamma_0}^{1,2}(\Omega, \mathbb{R}^d)$ .

**Proof.** It is not difficult to see that

$$\begin{aligned} & \frac{1}{2} \mathbf{d}^2(v, \mathbb{S}_{0,\Gamma_0}^\Phi) \\ &= \inf_{w \in W_{0,\Gamma_0}^{1,2}(\Omega, \mathbb{R}^d)} \sup_{\substack{\lambda_i \in \mathbb{R} \\ i=1,2,\dots,N}} \left\{ \frac{1}{2} \|\nabla(v-w)\|^2 + \sum_{i=1}^N \lambda_i \int_{\Omega} \phi_i \operatorname{div} w dx \right\} \\ &= \inf_{w \in W_{0,\Gamma_0}^{1,2}(\Omega, \mathbb{R}^d)} \sup_{\substack{\lambda_i \in \mathbb{R} \\ i=1,2,\dots,N}} \left\{ \frac{1}{2} \|\nabla w\|^2 + \sum_{i=1}^N \lambda_i \int_{\Omega} \phi_i \operatorname{div}(v-w) dx \right\} \\ &= \inf_{w \in W_{0,\Gamma_0}^{1,2}(\Omega, \mathbb{R}^d)} \sup_{\lambda_i \in \mathbb{R}^N} L(w, \lambda), \end{aligned} \quad (10)$$

where

$$L(w, \lambda) := \frac{1}{2} \|\nabla w\|^2 - \sum_{i=1}^N \lambda_i \int_{\Omega} \phi_i \operatorname{div} w dx + \lambda \cdot \mathbf{b}(\operatorname{div} v)$$

and  $\lambda = \{\lambda_1, \lambda_2, \dots, \lambda_N\}$ . Evidently,

$$\frac{1}{2}d^2(v, \mathbb{S}_{0,\Gamma_0}^\Phi) \geq \sup_{\lambda \in \mathbb{R}^N} \inf_{w \in W_{0,\Gamma_0}^{1,2}(\Omega, \mathbb{R}^d)} L(\lambda, w). \quad (11)$$

Consider an auxiliary variational Problem  $\mathcal{P}_\lambda$ . The problem is to find  $u_\lambda \in W_{0,\Gamma_0}^{1,2}(\Omega, \mathbb{R}^d)$  such that the functional  $L(w, \lambda)$  attains infimum

$$\inf \mathcal{P}_\lambda := \inf_{w \in W_{0,\Gamma_0}^{1,2}(\Omega, \mathbb{R}^d)} \left\{ \frac{1}{2} \|\nabla w\|^2 - \sum_{i=1}^N \lambda_i \int_{\Omega} \phi_i \operatorname{div} w \, dx \right\} + \lambda \cdot \mathbf{b}(\operatorname{div} v).$$

Due to well known results of convex analysis, there exists a unique minimizer  $u_\lambda$  of this problem, which satisfies the integral identity

$$\int_{\Omega} \nabla u_\lambda : \nabla w \, dx = \sum_{i=1}^N \lambda_i \int_{\Omega} \phi_i \operatorname{div} w \, dx \quad \forall w \in W_{0,\Gamma_0}^{1,2}(\Omega, \mathbb{R}^d)$$

and has the form  $u_\lambda = \sum_{i=1}^N \lambda_i u^{(i)}$ , where

$$\int_{\Omega} \nabla u^{(i)} : \nabla w \, dx = \int_{\Omega} \phi_i \operatorname{div} w \, dx \quad \forall w \in W_{0,\Gamma_0}^{1,2}(\Omega, \mathbb{R}^d). \quad (12)$$

Therefore, for  $i, j = 1, 2, \dots, N$  we have equivalent representations of the coefficients

$$a_{ij} = \int_{\Omega} \phi_i \operatorname{div} u^{(j)} \, dx, \quad b_i = \int_{\Omega} \nabla u^{(i)} : \nabla v \, dx. \quad (13)$$

Since

$$\|\nabla u_\lambda\|^2 = \sum_{i,j=1}^N a_{ij} \lambda_i \lambda_j = \mathbb{A} \lambda \cdot \lambda$$

and

$$\sum_{i=1}^N \lambda_i \int_{\Omega} \phi_i \operatorname{div} u_\lambda \, dx = \sum_{i,j=1}^N \lambda_i \lambda_j \int_{\Omega} \phi_i \operatorname{div} u^{(j)} \, dx = \mathbb{A} \lambda \cdot \lambda,$$

we find that

$$\inf \mathcal{P}_\lambda = -\frac{1}{2} \mathbb{A} \lambda \cdot \lambda + \lambda \cdot \mathbf{b}(\operatorname{div} v). \quad (14)$$

Hence

$$\begin{aligned} & \frac{1}{2} d^2(v, \mathbb{S}_{0, \Gamma_0}^\Phi) \\ & \geq \sup_{\lambda \in \mathbb{R}^N} \left\{ -\lambda \cdot \mathbf{b}(\operatorname{div} v) - \frac{1}{2} \mathbb{A} \lambda \cdot \lambda \right\} = \frac{1}{2} \mathbb{A}^{-1} \mathbf{b}(\operatorname{div} v) \cdot \mathbf{b}(\operatorname{div} v) \quad (15) \end{aligned}$$

and maximum is attained for

$$\lambda = \lambda^* := \mathbb{A}^{-1} \mathbf{b}(\operatorname{div} v).$$

Notice that the inverse matrix  $\mathbb{A}^{-1}$  exists. This fact follows from linear independence of the system  $\{\phi_i\}$ , which yields linear independence of the tensor functions  $\{\nabla u^{(i)}\}$  in  $U := L^2(\Omega, \mathbb{M}^{d \times d})$ . Indeed, assume that there exist real numbers  $\mu_i$ ,  $i = 1, 2, \dots, N$  not all equal to zero such that

$$\sum_{i=1}^N \mu_i \nabla u^{(i)} = 0.$$

Then (see (12)), for any  $w \in W_{0, \Gamma_0}^{1,2}(\Omega, \mathbb{R}^d)$  we have

$$\sum_{i=1}^N \mu_i \int_{\Omega} \nabla u^{(i)} : \nabla w \, dx = \int_{\Omega} \left( \sum_{i=1}^N \mu_i \phi_i \right) \operatorname{div} w \, dx = 0. \quad (16)$$

It is not difficult to see that (16) cannot be true for a system of linearly independent  $\phi_i$ . If  $\Gamma = \Gamma_0$ , then this fact follows from Theorem 1. Indeed, in this case the function  $g = \sum_{i=1}^N \mu_i \phi_i$  has zero mean and we can find  $w_g \in W_0^{1,2}(\Omega, \mathbb{R}^d)$  such that  $\operatorname{div} w_g = g$ . Then,

$$\int_{\Omega} \left( \sum_{i=1}^N \mu_i \phi_i \right)^2 \, dx = 0, \quad (17)$$

i.e.,  $\sum_{i=1}^N \mu_i \phi_i = 0$  almost everywhere and we arrive at the contradiction with linear independence of the system  $\{\phi_i\}$ .

If  $\Gamma_0 \subset \Gamma$ , then the same conclusion follows by means of similar but a bit more complicated arguments. Let  $\tilde{g} := g - \{g\}_{\Omega}$ . There exists  $w_{\tilde{g}} \in W_0^{1,2}(\Omega, \mathbb{R}^d)$  satisfying  $\operatorname{div} w_{\tilde{g}} = \tilde{g}$ . Let  $w_{\mathbf{1}} \in W^{1,2}(\Omega, \mathbb{R}^d)$  be such that  $\operatorname{div} w_{\mathbf{1}} = 1$  in  $\Omega$  (it is clear that there exist an infinite amount of

functions satisfying this condition). Consider the following auxiliary Stokes problem

$$\begin{aligned} \Delta \bar{w} &= \nabla \bar{p} && \text{in } \Omega, \\ \operatorname{div} \bar{w} &= 0 && \text{in } \Omega, \\ \bar{w} &= w_{\mathbb{1}} && \text{on } \Gamma_0, \\ \sigma_n &= 0 && \text{on } \Gamma \setminus \Gamma_0 \neq \emptyset. \end{aligned}$$

A solution to this problem exists and belongs to  $W^{1,2}(\Omega, \mathbb{R}^d)$ . We set

$$w_g := w_{\bar{g}} + \{g\} (w_{\mathbb{1}} - \bar{w}) \in W_{0,\Gamma_0}^{1,2}(\Omega, \mathbb{R}^d)$$

and find that  $\operatorname{div} w_g = g$ . Hence, we again arrive at (17) and conclude that  $\mathbb{A}$  is the Gram matrix for the system of tensor functions  $\{\nabla u^{(i)}\}$  containing linearly independent elements. It is positive definite and has an inverse  $\mathbb{A}^{-1}$ .

Now we establish the inequality inverse to (15). Let  $\bar{a}_{ij}$  denote entries of the matrix  $\mathbb{A}^{-1}$ . Then

$$\begin{aligned} \int_{\Omega} \phi_i \operatorname{div}(v - u_{\lambda^*}) dx &= \int_{\Omega} \phi_i \operatorname{div} v dx - \sum_{j=1}^N \lambda_j^* \int_{\Omega} \phi_i \operatorname{div} u^{(j)} dx \\ &= b_i - \sum_{j=1}^N \lambda_j^* a_{ij} = b_i - \sum_{j,k=1}^N \bar{a}_{jk} b_k a_{ij} \\ &= b_i - \sum_{k=1}^N b_k \sum_{j=1}^N \bar{a}_{kj} a_{ji} = b_i - \sum_{k=1}^N b_k \delta_{ki} = 0 \end{aligned}$$

Hence,

$$v - u_{\lambda^*} \in \mathbb{S}_{0,\Gamma_0}^{\Phi}. \quad (18)$$

Therefore,

$$\begin{aligned} \frac{1}{2} d^2(v, \mathbb{S}_{0,\Gamma_0}^{\Phi}) &\leq \frac{1}{2} \|\nabla u_{\lambda^*}\|^2 = \frac{1}{2} \sum_{i,j=1}^N \lambda_i^* \lambda_j^* \int_{\Omega} \nabla u^{(i)} : \nabla u^{(j)} dx \\ &= \frac{1}{2} \mathbb{A} \lambda^* \cdot \lambda^* = \frac{1}{2} \mathbb{A} \mathbb{A}^{-1} \mathbf{b}(\operatorname{div} v) \cdot \mathbb{A}^{-1} \mathbf{b}(\operatorname{div} v) \\ &= \frac{1}{2} \mathbb{A}^{-1} \mathbf{b}(\operatorname{div} v) \cdot \mathbf{b}(\operatorname{div} v). \end{aligned} \quad (19)$$

Now (9) follows from (15) and (19). Finally, we conclude that

$$\mathbf{P}_{\mathbb{S}_{0,\Gamma_0}^\Phi}(v) = v - u_{\lambda^*} = v - \sum_{i=1}^N \sum_{j=1}^N \bar{a}_{ij} b_j(v) u^{(i)}. \quad (20)$$

□

### §3. ESTIMATES BASED ON SOLUTIONS OF FINITE DIMENSIONAL PROBLEMS

From the practical point of view, it is preferable to replace  $u_i$  (exact solutions of boundary value problems) by solutions of some suitable finite dimensional problems. Let  $V_{0,\Gamma_0}^n \subset W_{0,\Gamma_0}^{1,2}$  be a finite dimensional space ( $\dim V_{0,\Gamma_0}^n = n$ ) satisfying the condition

$$V_{0,\Gamma_0}^n \cap \mathbb{S}_{0,\Gamma_0}^\Phi(\Omega, \mathbb{R}^d) \neq \emptyset. \quad (21)$$

By  $\mathbf{l}^n$  we denote a bounded operator mapping  $V_{0,\Gamma_0}$  to  $V_{0,\Gamma_0}^n$ . Particular forms of this operator can be different (they depend on the interpolation method used). In this paper, we do not discuss these questions in detail. The most important property of  $\mathbf{l}^n$  is that the computation of  $v_n := \mathbf{l}^n v$  is explicit and does not require large expenditures. Then, for any  $v \in V_{0,\Gamma_0}$  the quantity

$$\epsilon_n := \|\nabla(v - \mathbf{l}^n v)\|$$

is known. Since

$$\inf_{w \in \mathbb{S}_{0,\Gamma_0}^\Phi} \|\nabla(v - w)\| \leq \inf_{w \in \mathbb{S}_{0,\Gamma_0}^\Phi} \|\nabla(v_n - w)\| + \epsilon_n, \quad (22)$$

the estimation of  $\mathbf{d}^2(v, \mathbb{S}_{0,\Gamma_0}^\Phi)$  is reduced to the estimation of  $\mathbf{d}^2(v_n, \mathbb{S}_{0,\Gamma_0}^\Phi)$ .

In view of (21), there exists  $\hat{v}_n \in V_{0,\Gamma_0}^n \cap \mathbb{S}_{0,\Gamma_0}^\Phi(\Omega, \mathbb{R}^d)$ . Hence the problem of projecting  $v_n$  to this subset of  $\mathbb{S}_{0,\Gamma_0}^\Phi(\Omega, \mathbb{R}^d)$  is well posed. Moreover,

$$\begin{aligned} & \frac{1}{2} \mathbf{d}^2(v_n, \mathbb{S}_{0,\Gamma_0}^\Phi) \\ & \leq \inf_{w_n \in V_{0,\Gamma_0}^n(\Omega, \mathbb{R}^d)} \sup_{\substack{\lambda_i \in \mathbb{R} \\ i=1,2,\dots,N}} \left\{ \frac{1}{2} \|\nabla(v_n - w_n)\|^2 + \sum_{i=1}^N \lambda_i \int_{\Omega} \phi_i \operatorname{div} w_n dx \right\} \\ & = \inf_{w_n \in V_{0,\Gamma_0}^n(\Omega, \mathbb{R}^d)} \sup_{\substack{\lambda_i \in \mathbb{R} \\ i=1,2,\dots,N}} \left\{ \frac{1}{2} \|\nabla w_n\|^2 + \sum_{i=1}^N \lambda_i \int_{\Omega} \phi_i \operatorname{div}(v_n - w_n) dx \right\} \end{aligned}$$

$$= \inf_{w_n \in V_{0,\Gamma_0}^n(\Omega, \mathbb{R}^d)} \sup_{\lambda_i \in \mathbb{R}^N} L(w_n, \lambda) =: \kappa_n \leq \frac{1}{2} \|\nabla(v_n - \widehat{v}_n)\|^2. \quad (23)$$

On the other hand,

$$\kappa_n \geq \sup_{\lambda \in \mathbb{R}^N} \inf_{w_n \in V_{0,\Gamma_0}^n(\Omega, \mathbb{R}^d)} L(\lambda, w_n). \quad (24)$$

From (24) it follows that for any  $\lambda$

$$\kappa_n \geq \inf_{w_n \in V_{0,\Gamma_0}^n(\Omega, \mathbb{R}^d)} \left\{ \frac{1}{2} \|\nabla w_n\|^2 - \sum_{i=1}^N \lambda_i \int_{\Omega} \phi_i \operatorname{div} w_n dx \right\} + \lambda \cdot \mathbf{b}(\operatorname{div} v_n).$$

Select some  $\lambda$  and consider the auxiliary problem: find  $u_{\lambda,n} \in V_{0,\Gamma_0}^n(\Omega, \mathbb{R}^d)$  such that the functional  $L(w_n, \lambda) : V_{0,\Gamma_0}^n(\Omega, \mathbb{R}^d) \rightarrow \mathbb{R}$  attains infimum with respect to the first variable. The minimizer  $u_{\lambda,n}$  satisfies

$$\int_{\Omega} \nabla u_{\lambda,n} : \nabla w_n dx = \sum_{i=1}^N \lambda_i \int_{\Omega} \phi_i \operatorname{div} w_n dx \quad \forall w \in V_{0,\Gamma_0}^n(\Omega, \mathbb{R}^d),$$

where

$$u_{\lambda,n} = \sum_{i=1}^N \lambda_i u_n^{(i)}$$

and  $u_n^{(i)}$  are solutions of the finite dimensional problems

$$\int_{\Omega} \nabla u_n^{(i)} : \nabla w_n dx = \int_{\Omega} \phi_i \operatorname{div} w_n dx \quad \forall w_n \in V_{0,\Gamma_0}^n(\Omega, \mathbb{R}^d). \quad (25)$$

Hence

$$\nabla u_{\lambda,n} = \sum_{i=1}^N \lambda_i \nabla u_n^{(i)}$$

and

$$\|\nabla u_{\lambda,n}\|^2 = \sum_{i,j=1}^N a_{ij}^n \lambda_i \lambda_j = \mathbb{A}_n \lambda \cdot \lambda,$$

where

$$\mathbb{A}_n = \{a_{ij}^n\}, \quad a_{ij}^n = \int_{\Omega} \nabla u_n^{(i)} : \nabla u_n^{(j)} dx.$$

In view of (25), we have

$$\sum_{i=1}^N \lambda_i \int_{\Omega} \phi_i \operatorname{div} u_{\lambda, h} dx = \sum_{i,j=1}^N \lambda_i \lambda_j \int_{\Omega} \phi_i \operatorname{div} u_n^{(j)} dx = \mathbb{A}_n \boldsymbol{\lambda} \cdot \boldsymbol{\lambda}.$$

Therefore,

$$\kappa_n \geq \sup_{\boldsymbol{\lambda} \in \mathbb{R}^N} \left\{ \boldsymbol{\lambda} \cdot \mathbf{b}(\operatorname{div} v_n) - \frac{1}{2} \mathbb{A}_n \boldsymbol{\lambda} \cdot \boldsymbol{\lambda} \right\} = \frac{1}{2} \mathbb{A}_n^{-1} \mathbf{b}(\operatorname{div} v_n) \cdot \mathbf{b}(\operatorname{div} v_n) \quad (26)$$

and maximum is attained if

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}_n^* := \mathbb{A}_n^{-1} \mathbf{b}(\operatorname{div} v_n).$$

Here we assume that the matrix  $\mathbb{A}_n$  is invertible. Notice that the functions  $u_n^{(i)}$  are known (they are solutions of finite dimensional problems). Therefore, we do not need the argumentation used in the previous section and can verify the invertibility of  $\mathbb{A}_n$  directly.

In view of (25),  $\int_{\Omega} \phi_i \operatorname{div} u_n^{(j)} dx = a_{ij}^n$  and we find that

$$\begin{aligned} \int_{\Omega} \phi_i \operatorname{div}(v_n - u_{\lambda_n^*}) dx &= \int_{\Omega} \phi_i \operatorname{div} v_n dx - \sum_{j=1}^N (\lambda_n^*)_j \int_{\Omega} \phi_i \operatorname{div} u_n^{(j)} dx \\ &= b_i - \sum_{j=1}^N (\lambda_n^*)_j a_{ij}^n = b_i - \sum_{j,k=1}^N \bar{a}_{jk}^n b_k a_{ij}^n = 0. \end{aligned} \quad (27)$$

By (23) and (27), we obtain

$$\begin{aligned} \kappa_n &= \inf_{w_n \in V_{0,\Gamma_0}^n(\Omega, \mathbb{R}^d)} \sup_{\substack{\lambda_i \in \mathbb{R} \\ i=1,2,\dots,N}} L(w_n, \boldsymbol{\lambda}) \leq \sup_{\substack{\lambda_i \in \mathbb{R} \\ i=1,2,\dots,N}} L(u_{\lambda_n^*}, \boldsymbol{\lambda}) \\ &= \sup_{\substack{\lambda_i \in \mathbb{R} \\ i=1,2,\dots,N}} \left\{ \frac{1}{2} \|\nabla u_{\lambda_n^*}\|^2 + \sum_{i=1}^N \lambda_i \int_{\Omega} \phi_i \operatorname{div}(v - u_{\lambda_n^*}) dx \right\} \\ &= \frac{1}{2} \|\nabla u_{\lambda_n^*}\|^2 = \frac{1}{2} \mathbb{A}_n \boldsymbol{\lambda}_n^* \cdot \boldsymbol{\lambda}_n^* = \frac{1}{2} \mathbb{A}_n \mathbb{A}_n^{-1} \mathbf{b}(\operatorname{div} v) \cdot \mathbb{A}_n^{-1} \mathbf{b}(\operatorname{div} v) \\ &= \frac{1}{2} \mathbb{A}_n^{-1} \mathbf{b}(\operatorname{div} v) \cdot \mathbf{b}(\operatorname{div} v). \end{aligned} \quad (28)$$

Now (26) and (28) imply

$$\kappa_n = \frac{1}{2} \mathbb{A}_n^{-1} \mathbf{b}(\operatorname{div} v_n) \cdot \mathbf{b}(\operatorname{div} v_n) \quad (29)$$

and (22) yields the desired estimate

$$d(v, \mathbb{S}_{0, \Gamma_0}^\Phi) \leq (\mathbb{A}_n^{-1} \mathbf{b}(\operatorname{div} v_n) \cdot \mathbf{b}(\operatorname{div} v_n))^{1/2} + \epsilon_n. \quad (30)$$

#### §4. ESTIMATES OF THE DISTANCE TO $\mathbb{S}_{0, \Gamma_0}(\Omega, \mathbb{R}^d)$

In order to deduce an upper bound of the distance to  $\mathbb{S}_{0, \Gamma_0}(\Omega, \mathbb{R}^d)$ , we apply ideas of domain decomposition (see also [18, 19, 21] and some other publications). Let

$$\overline{\Omega} = \bigcup_{i=1}^N \overline{\Omega}_i, \quad (31)$$

where  $\Omega_i$  are connected Lipschitz subdomains for which the respective constants  $\mathbb{C}_{\Omega_i}$  in (2) (or some suitable majorants of them) are known.

##### 4.1. Theorem 1 for decomposed domains.

**Lemma 2.** *Let  $\Omega$  satisfy (31) and*

$$f = \sum_{i=1}^N f_i, \quad f_i \in L^2(\Omega), \quad \{f_i\}_\Omega = 0, \quad (32)$$

where  $\operatorname{supp} f_i \subset \Omega_i \subset \Omega$  for all  $i = 1, 2, \dots, N$ .

*There exists  $v_f \in W_0^{1,2}(\Omega_i, \mathbb{R}^d)$  such that  $\operatorname{div} v_f = f$  and*

$$\|\nabla v_f\| \leq \sum_{i=1}^N \mathbb{C}_{\Omega_i} \|f_i\|_\Omega. \quad (33)$$

**Proof.** For any  $f_i$  satisfying (32), we have  $v_{f_i} \in W_0^{1,2}(\Omega_i, \mathbb{R}^d)$  such that

$$\operatorname{div} v_{f_i} = f_i \quad \text{in } \Omega_i$$

and

$$\|\nabla v_{f_i}\|_{\Omega_i} \leq \mathbb{C}_{\Omega_i} \|f_i\|_{\Omega_i}.$$

We extend  $v_{f_i}$  to  $\Omega$  by zero and set  $v_f = \sum_{i=1}^N v_{f_i}$ . Then,  $\operatorname{div} v_f = f$  and

$$\|\nabla v_f\| \leq \sum_{i=1}^N \|\nabla v_{f_i}\|_{\Omega_i} = \sum_{i=1}^N \mathbb{C}_{\Omega_i} \|f_i\|_{\Omega_i}.$$

Thus, we obtain (33).  $\square$

**Corollary 1.** *Let  $v \in \mathbb{S}_{0,\Gamma_0}^\Phi$  and  $\phi_i(x)$  be a collection of bounded functions such that*

$$\text{supp } \phi_i = \Omega_i \subset \Omega \quad \text{and} \quad \sum_{i=1}^N \phi_i(x) = 1. \quad (34)$$

*Set  $f_i = \phi_i \operatorname{div} v$  and  $f = \sum_{i=1}^N f_i = \operatorname{div} v$ . Since  $v \in \mathbb{S}_{0,\Gamma_0}^\Phi$ , we see that*

$$\int_{\Omega} f_i dx = \int_{\Omega} \phi_i \operatorname{div} v dx = 0 \quad i = 1, 2, \dots, N.$$

*Lemma 2 guarantees existence of  $v_f \in W_0^{1,2}(\Omega_i, \mathbb{R}^d)$  such that*

$$\operatorname{div} v_f = f \quad \text{and} \quad \|\nabla v_f\| \leq \sum_{i=1}^N \mathbb{C}_{\Omega_i} \|\phi_i \operatorname{div} v\|_{\Omega}.$$

*Then  $v_0 = v - v_f \in \mathbb{S}_{0,\Gamma_0}$  and*

$$\|\nabla(v - v_0)\| \leq \sum_{i=1}^N \mathbb{C}_{\Omega_i} \|\phi_i \operatorname{div} v\|_{\Omega}.$$

*We conclude that the projection of  $v \in \mathbb{S}_{0,\Gamma_0}^\Phi$  meets the estimate*

$$\|\nabla(v - \mathbf{P}_{\mathbb{S}_{0,\Gamma_0}} v)\| \leq \sum_{i=1}^N \mathbb{C}_{\Omega_i} \|\phi_i \operatorname{div} v\|_{\Omega}. \quad (35)$$

**4.2. Distance to  $\mathbb{S}_{0,\Gamma_0}(\Omega, \mathbb{R}^d)$ .** Now we will use Lemma 2 and deduce an estimate of the distance between  $v \in W_{0,\Gamma_0}^{1,2}$  and the set of divergence free fields  $\mathbb{S}_{0,\Gamma_0}(\Omega, \mathbb{R}^d)$ , which is based on local constants  $\mathbb{C}_{\Omega_i}$ .

**Theorem 2.** *Let  $v \in W_{0,\Gamma_0}^{1,2}$  and  $\phi_i$  satisfy (34). Then,*

$$\begin{aligned} d(v, \mathbb{S}_{0,\Gamma_0}(\Omega, \mathbb{R}^d)) &\leq \left( \mathbb{A}^{-1} \mathbf{b}(\operatorname{div} v) \cdot \mathbf{b}(\operatorname{div} v) \right)^{1/2} \\ &\quad + \sum_{i=1}^N \mathbb{C}_{\Omega_i} \left( \|\phi_i \operatorname{div} v\|^2 - 2 \mathbb{A}^{-1} \mathbf{b}(\operatorname{div} v) \cdot \boldsymbol{\Theta}^{(i)} \right. \\ &\quad \left. + \mathbb{A}^{-1} \mathbf{D}^{(i)} \mathbb{A}^{-1} \mathbf{b}(\operatorname{div} v) \cdot \mathbf{b}(\operatorname{div} v) \right)^{1/2}, \quad (36) \end{aligned}$$

where

$$\begin{aligned}\Theta^{(i)} &= \{\Theta_j^{(i)}\}, \quad \{\Theta_j^{(i)}\} := \int_{\Omega} \phi_i^2 \operatorname{div} v \operatorname{div} u^{(j)} dx, \\ \mathbf{D}^{(i)} &= \{D_{jk}^{(i)}\}, \quad D_{jk}^{(i)} := \int_{\Omega} \phi_i^2 \operatorname{div} u^{(j)} \operatorname{div} u^{(k)} dx.\end{aligned}$$

**Proof.** In view of (18),  $\tilde{v} = v - u_{\lambda^*} \in \mathbb{S}_{0,\Gamma_0}^{\Phi}$ . We use (35) and find that

$$\begin{aligned}\|\nabla(v - P_{\mathbb{S}_{0,\Gamma_0}} v)\| &= \inf_{v_0 \in \mathbb{S}_{0,\Gamma_0}} \|\nabla(v - v_0)\| \\ &\leq \|\nabla u_{\lambda^*}\| + \inf_{v_0 \in \mathbb{S}_{0,\Gamma_0}} \|\nabla(\tilde{v} - v_0)\| \\ &\leq (\mathbb{A}^{-1} \mathbf{b}(\operatorname{div} v) \cdot \mathbf{b}(\operatorname{div} v))^{1/2} \\ &\quad + \sum_{i=1}^N \mathbb{C}_{\Omega_i} \|\phi_i (\operatorname{div} v - \operatorname{div} u_{\lambda^*})\|_{\Omega}.\end{aligned}\tag{37}$$

Consider the last term in the right hand side. We have

$$\begin{aligned}\int_{\Omega} \phi_i^2 \operatorname{div} v \operatorname{div} u_{\lambda^*} dx &= \int_{\Omega} \phi_i^2 \operatorname{div} v \sum_{j=1}^N \lambda_j^* \operatorname{div} u^{(j)} dx \\ &= \sum_{j=1}^N \lambda_j^* \int_{\Omega} \phi_i^2 \operatorname{div} v \operatorname{div} u^{(j)} dx \\ &= \sum_{j,k=1}^N \bar{a}_{jk} b_k \Theta_j^{(i)} = \mathbb{A}^{-1} \mathbf{b}(\operatorname{div} v) \cdot \Theta^{(i)}\end{aligned}$$

and

$$\begin{aligned}\int_{\Omega} |\phi_i \operatorname{div} u_{\lambda^*}|^2 dx &= \sum_{j,k=1}^N \lambda_j^* \lambda_k^* \int_{\Omega} \phi_i^2 \operatorname{div} u^{(k)} \operatorname{div} u^{(j)} dx \\ &= \sum_{j,k=1}^N D_{jk}^{(i)} \lambda_j^* \lambda_k^* = \mathbf{D}^{(i)} \lambda^* \cdot \lambda^* \\ &= \mathbb{A}^{-1} D^{(i)} \mathbb{A}^{-1} \mathbf{b}(\operatorname{div} v) \cdot \mathbf{b}(\operatorname{div} v).\end{aligned}$$

Hence

$$\begin{aligned} \int_{\Omega} \phi_i^2 (\operatorname{div} v - \operatorname{div} u_{\lambda^*})^2 dx &= \|\phi_i \operatorname{div} v\|^2 - 2\mathbb{A}^{-1} \mathbf{b}(\operatorname{div} v) \cdot \boldsymbol{\Theta}^{(i)} \\ &\quad + \mathbb{A}^{-1} \mathbf{D}^{(i)} \mathbb{A}^{-1} \mathbf{b}(\operatorname{div} v) \cdot \mathbf{b}(\operatorname{div} v). \end{aligned} \quad (38)$$

Now (36) follows from (37) and (38).  $\square$

**Remark 1.** Notice that  $\mathbf{b}(\operatorname{div} v) = 0$  for any  $v \in \mathbb{S}_{0,\Gamma_0}^{\Phi}$ . Hence in this case (36) is reduced to (35).

**4.3. Particular case.** Consider a particular but important case, where  $\Omega_i$  are disjoint sets:

$$\Omega_i \cap \Omega_j = \emptyset \quad \text{if } i \neq j$$

and  $\phi_i(x)$  coincides with the characteristic function

$$\phi_i(x) := \begin{cases} 1 & \text{if } x \in \Omega_i, \\ 0 & \text{if } x \notin \Omega_i. \end{cases} \quad (39)$$

In this case, instead of (33), we have the estimate

$$\|\nabla v_f\|^2 \leq \sum_{i=1}^N \mathbb{C}_{\Omega_i}^2 \|f_i\|^2, \quad (40)$$

where we set

$$f_i = \begin{cases} \operatorname{div} v & \text{in } \Omega_i, \\ 0 & \text{in } \Omega \setminus \Omega_i. \end{cases}$$

Now the space  $\mathbb{S}_{0,\Gamma_0}^{\Phi}(\Omega, \mathbb{R}^d)$  contains vector valued functions with zero mean divergence in the subdomains  $\Omega_i$ :

$$\mathbb{S}_{0,\Gamma_0}^{\Phi}(\Omega, \mathbb{R}^d) = \left\{ w \in W_{0,\Gamma_0}^{1,2}(\Omega, \mathbb{R}^d) \mid \{\operatorname{div} w\}_{\Omega_i} = 0 \quad \forall i = 1, 2, \dots, N \right\}.$$

For  $v \in \mathbb{S}_{0,\Gamma_0}^{\Phi}(\Omega, \mathbb{R}^d)$ , we have the estimate

$$\|\nabla(v - \mathbf{P}_{\mathbf{s}_{0,\Gamma_0}} v)\|^2 \leq \sum_{i=1}^N \mathbb{C}_{\Omega_i}^2 \|\operatorname{div} v\|_{\Omega_i}^2. \quad (41)$$

**Theorem 3.** *Let  $v \in W_{0,\Gamma_0}^{1,2}(\Omega, \mathbb{R}^d)$  and  $\phi_i$  satisfy (39). Then,*

$$\begin{aligned} d^2(v, S_{0,\Gamma_0}^{1,2}) &\leq \alpha \mathbb{A}^{-1} \mathbf{b}(\operatorname{div} v) \cdot \mathbf{b}(\operatorname{div} v) \\ &\quad + \alpha' \sum_{i=1}^N \mathbb{C}_{\Omega_i}^2 \left( \|\operatorname{div} v\|_{\Omega_i}^2 - 2 \mathbb{A}^{-1} \mathbf{b}(\operatorname{div} v) \cdot \boldsymbol{\Theta}^{(i)} \right. \\ &\quad \left. + \mathbb{A}^{-1} \mathbf{D}^{(i)} \mathbb{A}^{-1} \mathbf{b}(\operatorname{div} v) \cdot \mathbf{b}(\operatorname{div} v) \right), \end{aligned} \quad (42)$$

where  $\alpha, \alpha' \geq 1$  are two conjugate numbers ( $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$ ) and

$$\begin{aligned} \Theta_j^{(i)} &= \int_{\Omega_i} \operatorname{div} v \operatorname{div} u^{(j)} dx, \quad D_{jk}^{(i)} = \int_{\Omega_i} \operatorname{div} u^{(j)} \operatorname{div} u^{(k)} dx, \\ a_{ij} &= \int_{\Omega_i} \operatorname{div} u_j dx, \quad b_i = \int_{\Omega_i} \operatorname{div} v dx = |\Omega_i| \{ \operatorname{div} v \}_{\Omega_i}. \end{aligned}$$

**Proof.** The function

$$\tilde{v} = v - u_{\lambda^*} = v - \sum_{i=1}^N \lambda_i^* u^{(i)}$$

belongs to the set  $\mathbb{S}_{0,\Gamma_0}^\Phi(\Omega, \mathbb{R}^d)$ . Therefore,

$$\begin{aligned} \|\nabla(v - \mathbf{P}_{\mathbb{S}_{0,\Gamma_0}} v)\|^2 &= \inf_{v_0 \in \mathbb{S}_{0,\Gamma_0}} \|\nabla(v - v_0)\|^2 \\ &\leq \alpha \|\nabla u_{\lambda^*}\|^2 + \inf_{v_0 \in \mathbb{S}_{0,\Gamma_0}} \alpha' \|\nabla(\tilde{v} - v_0)\|^2 \\ &\leq \alpha \mathbb{A}^{-1} \mathbf{b}(\operatorname{div} v) \cdot \mathbf{b}(\operatorname{div} v) \\ &\quad + \alpha' \sum_{i=1}^N \mathbb{C}_{\Omega_i}^2 \|\operatorname{div}(v - u_{\lambda^*})\|_{\Omega_i}^2. \end{aligned} \quad (43)$$

Now (38), (40), and (43) yield (42).  $\square$

**Remark 2.** If  $v \in \mathbb{S}_{0,\Gamma_0}^\Phi$  then  $\mathbf{b}(\operatorname{div} v) = 0$  and we can tend  $\alpha$  to  $+\infty$ . We see that (42) is reduced to (41).

## §5. ESTIMATES OF $\mathbb{C}_\Omega$

First, we notice that an estimate of the distance to the set  $\mathbb{S}_{0,\Gamma}^\Phi$  implies an analog of Theorem 1 and a certain related inf-sup condition.

**Lemma 3.** *For any  $f \in L^2(\Omega)$  with zero mean, there exists  $w_f \in V_0$  such that*

$$\int_{\Omega} \phi_i (\operatorname{div} w_f - f) dx = 0 \quad \forall \phi_i \in \Phi \quad (44)$$

and

$$\|\nabla w_f\|^2 = \mathbb{A}^{-1} \mathbf{b}(f) \cdot \mathbf{b}(f), \quad (45)$$

where  $b_i = \int_{\Omega} \phi_i f dx$ .

**Proof.** There exists  $v_f \in V_0$  such that  $\operatorname{div} v_f = f$ . In view of Lemma 1, there exists  $v_0 \in \mathbb{S}_{0,\Gamma}^{\Phi}$  such that

$$\|\nabla(v_f - v_0)\|^2 = \mathbb{A}^{-1} \mathbf{b}(\operatorname{div} v_f) \cdot \mathbf{b}(\operatorname{div} v_f) \quad (46)$$

and

$$\int_{\Omega} \phi_i \operatorname{div}(v_f - v_0) dx = \int_{\Omega} \phi_i f dx \quad (47)$$

Set  $w_f = v_f - v_0$  (notice that  $\operatorname{div} w_f \neq f$ ). We see that (44) and (45) follow from (46) and (47).  $\square$

**Remark 3.** Lemma 3 imposes much weaker conditions on  $w_f$  (in comparison with Theorem 1), namely, instead of  $\operatorname{div} w_f = f$  it is required that  $\operatorname{div} w_f - f$  must be orthogonal to any function in the set  $\Phi$ . As a result,  $\|\nabla w_f\|$  is explicitly defined by (45) without using the constant  $c_{\Omega}$ .

Let  $|\mathbb{A}^{-1}| \leq \gamma_A$ . Then

$$\mathbb{A}^{-1} \mathbf{b}(f) \cdot \mathbf{b}(f) \leq \gamma_A |\mathbf{b}(f)|^2 \leq \gamma_A \sum_{i=1}^N \|\phi_i\|^2 \|f\|^2 \leq N \gamma_A \|f\|^2$$

and (45) infers the estimate

$$\|\nabla w_f\| \leq C_A \|f\|, \quad (48)$$

where  $C_A = \sqrt{N \gamma_A}$ .

Assume that  $f \in \operatorname{Lin} \Phi$  and  $\Phi$  is formed by an orthogonal system. Then

$$f = \sum_i \lambda_i \phi_i, \quad \|f\|^2 = \sum_i \lambda_i^2 = \boldsymbol{\lambda} \cdot \boldsymbol{\lambda} \quad b_i = \int_{\Omega} \phi_i f dx = \lambda_i.$$

We have

$$\mathbb{A}^{-1} \mathbf{b}(f) \cdot \mathbf{b}(f) = \sum_{i,j=1}^N \bar{a}_{ij} b_i b_j = \sum_{ij=1}^N \bar{a}_{ij} \lambda_i \lambda_j \leq \gamma_A \|f\|^2.$$

Hence in this case,  $C_A = \sqrt{\gamma_A}$ .

**Remark 4.** Let  $\phi \in \text{Lin} \Phi$ . In view of Lemma 3, there exists  $w_\phi \in V_0$  such that

$$\int_{\Omega} \phi_i (\text{div } w_\phi - \phi) dx = 0 \quad i = 1, 2, \dots, N \quad (49)$$

and  $\|\nabla w_\phi\|^2 = \mathbb{A}^{-1} \mathbf{b}(\phi) \cdot \mathbf{b}(\phi)$ . By (49) we conclude that

$$\int_{\Omega} \phi \text{div } w_\phi dx = \|\phi\|^2$$

and obtain an abridged form of the inf-sup condition

$$\inf_{\substack{\phi \in \text{Lin} \Phi \\ \phi \neq 0}} \sup_{\substack{w \in V_0 \\ w \neq 0}} \frac{\int_{\Omega} \phi \text{div } w dx}{\|\phi\| \|\nabla w\|} \geq \inf_{\substack{\phi \in \text{Lin} \Phi \\ \phi \neq 0}} \frac{\int_{\Omega} \phi \text{div } w_\phi dx}{\|\phi\| \|\nabla w_\phi\|} = \inf_{\substack{\phi \in \text{Lin} \Phi \\ \phi \neq 0}} \frac{\|\phi\|}{\|\nabla w_\phi\|},$$

which generates the problem

$$\inf_{\substack{\phi \in \text{Lin} \Phi \\ \phi \neq 0}} \frac{\|\phi\|^2}{\mathbb{A}^{-1} \mathbf{b}(\phi) \cdot \mathbf{b}(\phi)} =: (c_\Omega^\Phi)^2. \quad (50)$$

It is clear that  $c_\Omega^\Phi$  defined by (50) is larger than  $c_\Omega$ . Hence  $\frac{1}{c_\Omega^\Phi}$  generates a minorant of  $\mathbb{C}_\Omega$ . If  $\Phi$  is the orthonormal system, then  $c_\Omega^\Phi = \frac{1}{\gamma_A}$ .

In order to deduce a majorant of  $\mathbb{C}_\Omega$  we recall that  $\mathbb{C}_\Omega$  is the best (minimal) constant in (3) (or in (4) in the case of functions vanishing on a part of the boundary). Let us estimate the right hand side of (43) and represent it in the form (3).

Let

$$m := \max_i |\Omega_i|^{1/2} \quad \text{and} \quad c := \max_i \mathbb{C}_{\Omega_i}.$$

We have

$$\mathbb{A}^{-1} \mathbf{b}(\text{div } v) \cdot \mathbf{b}(\text{div } v) \leq \gamma_A |\mathbf{b}(\text{div } v)|^2 \leq \gamma_A m^2 \|\text{div } v\|^2. \quad (51)$$

For any  $i = 1, 2, \dots, N$ ,

$$|\lambda_i^*|^2 \leq \left( \sum_{j=1}^N \bar{a}_{ij} \int_{\Omega_j} \operatorname{div} v dx \right)^2 \leq \left( \sum_{j=1}^N |\bar{a}_{ij}| |\Omega_j|^{1/2} \|\operatorname{div} v\|_{\Omega_j} \right)^2 \leq \rho_i^2 \|\operatorname{div} v\|^2,$$

where

$$\rho_i^2 := \sum_{j=1}^N |\bar{a}_{ij}|^2 |\Omega_j|.$$

Then

$$|\boldsymbol{\lambda}^*| \leq |\boldsymbol{\rho}| \|\operatorname{div} v\|, \quad (52)$$

where  $\boldsymbol{\rho} \in \mathbb{R}^N$  is the vector with the components  $\rho_i$ .

Next, let  $\beta$  and  $\beta'$  be two conjugate positive numbers. Since

$$\begin{aligned} \|\operatorname{div}(v - u_{\boldsymbol{\lambda}^*})\|_{\Omega_i}^2 &\leq \beta \|\operatorname{div} v\|_{\Omega_i}^2 + \beta' \|\operatorname{div} u_{\boldsymbol{\lambda}^*}\|_{\Omega_i}^2 \\ &= \beta \|\operatorname{div} v\|_{\Omega_i}^2 + \beta' \sum_{j,k} \lambda_j^* \lambda_k^* \int_{\Omega_i} \operatorname{div} u^{(j)} \operatorname{div} u^{(k)} dx, \end{aligned}$$

we find that

$$\begin{aligned} &\sum_{i=1}^N \|\operatorname{div}(v - u_{\boldsymbol{\lambda}^*})\|_{\Omega_i}^2 \\ &= \beta \|\operatorname{div} v\|^2 + \beta' \sum_{j,k} \lambda_j^* \lambda_k^* \sum_i \int_{\Omega_i} \operatorname{div} u^{(j)} \operatorname{div} u^{(k)} dx \\ &= \beta \|\operatorname{div} v\|^2 + \beta' \sum_{j,k} \lambda_j^* \lambda_k^* \int_{\Omega} \operatorname{div} u^{(j)} \operatorname{div} u^{(k)} dx \\ &\leq \beta \|\operatorname{div} v\|^2 + \beta' \sum_{j,k} D_{jk} \lambda_j^* \lambda_k^* \\ &\leq \beta \|\operatorname{div} v\|^2 + \beta' |\mathbb{D}| |\boldsymbol{\lambda}^*|^2 \leq (\beta + \beta' |\mathbb{D}| |\boldsymbol{\rho}|^2) \|\operatorname{div} v\|^2, \end{aligned} \quad (53)$$

where  $|\mathbb{D}|$  denotes the norm of the matrix  $\mathbb{D}$  with the entries

$$D_{jk} = \int_{\Omega} \operatorname{div} u^{(j)} \operatorname{div} u^{(k)} dx.$$

Now (43), (51), (52), and (53) imply the estimate

$$\|\nabla(v - \mathbf{P}_{\mathbf{s}_{0,\Gamma_0}} v)\|^2 \leq (\alpha m^2 \gamma_A + \alpha' c^2 \beta + \alpha' \beta' c^2 |\mathbb{D}| |\boldsymbol{\rho}|^2) \|\operatorname{div} v\|^2.$$

Minimization with respect to  $\alpha$  and  $\beta$  yields

$$d(v, \mathbb{S}_{0, \Gamma_0}(\Omega, \mathbb{R}^d)) \leq \mathbb{C} \|\operatorname{div} v\|, \quad (54)$$

where

$$\mathbb{C} = m\gamma_A^{1/2} + c \left(1 + |\rho| |\mathbb{D}|^{1/2}\right). \quad (55)$$

This quantity gives an upper bound of  $\mathbb{C}_\Omega$ .

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