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**$L\log L$ -INTEGRABILITY OF THE VELOCITY  
GRADIENT FOR STOKES SYSTEM WITH DRIFTS IN  
 $L_\infty(BMO^{-1})$**

ABSTRACT. For any weak solution of the Stokes system with drifts in  $L_\infty(BMO^{-1})$ , we prove a reverse Hölder inequality and  $L\log L$ -higher integrability of the velocity gradients.

§1. INTRODUCTION

Let us consider the following 3D Stokes system with drift

$$\begin{aligned}\partial_t v + b \cdot \nabla v - \Delta v + \nabla q &= 0, \\ \operatorname{div} v &= 0,\end{aligned}\tag{1.1}$$

where  $b$  is a given vector field and  $v$  and  $q$  are unknown velocity field and pressure.

Our interest in (1.1) is related to possible regularity improvements in the Navier-Stokes borderline case  $b \in L_\infty(BMO^{-1})$ , at least in the size of a possible singular set. Hence we assume throughout this note that

$$\operatorname{div} b = 0.\tag{1.2}$$

There are different definitions of the space  $BMO^{-1}$ , see for example Koch & Tataru [10]. In our 3D case, it is convenient to use the following one: there exists a tensor  $d \in BMO$  such that

$$b = \operatorname{div} d\tag{1.3}$$

in the sense of distributions, while condition (1.2) implies its skew-symmetry. Equivalently, there exists a divergence free field  $\omega \in BMO$  such that  $b = \operatorname{rot} \omega$ . Then  $d_{ij} = \epsilon_{ijk} \omega_k$ , where  $(\epsilon_{ijk})$  is the Levi-Civita tensor.

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*Key words and phrases:* Stokes system with drift, reverse Hölder inequality, higher integrability.

The authors are indebted to Piotr Hajlasz in relation to local maximal functions. J. Burczak was supported by MNiSW "Mobilność Plus" grant 1289/MOB/IV/2015/0. G. Seregin was supported in parts by the grant RFBR 17-01-00099-a and by the Program of the Presidium of the Russian Academy of Sciences No.1 "Fundamental Mathematics and its Applications" under grant PRAS-18-01.

The relationship between  $b$  and  $d$  shows that one may recast (1.1) as a generalised Stokes system with the main part  $A = Id + D$ , where  $D = (D_{ijkl})$  with  $D_{ijkl} = \delta_{ik}d_{jl} \in L_\infty(BMO)$ . A general  $A \in L_\infty(BMO)$  is naturally too rough even to define a standard weak solution. But here skew-symmetry comes again to our aid. Namely, we have the following estimate

$$\int_{\mathbb{R}^n} (D \nabla u) : \nabla v \, dx \leq c \|d\|_{BMO} \|\nabla u\|_2 \|\nabla v\|_2 \quad (1.4)$$

for any  $u, v \in C_0^\infty(\mathbb{R}^3)$ , which can be deduced from the results of Maz'ya & Verbitsky [12]. A related discussion may be found in Silvestre, Šverák, Zlatoš & coauthor [16]. We give a straightforward proof of (1.4) in the Appendix I for completeness.

It is important to keep in mind that over the entirety of this note, while we refer to  $b \in L_\infty(BMO^{-1})$  satisfying (1.2), we automatically consider (1.3) with the related  $D$ .

Among other interesting cases, in which the system (1.1) plays an important part, there is the question about potential Type I blowup of solutions to the Navier-Stokes system, compare the recent paper [14] by Schonbek & coauthor about a Liouville-type theorem via duality.

For the account of the achievable regularity results for the scalar version of the problem (1.1) with the structural restriction (1.2) but with no pressure, i.e.

$$\partial_t u + b \cdot \nabla u - \Delta u = 0, \quad \operatorname{div} b = 0,$$

we refer to [16]. The essence of its results reads: among  $L_\infty(X)$  spaces for  $b$ ,  $X = BMO^{-1}$  is the widest one, where local ‘deep’ regularity results for  $u$  are available (e.g. Harnack inequality) and the choice of  $BMO^{-1}$  is close to being sharp. See also Nazarov & Ural'tseva [13] for  $b$  in space-time Morrey spaces on the same scale and Liskevich & Zhang [11] for similar results under a ‘form boundedness assumption’ on  $b$ . One should in addition mention Friedlander & Vicol [4], where Hölder continuity of solutions to the related Cauchy problem was proved, with  $b \in L_\infty(BMO^{-1})$ .

In relation to the full system (1.1-1.2), the current best result for the associated Cauchy problem is Silvestre & Vicol [18]. The authors show for  $b \in L_p(M^\beta)$ , a Lebesgue-Morrey scale of spaces, that there exists a  $C(C^\alpha)$  solution. However, for the endpoint of this scale, i.e. for  $L_\infty(M^{-1})$ ,  $M^{-1} \supset L^3$ , in order to conclude with the same result, an additional smallness assumption is needed (which is automatically satisfied for  $C(L^3)$ , but not

for  $L_\infty(L^3)$ ). For the local setting, we refer to Zhang [22], where  $b$  must belong to a certain Kato class.

Let us conclude with two remarks. Firstly, as already seen above, for a scale of spaces, the regularity results in the endpoint case  $L_\infty(X)$  are substantially more difficult and even likely not always to hold. Secondly, the result of Escauriaza, Šverák & coauthor [3], where  $b = v \in L_\infty(L_3)$  suffices to obtain regularity, utilises essentially the nonlinear structure. Hence to study regularity of solutions to (1.1) with (1.2), even with  $L_\infty(L_3)$ , one needs different ideas.

## §2. MAIN RESULTS

We write  $B(x_0, R)$  for the ball with radius  $R$  centred at  $x_0 \in \mathbb{R}^3$ .  $Q(z_0, R) = B(x_0, R) \times (t_0 - R^2, t_0)$  is the (parabolic) cylinder with its centre  $z_0 = (x_0, t_0)$ , where  $t_0 \in \mathbb{R}$ . For an open set  $\Omega \subset \mathbb{R}^3$  and an interval  $]T_1, T_2[$ , we write  $Q_{T_1, T_2} = \Omega \times ]T_1, T_2[$ .

We use standard function spaces:  $L_\infty(]T_1, T_2[; L_2(\Omega)) = L_{2, \infty}(Q_{T_1, T_2})$ ,  $W_2^{1,0}(Q_{T_1, T_2}) = \{v, \nabla v \in L_2(Q_{T_1, T_2})\}$ , etc.

In what follows we always adopt the following convention

$$\Gamma(z, \rho) = \|b\|_{L_\infty(t-\rho^2, t; BMO^{-1}(B(x, \rho)))} = \|d\|_{L_\infty(t-\rho^2, t; BMO(B(x, \rho)))}, \quad (2.1)$$

where  $d$  is related with  $b$  via (1.3). Naturally, the right-hand side of (2.1) is merely a seminorm for  $d$ , but the right-hand side is a proper norm for  $b$ , see e.g. [10].

Where there is no danger of confusion, we may sometimes suppress certain indices.

**Definition 2.1** (Weak solution). *Let us fix a space-time domain  $Q_{T_1, T_2}$ . A pair  $v = (v_i)$  and  $q$  is a weak solution to (1.1) on  $Q_{T_1, T_2}$  if and only if*

- (i)  $v \in L_{2, \infty}(Q_{T_1, T_2}) \cap W_2^{1,0}(Q_{T_1, T_2})$  and  $q \in L_2(Q_{T_1, T_2})$ ;
- (ii)  $v$  and  $q$  satisfy (1.1) in the sense of distributions on  $Q_{T_1, T_2}$ .

**Remark 2.2.** *The regularity classes appearing in Definition 2.1, in particular  $L_2$  for the pressure  $q$ , agree with the existence result for the Cauchy problem for (1.1) with a solenoidal drift  $b \in L_\infty(BMO^{-1})$ , see Appendix II.*

**Remark 2.3.** *Any weak solution to (1.1–1.2) on  $Q_{T_1, T_2}$  satisfies the following local energy identity*

$$\int_{\Omega} \varphi |v(x, t)|^2 dx + 2 \int_0^t \int_{\Omega} \varphi |\nabla v|^2 dx dt'$$

$$= \int_0^t \int_{\Omega} (|v|^2 (\partial_t + \Delta) \varphi - 2D\nabla v : v \otimes \nabla \varphi + 2qv \cdot \nabla \varphi) dx dt'$$

for any  $t \in ]T_1, T_2[$  and any non-negative  $\varphi \in C_0^\infty(Q_{T_1, T_2+1})$ .

The above remark follows from (1.4) and standard duality arguments. Observe that it renders a notion of a suitable weak solution redundant in our setting.

Our first result is as follows.

**Proposition 2.4.** *For any  $l \in ]6/5, 2[$ , any weak solution  $v$  and  $q$  to (1.1–1.2) on  $Q_{T_1, T_2}$  satisfies*

$$\begin{aligned} \frac{1}{|Q(\rho)|} \int_{Q(z_0, \rho)} |\nabla v|^2 dz \\ \leq C(l)(\Gamma^5(z_0, 2\rho) + 1) \left( \frac{1}{|Q(2\rho)|} \int_{Q(z_0, 2\rho)} |\nabla v|^l dz \right)^{\frac{2}{l}} \\ + C \left( \frac{1}{|Q(2\rho)|} \int_{Q(z_0, 2\rho)} |q| dz \right)^2 \quad (2.2) \end{aligned}$$

on any  $Q(z_0, 2\rho) \subset Q_{T_1, T_2}$ , with constants  $C(l)$  and  $C$ .

A simple consequence of Proposition 2.4 is as follows.

**Remark 2.5.** *Let  $b \in L_\infty(\mathbb{R}; BMO^{-1}(\mathbb{R}^3))$  satisfy (1.2). Then any weak solution to (1.1) on  $\mathbb{R}^3 \times \mathbb{R}$  vanishes.*

Indeed, let  $\Gamma_\infty = \|b\|_{L_\infty(\mathbb{R}; BMO^{-1}(\mathbb{R}^3))}$ ,  $h = |\nabla v|^s$  and  $M$  denote the (centred) maximal function with respect to parabolic cylinders (they satisfy the ‘doubling’ assumptions on families of open sets, needed to provide the usual maximal function theory, compare Stein [21], §I.1). Proposition 2.4 gives

$$M(h^{\frac{2}{s}})(z) \leq C(s, \Gamma_\infty) M^{\frac{2}{s}}(h)(z) + CM^2(q)(z).$$

The strong  $L_p$  estimates for  $M$  imply

$$\int_{\mathbb{R}^4} M(h^{\frac{2}{s}}) dz \leq C(s, \Gamma_\infty) \int_{\mathbb{R}^4} h^{\frac{2}{s}} dz + C \int_{\mathbb{R}^4} |q|^2 dz$$

$$= C(s, \Gamma_\infty) \int_{\mathbb{R}^4} |\nabla v|^2 dz + C \int_{\mathbb{R}^4} |q|^2 dz \leq C.$$

This means that both  $M(h_s^{\frac{2}{s}})$  and  $h_p^{\frac{2}{p}}$  are integrable. On the full space it yields that  $h_s^{\frac{2}{s}} \equiv 0$ , compare [21], §I.8.14. Therefore  $v$  can only be time-dependant, but then our assumption  $v \in L_{2,\infty}$  implies  $v \equiv 0$ .

Our main result reads

**Theorem 2.6.** *Let  $b$  satisfy (1.2). Then, there exists a number  $C$ , such that any weak solution  $v$  and  $q$  to (1.1) in  $Q_{T_1, T_2}$  satisfies*

$$\begin{aligned} \int_{Q(z_0, r)} |\nabla v|^2 \log \left( 1 + \frac{|\nabla v|^2}{(|\nabla v|^2)_{z_0, r}} \right) dz \\ \leq C(1 + \Gamma^5(z_0, 5r)) \int_{Q(z_0, 5r)} |\nabla v|^2 dz + C \int_{Q(z_0, 5r)} |q|^2 dz \end{aligned}$$

for any  $Q(z_0, 5r) \Subset Q_{T_1, T_2}$ .

Here,  $(f)_{z_0, r}$  is the mean value of function  $f$  over the parabolic cylinder  $Q(z_0, r)$ .

We would like to notice that, in [2], the authors claim even a stronger result about higher integrability of the velocity gradient.

### §3. PROOF OF PROPOSITION 2.4

Over this proof, we will refer at certain times to [15]. Let us thence initially observe, that however it deals with the case  $b = v$ , all the computations are in fact performed there for (1.1–1.2).

For an  $x_0 \in \mathbb{R}^3$  and  $r < R$ , let  $\varphi_{x_0, r, R}(x)$  be a radial nonnegative smooth space cut-off function, such that

$$\varphi_{x_0, r, R} \equiv 1 \text{ on } B(x_0, r), \quad \varphi_{x_0, r, R} \equiv 0 \text{ outside } B(x_0, R),$$

$$|\nabla^i \varphi_{x_0, r, R}| \leq \frac{C_i}{(R-r)^i}.$$

Let us introduce the related mean value of a function  $f$

$$f_{x_0, r, R}(t) = \int_{B(x_0, R)} f(x, t) \varphi_{x_0, r, R}^2(x) dx \left( \int_{B(x_0, R)} \varphi_{x_0, r, R}^2(x) dx \right)^{-1}.$$

We will also need a smooth nonnegative time cut-off function  $\chi_{t_0,r,R}(t)$  with the following properties

$$\begin{aligned} \chi_{t_0,r,R}(t) &\equiv 1 \text{ for } t \leq t_0 - R^2, & \chi_{t_0,r,R}(t) &\equiv 0 \text{ for } t \geq t_0 - r^2, \\ |\partial_t \chi_{t_0,r,R}(t)| &\leq \frac{C}{R^2 - r^2} \leq \frac{2C}{(R - r)^2}. \end{aligned}$$

Together, let us write for brevity

$$\eta_{z_0,r,R}(x, t) = \chi_{t_0,r,R}(t) \varphi_{x_0,r,R}(x).$$

Finally, for a function  $f$  let us denote the oscillations at  $z = (x, t)$  as follows

$$\widehat{f}(z) = f(z) - f_{x_0,r,R}(t), \quad \bar{f}(z) = f(z) - [f]_{x_0,R}(t),$$

where  $[f]_{x_0,R}$  is the mean value of  $f$  over the ball  $B(x_0, R)$ .

Keeping in mind Remark 2.3, it is straightforward to conclude that Lemma 2.1 of [15] (compare also Lemma 2.3 of [16]) holds in our case in the following form.

**Lemma 3.1.** *Let  $b \in L_\infty(T_1, T_2; BMO^{-1}(\Omega))$  satisfy (1.2). Consider any weak solution  $v$  and  $q$  of (1.1) on  $Q_{T_1, T_2}$ . Let  $Q(z_0, R) \Subset Q_{T_1, T_2}$ . Then for any  $t \in (t_0 - R^2, t_0)$*

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} |\widehat{v}(x, t)|^2 \eta_{z_0,r,R}^2(x, t) dx + \int_{t_0 - R^2}^t \int_{\Omega} |\nabla v|^2 \eta_{z_0,r,R}^2 dx dt' \\ & \leq \int_{t_0 - R^2}^t \int_{\Omega} \left( \frac{1}{2} |\widehat{v}|^2 (\Delta + \partial_{t'}) \eta_{z_0,r,R}^2 - \bar{d}_{jl} v_{i,l} \widehat{v}_i (\eta_{z_0,r,R}^2)_{,j} q \widehat{v} \cdot \nabla \eta_{z_0,r,R}^2 \right) dx dt'. \end{aligned} \tag{3.1}$$

Let us assume that  $Q(x_0, R_1) \Subset Q_{T_1, T_2}$  with  $R < R_1$  fixed. Recall that by definition  $\Gamma(z_0, R_1) = \|d\|_{L_\infty(t_0 - R_1^2, t_0; BMO(B(x_0, R_1)))}$ . Identically as in [15] its Lemma 2.1 implies (2.7) there, we obtain from (3.1) that for any  $s \in (1, 6/5)$

$$\begin{aligned} & \sup_{t \in ]t_0 - R^2, t_0[} \frac{1}{2} \int_{\Omega} |\widehat{v}(x, t)|^2 \eta_{z_0,r,R}^2(x, t) dx + \int_{t_0 - R^2}^{t_0} \int_{\Omega} |\nabla v|^2 \eta_{z_0,r,R}^2 dz \\ & \leq \frac{C}{R - r} \int_{Q(z_0, R)} |q| |\widehat{v}| \chi_{t_0,r,R}^2 \varphi_{x_0,r,R} dz \end{aligned}$$

$$\begin{aligned}
 & + C(s) \left( \frac{\Gamma(z_0, R_1) R^{\frac{3}{s'}}}{R-r} + \frac{R^{1+\frac{3}{s'}}}{(R-r)^2} \right) \left( \int_{Q(z_0, R)} |\nabla v|^2 dz \right)^{\frac{1}{2}} \\
 & \quad \times \left( \int_{t_0-R^2}^{t_0} \left( \int_{B(x_0, R)} |\widehat{v}(x, t)|^{\frac{2s}{2-s}} dx \right)^{\frac{2-s}{s}} dt \right)^{\frac{1}{2}} \\
 & \leq \frac{C}{R-r} \int_{Q(z_0, R)} |q| |\widehat{v}| \chi_{t_0, r, R}^2 \varphi_{x_0, r, R} dz + C(s) \frac{(\Gamma(z_0, R_1) + 1) R^{1+\frac{3}{s'}}}{(R-r)^2} \\
 & \quad \times \left( \int_{Q(z_0, R)} |\nabla v|^2 dz \right)^{\frac{1}{2}} \left( \int_{t_0-R^2}^{t_0} \left( \int_{B(x_0, R)} |\widehat{v}(x, t)|^{\frac{2s}{2-s}} dx \right)^{\frac{2-s}{s}} dt \right)^{\frac{1}{2}}. \tag{3.2}
 \end{aligned}$$

We deal with the pressure part also in a similar way as in [15], pp. 332–333. Again, as in the case of (3.1), the only difference is our use of a cut-off function between any  $r < R$ , as opposed to a cutoff between  $R$  and  $2R$  in [15]. Nevertheless, let us present details for clarity. Since  $\operatorname{div} v = 0$ , (1.1) implies that for any  $\varphi \in C_0^\infty(\Omega)$  and a. a.  $t \in ]T_1, T_2[$

$$\int_{\Omega} q(x, t) \Delta \varphi(x) dx = \int_{\Omega} \bar{d}_{jl}(x, t) v_{i,l}(x, t) \varphi_{,ij}(x) dx.$$

Define  $q_G$  as the solution to the related very weak homogenous boundary problem in  $B(x_0, R_1)$ :

$$\int_{B(x_0, R_1)} q_G(x, t) \Delta \varphi(x) dx = \int_{B(x_0, R_1)} \bar{d}_{jl}(x, t) v_{i,l}(x, t) \varphi_{,ij}(x) dx$$

for all  $\varphi \in W_{\frac{2s}{2-s}}^2(B(x_0, R_1))$  satisfying boundary condition  $\varphi(x, t) = 0$  as  $x \in \partial B(x_0, R_1)$ . The dual estimate implies then for a.a.  $t$

$$\begin{aligned}
 & \left( \int_{B(x_0, R_1)} |q_G(x, t)|^{\frac{2s}{3s-2}} dx \right)^{\frac{3s-2}{2s}} \\
 & \leq C(s) R_1^{\frac{3}{s'}} \Gamma(z_0, R_1) \left( \int_{B(x_0, R_1)} |\nabla v(x, t)|^2 dx \right)^{\frac{1}{2}} \tag{3.3}
 \end{aligned}$$

(compare (2.11) of [15]). The remainder  $q_H = q - q_G$  is harmonic on  $B(x_0, R_1)$ . Since  $R < R_1$ , we have then

$$\begin{aligned} \|q_H(\cdot, t)\|_{L_\infty(B(x_0, R))} &\leq \frac{C}{(R_1 - R)^3} \int_{B(x_0, R_1)} |q_H(x, t)| dx \\ &\leq \frac{C}{(R_1 - R)^3} \int_{B(x_0, R_1)} (|q(x, t)| + |q_G(x, t)|) dx. \end{aligned}$$

Use of (3.3) above implies

$$\begin{aligned} \|q_H(\cdot, t)\|_{L_\infty(B(x_0, R))} &\leq \frac{C}{(R_1 - R)^3} \int_{B(x_0, R_1)} |q(x, t)| dx \\ &\quad + \frac{C(s)\Gamma(z_0, R_1)R_1^{\frac{3}{2}}}{(R_1 - R)^3} \left( \int_{B(x_0, R_1)} |\nabla v(x, t)|^2 dx \right)^{\frac{1}{2}}. \end{aligned} \quad (3.4)$$

We intend to use the above formulas to estimate the pressure part of (3.2). Before that, since  $q = q_G + q_H$ , we rewrite it as follows

$$\begin{aligned} &\frac{C}{R - r} \int_{Q(z_0, R)} |q| |\widehat{v}| \chi_{t_0, r, R}^2 \varphi_{x_0, r, R} dz \\ &\leq \frac{C}{R - r} \int_{t_0 - R^2}^{t_0} \left( \int_{B(x_0, R)} |q_G(x, t)|^{\frac{2s}{3s-2}} dx \right)^{\frac{3s-2}{2s}} \left( \int_{B(x_0, R)} |\widehat{v}(x, t)|^{\frac{2s}{2-s}} dx \right)^{\frac{2-s}{2s}} dt \\ &\quad + \frac{C}{R - r} \int_{t_0 - R^2}^{t_0} \|q_H(\cdot, t)\|_{L_\infty(B(x_0, R))} \left( \int_{B(x_0, R)} |\widehat{v}(x, t) \eta_{z_0, r, R}(x, t)| dx \right) dt \\ &= I + II. \end{aligned} \quad (3.5)$$

We estimate  $I$  using (3.3)



$$I \leq \frac{C(s)R_1^{\frac{3}{s'}}\Gamma(z_0, R_1)}{R-r} \left( \int_{Q(z_0, R_1)} |\nabla v|^2 dz \right)^{\frac{1}{2}} \\ \times \left( \int_{t_0-R_1^2}^{t_0} \left( \int_{B(x_0, R)} |\widehat{v}(x, t)|^{\frac{2s}{2-s}} dx \right)^{\frac{2-s}{s}} dt \right)^{\frac{1}{2}}$$

and  $II$  using (3.4) and next the Hölder inequality

$$II \leq \frac{C}{R-r} \int_{t_0-R^2}^{t_0} \left( \frac{1}{(R_1-R)^3} \int_{B(x_0, R_1)} |q(x, t)| dx \right) \\ \times \left( \int_{B(x_0, R)} |\widehat{v}(x, t)\eta_{t_0, r, R}(x, t)| dx \right) dt \\ + \frac{C}{R-r} \int_{t_0-R^2}^{t_0} \frac{C(s)\Gamma(z_0, R_1)R_1^{\frac{3}{2}}}{(R_1-R)^3} \left( \int_{B(x_0, R_1)} |\nabla v(x, t)|^2 dx \right)^{\frac{1}{2}} \\ \times \left( \int_{B(x_0, R)} |\widehat{v}(x, t)| dx \right) dt \\ \leq \sup_{t \in ]t_0-R^2, t_0[} \left( \int_{B(x_0, R)} |\widehat{v}(x, t)|^2 \eta_{t_0, r, R}^2(x, t) dx \right)^{\frac{1}{2}} \frac{C}{R-r} \frac{R^{\frac{3}{2}}}{(R_1-R)^3} \\ \int_{Q_{R_1}(z_0)} |q| dz + \frac{C}{R-r} \frac{C(s)\Gamma(z_0, R_1)R_1^{\frac{3}{2}}}{(R_1-R)^3} R^{\frac{3}{2}+\frac{3}{s'}} \left( \int_{Q(z_0, R_1)} |\nabla v|^2 dz \right)^{\frac{1}{2}} \\ \times \left( \int_{t_0-R_1^2}^{t_0} \left( \int_{B(x_0, R)} |\widehat{v}(x, t)|^{\frac{2s}{2-s}} dx \right)^{\frac{2-s}{s}} dt \right)^{\frac{1}{2}}.$$

Finally, applying the above estimates of  $I$  and  $II$  to (3.5), we control the pressure term in (3.2) and arrive, after absorbing the sup term into the

left-hand side, at

$$\begin{aligned}
& \sup_{t \in ]t_0 - r^2, t_0[} \frac{1}{4} \int_{B(x_0, r)} |\widehat{v}(x, t)|^2 dx + \int_{Q(z_0, r)} |\nabla v|^2 dz \\
& \leq C(s)(\Gamma(z_0, R_1) + 1) \left( \frac{R^{1+\frac{3}{s}}}{(R-r)^2} + \frac{R_1^{\frac{3}{s}}}{R-r} + \frac{1}{R-r} \frac{R_1^{\frac{3}{2}}}{(R_1-R)^3} R^{\frac{3}{2}+\frac{3}{s}} \right) \\
& \quad \times \left( \int_{Q(z_0, R_1)} |\nabla v|^2 dz \right)^{\frac{1}{2}} \left( \int_{t_0 - R_1^2}^{t_0} \left( \int_{B(x_0, R_1)} |\widehat{v}(x, t)|^{\frac{2s}{2-s}} dx \right)^{\frac{2-s}{s}} dt \right)^{\frac{1}{2}} \\
& \quad + \frac{C}{(R-r)^2} \frac{R^3}{(R_1-R)^6} \left( \int_{Q(z_0, R_1)} |q| dz \right)^2.
\end{aligned} \tag{3.6}$$

Choosing  $R = \frac{R_1+r}{2}$  we have

$$\begin{aligned}
& \sup_{t \in (t_0 - r^2, t_0)} \frac{1}{4} \int_{B(x_0, r)} |\widehat{v}(x, t)|^2 dx + \int_{Q(z_0, r)} |\nabla v|^2 dz \\
& \leq C(s)(\Gamma(z_0, R_1) + 1) R_1^{\frac{3}{s}-1} \frac{R_1^4}{(R_1-r)^4} \left( \int_{Q(z_0, R_1)} |\nabla v|^2 dz \right)^{\frac{1}{2}} \\
& \quad \times \left( \int_{t_0 - R_1^2}^{t_0} \left( \int_{B(x_0, R_1)} |\widehat{v}(x, t)|^{\frac{2s}{2-s}} dx \right)^{\frac{2-s}{s}} dt \right)^{\frac{1}{2}} \\
& \quad + C \frac{R_1^3}{(R_1-r)^8} \left( \int_{Q(z_0, R_1)} |q| dz \right)^2,
\end{aligned} \tag{3.7}$$

valid for any  $R_1 > r$ . The estimate (3.7) counterparts (2.13) of [15].

We will use (3.7) twofold. Before doing so, observe that the Sobolev and Hölder inequalities yield for

$$l = \frac{6s}{12-7s} \in ]1, 2[ \tag{3.8}$$

the inequality

$$\int_{t_0 - r^2}^{t_0} \left( \int_{B(x_0, r)} |\widehat{v}(x, t)|^{\frac{2s}{2-s}} dx \right)^{\frac{2-s}{s}} dt$$

$$\leq C(s) r^{\frac{2(l-1)}{l}} \sup_{t \in ]t_0 - r^2, t_0[} \left( \int_{B(x_0, r)} |\widehat{v}(x, t)|^2 dx \right)^{\frac{1}{2}} \left( \int_{Q(z_0, r)} |\nabla v|^l dz \right)^{\frac{1}{l}}, \quad (3.9)$$

compare estimate of  $I_*$  on p.335 of [15] ( $l$  is denoted as  $r$  there).

Let us return to (3.7). Firstly, using the Poincaré-Sobolev inequality

$$\left( \int_{B(x_0, R_1)} |\widehat{v}(x, t)|^{\frac{2s}{2-s}} dx \right)^{\frac{2-s}{s}} \leq C R_1^{\frac{6-4s}{s}} \int_{B(x_0, R_1)} |\nabla v(x, t)|^2 dx,$$

we estimate only the evolutionary part of (3.7) to get

$$\begin{aligned} & \sup_{t \in ]t_0 - r^2, t_0[} \frac{1}{4} \int_{B(x_0, r)} |\widehat{v}(x, t)|^2 dx \\ & \leq C(s) (\Gamma(z_0, R_1) + 1) \frac{R_1^4}{(R_1 - r)^4} \left( \int_{Q(z_0, R_1)} |\nabla v|^2 dz \right) \\ & \quad + C \frac{R_1^3}{(R_1 - r)^8} \left( \int_{Q(z_0, R_1)} |q| dz \right)^2. \quad (3.10) \end{aligned}$$

The above estimate in the sup term of (3.9) yields

$$\begin{aligned} & \int_{t_0 - r^2}^{t_0} \left( \int_{B(x_0, r)} |\widehat{v}(x, t)|^{\frac{2s}{2-s}} dx \right)^{\frac{2-s}{s}} dt \\ & \leq C(s) (\Gamma^{\frac{1}{2}}(z_0, R_1) + 1) R_1^{\frac{2(l-1)}{l}} \left( \frac{R_1^2}{(R_1 - r)^2} \left( \int_{Q(z_0, R_1)} |\nabla v|^2 dz \right)^{\frac{1}{2}} \right. \\ & \quad \left. + C \frac{R_1^{\frac{3}{2}}}{(R_1 - r)^4} \left( \int_{Q(z_0, R_1)} |q| dz \right) \right) \left( \int_{Q(z_0, R_1)} |\nabla v|^l dz \right)^{\frac{1}{l}}. \quad (3.11) \end{aligned}$$

Secondly, let us rewrite (3.7) for any  $r > \rho$  in place of  $R_1 > r$ , dropping this time the evolutionary term

$$\begin{aligned} \int_{Q(z_0, \rho)} |\nabla v|^2 dz &\leq C(s)(\Gamma(z_0, r) + 1)r^{\frac{3}{s}-1} \frac{r^4}{(r-\rho)^4} \left( \int_{Q(z_0, r)} |\nabla v|^2 dz \right)^{\frac{1}{2}} \\ &\times \left( \int_{t_0-r^2}^{t_0} \left( \int_{B(x_0, r)} |\widehat{v}(x, t)|^{\frac{2s}{2-s}} dx \right)^{\frac{2-s}{s}} dt \right)^{\frac{1}{2}} + C \frac{r^3}{(r-\rho)^8} \left( \int_{Q(z_0, r)} |q| dz \right)^2 \end{aligned} \quad (3.12)$$

and use for its right-hand side (3.11). Together with choosing  $r = \frac{R_1 + \rho}{2}$  we arrive at

$$\begin{aligned} \int_{Q(z_0, \rho)} |\nabla v|^2 dz &\leq \frac{1}{2} \int_{Q(z_0, R_1)} |\nabla v|^2 dz \\ &+ C(s)(\Gamma^5(z_0, R_1) + 1)R_1^{4(\frac{3}{s}-1)} \frac{R_1^{20}}{(R_1 - \rho)^{20}} \left( \int_{Q(z_0, R_1)} |\nabla v|^l dz \right)^{\frac{2}{l}} \\ &+ C \frac{R_1^3}{(R_1 - \rho)^8} \left( \int_{Q(z_0, R_1)} |q| dz \right)^2, \end{aligned} \quad (3.13)$$

valid for any  $R_1 > \rho$  such that  $Q(x_0, R_1) \Subset Q_{T_1, T_2}$ .

In order to deal with the first term on the right-hand side of (3.13), let us use the following lemma.

**Lemma 3.2.** *For  $0 \leq t_0 < t_1$ , let  $h : [t_0, t_1] \rightarrow \mathbb{R}$  be a nonnegative bounded function. Suppose that there exists  $\delta \in [0, 1)$  such that for any  $t_0 \leq t < s \leq t_1$  the following inequality is valid:*

$$h(t) \leq \delta h(s) + \sum_{i=1}^N \frac{A_i(s)}{(s-t)^{\alpha_i}},$$

in which  $\alpha_i \geq 0$ ,  $A_i : [t_0, t_1] \rightarrow \mathbb{R}$  is a bounded increasing function,  $i = 1, \dots, N$ . Then, there exists a constant  $C_\delta$  such that for any  $t_0 \leq t < s \leq t_1$

$$h(t) \leq C_\delta \sum_{i=1}^N \frac{A_i(s)}{(s-t)^{\alpha_i}}.$$

The proof is the same as in the classical case of constant  $A_i$ 's, see p. 161 of Giaquinta [5].

Invoking Lemma 3.2 with

$$\begin{aligned} h(\rho) &= \int_{Q_\rho(z_0)} |\nabla v|^2 dz, \\ A_1(\rho) &= C(s)(\Gamma^5(z_0, \rho) + 1)\rho^{20+4(\frac{3}{s^*}-\frac{1}{t})} \left( \int_{Q_\rho(z_0)} |\nabla v|^l dz \right)^{\frac{2}{t}}, \quad \alpha_1 = 20, \\ A_2(\rho) &= \rho^3 \left( \int_{Q_\rho(z_0)} |q| dz \right)^2, \quad \alpha_2 = 8, \end{aligned}$$

we dispose of the first term on the right-hand side of (3.13). Consequently, choosing  $R_1 = 2\rho$  we have

$$\begin{aligned} & \frac{1}{|Q(\rho)|} \int_{Q(z_0, \rho)} |\nabla v|^2 dz \\ & \leq C(s)(\Gamma^5(z_0, 2\rho) + 1)\rho^{4(-\frac{1}{p}+\frac{3}{s^*})}\rho^{-5}\rho^{\frac{10}{p}} \left( \frac{1}{|Q(2\rho)|} \int_{Q(z_0, 2\rho)} |\nabla v|^l dz \right)^{\frac{2}{t}} \\ & \quad + C \left( \frac{1}{|Q(2\rho)|} \int_{Q(z_0, 2\rho)} |q| dz \right)^2, \end{aligned}$$

which in tandem with (3.8) and  $s \in (1, 6/5)$  implies (2.2). Proposition 2.4 is proven.

#### §4. PROOF OF THEOREM 2.6

For simplicity of the Calderón-Zygmund argument below, let us use in what follows both the usual (parabolic) cylinders  $Q(z_0, R) = B(x_0, R) \times ]t_0 - R^2, t_0[$  and the related (parabolic) cubes  $C(z_0, R) = \{\max_{i=1,2,3} |x^i - x_0^i| < R\} \times ]t_0 - R^2, t_0[$ .

Let us introduce

**Definition 4.1** (Local maximal function). *Let  $G \subset \mathbb{R}^d$  be a fixed open set and  $f \in L_1(G)$ . The local maximal function  $M_G$  is given by*

$$(M_G f)(z) = \sup \left\{ (|f|)_C : \text{cubes } C \text{ such that } z \in C \subset G \right\},$$

where  $(g)_\omega$  denotes the mean value of  $g$  in  $\omega$ .

The following is true

**Lemma 4.2.** *Let  $C_0$  be a parabolic cube. Then*

$$2^{-9} \int_{C_0} (M_{C_0} f) dz \leq \int_{C_0} |f| \log \left( e + \frac{|f|}{(|f|)_{C_0}} \right) dz \leq 2^9 \int_{C_0} (M_{C_0} f) dz.$$

This result is classical in the case of the centred maximal function  $M$  on  $\mathbb{R}^d$ , under an additional restriction that  $f$  is compactly supported, see Theorem 1 of Stein [19]. Lifting the compact support assumption by using the local maximal function  $M_G$  seems virtually untapped in applications for PDEs, despite being apparently useful (in our case, trying to produce compactly supported functions, one may try to e.g. double-localise the estimates, which results in a scaling mismatch on the whole space). A range of results closely related to Lemma 4.2 can be found in works by Iwaniec with coauthors, e.g. [1, 6–8]. Since these papers are inspired however more by geometry-related considerations, the needed by us result seems not to be explicitly stated there. Let us therefore present the proof of Lemma 4.2, emphasising that it was essentially provided to us by Piotr Hajłasz. To this end we need the following Calderón-Zygmund decomposition on cubes

**Lemma 4.3.** *Let  $C_0$  be a parabolic cube and  $f \in L_1(C_0)$ . Fix any  $t \geq (|f|)_{C_0}$ . Then there exists a sequence of pairwise disjoint parabolic cubes  $\{C^i\}$ ,  $C^i \subset C_0$ ,  $i \in \mathbb{N}$ , such that*

$$|f| \leq t \quad \text{almost everywhere on } C_0 \setminus \bigcup_{i \in \mathbb{N}} C^i \quad (4.1a)$$

$$t < (|f|)_{C^i} \leq 2^8 t. \quad (4.1b)$$

The only difference from the classical proof as in Stein [20] §I.3.2 is a bigger constant of (4.1b), related to the parabolicity of cubes.

**Proof of Lemma 4.2.** Let us define  $E_t = \{z \in C_0 : (M_{C_0} f)(z) > t\}$ . In the setting of Lemma 4.3, the left inequality of (4.1b) implies  $\bigcup_{i \in \mathbb{N}} C^i \subset E_t$ .

Hence

$$\mu(E_t) \geq \sum_{i \in \mathbb{N}} \mu(C^i) \geq 2^{-8} \sum_{i \in \mathbb{N}} \frac{1}{t} \int_{C^i} |f| dz = 2^{-8} \frac{1}{t} \int_{\bigcup_{i \in \mathbb{N}} C^i} |f| dz,$$

with the latter inequality given by the right inequality of (4.1b). Since Lemma 4.3 implies also that  $\bigcup_{i \in \mathbb{N}} C^i \supset \{z \in C_0 : |f| > t\}$  up to a zero-measure set (considering (4.1a) and complements), we have in tandem with the above inequality that

$$\mu(E_t) \geq 2^{-8} \frac{1}{t} \int_{\{z \in C_0 : |f| > t\}} |f| dz, \quad (4.2)$$

valid for any  $t \geq (|f|)_{C_0} =: \Lambda$ . It holds

$$\begin{aligned} 2^8 \int_{C_0} M_{C_0} f dz &= 2^8 \int_0^\infty \mu(E_t) dt \geq \Lambda \mu(E_\Lambda) + \int_\Lambda^\infty \mu(E_t) dt \\ &\geq \int_{\{z \in C_0 : |f| > \Lambda\}} |f| dz + \int_\Lambda^\infty \frac{1}{t} \left( \int_{\{z \in C_0 : |f| > t\}} |f| dz \right) dt, \end{aligned}$$

see (4.2) for the last inequality. We estimate the last integral above with help of the Tonelli theorem and find that

$$\begin{aligned} 2^8 \int_{C_0} M_{C_0} f dz &\geq \int_{\{z \in C_0 : |f| > \Lambda\}} |f| dz + \int_{\{z \in C_0 : |f| > \Lambda\}} |f| \log \frac{|f|}{\Lambda} dz \\ &\geq 2^{-1} \int_{\{z \in C_0 : |f| > \Lambda\}} |f| \log \left( e + \frac{|f|}{\Lambda} \right) dz. \end{aligned}$$

Since also

$$\begin{aligned} 2^9 \int_{C_0} M_{C_0} f dz &\geq 2^9 \int_{\{z \in C_0 : |f| \leq \Lambda\}} |f| dz \\ &\geq \int_{\{z \in C_0 : |f| \leq \Lambda\}} |f| \log(e+1) dz \geq \int_{\{z \in C_0 : |f| \leq \Lambda\}} |f| \log \left( e + \frac{|f|}{\Lambda} \right) dz, \end{aligned}$$

we have the right (less standard) inequality of the thesis. The remaining left inequality follows in fact from the original [19]. Indeed, also for the local maximal function, one has the usual weak-type estimate (i.e. a practical reverse to (4.2))

$$\mu(E_t) \leq 2^8 \frac{1}{t} \int_{\{z \in C_0 : |f| > \frac{t}{2}\}} |f|,$$

by the Vitali covering of  $E_t$ . Along the previous lines utilising (4.2), with inequalities reversed, we prove now the remaining left inequality of Lemma 4.2 (in fact, not needed for our further purposes).  $\square$

Let us return to the proof of Theorem 2.6. We fix a parabolic cube  $C_1 = C(z_1, R_1)$  such that  $C'_1 = C(z_1, 3R_1) \Subset Q_{T_1, T_2}$ . Proposition 2.4, rewritten for cubes, yields for  $\Gamma_1 = \Gamma(z_0, 2R_1)$

$$\begin{aligned} \frac{1}{|C(\rho)|} \int_{C(z_0, \rho)} |\nabla v|^2 dz &\leq c(l)(\Gamma_1^5 + 1) \left( \frac{1}{|C(2\sqrt{2}\rho)|} \int_{C(z_0, 2\sqrt{2}\rho)} |\nabla v|^l dz \right)^{\frac{2}{l}} \\ &\quad + c \left( \frac{1}{|C(2\sqrt{2}\rho)|} \int_{C(z_0, 2\sqrt{2}\rho)} |q| dz \right)^2 \end{aligned}$$

for all  $C(z_0, \rho) \subset C_1$ .

Since all the domains of integration of the right-hand remain within in  $C(z_1, 2\sqrt{2}R_1)$ , we can introduce there into integrals a smooth function  $\psi$  such that  $\psi \equiv 1$  on  $C(z_1, 2\sqrt{2}R_1)$  and  $\psi \equiv 0$  outside  $C'_1$ . Hence

$$\begin{aligned} \frac{1}{|C(\rho)|} \int_{C(z_0, \rho)} |\nabla v|^2 dz &\leq c(l)(\Gamma_1^5 + 1) \left( \frac{1}{|C(2\sqrt{2}\rho)|} \int_{C(z_0, 2\sqrt{2}\rho)} |\nabla v|^l \psi dz \right)^{\frac{2}{l}} \\ &\quad + c \left( \frac{1}{|C(2\sqrt{2}\rho)|} \int_{C(z_0, 2\sqrt{2}\rho)} |q| \psi dz \right)^2 \end{aligned}$$

for all  $C(z_0, \rho) \subset C_1$ .

Recalling Definiton 4.1 we have then

$$\begin{aligned} M_{C_1}(|\nabla v|^2)(z) &\leq c(l)(\Gamma_1^5 + 1) M_{C(z_1, 2\sqrt{2}R_1)}^{\frac{2}{l}}(|\nabla v|^l \psi)(z) + c M_{C(z_1, 2\sqrt{2}R_1)}^2(|q| \psi)(z) \\ &\leq c(l)(\Gamma_1^5 + 1) M_{\mathbb{R}^4}^{\frac{2}{l}}(|\nabla v|^l \psi)(z) + c M_{\mathbb{R}^4}^2(|q| \psi)(z) \end{aligned}$$

and consequently

$$\int_{C_1} M_{C_1}(|\nabla v|^2) dz \leq c(l)(\Gamma_1^5 + 1) \int_{\mathbb{R}^4} M_{\mathbb{R}^4}^{\frac{2}{l}}(|\nabla v|^l \psi) dz + c \int_{\mathbb{R}^4} M_{\mathbb{R}^4}^2(|q| \psi) dz.$$



Observe that  $M_{\mathbb{R}^4}$  is the usual non-centred maximal function with respect to parabolic cubes. Since it enjoys the strong  $L_p$ -property, compare [21] §I.3.1, the above inequality implies

$$\begin{aligned} \int_{C_1} M_{C_1}(|\nabla v|^2) dz &\leq c(l)(\Gamma_1^5 + 1) \int_{\mathbb{R}^4} |\nabla v|^2 \psi^{\frac{2}{7}} dz + c \int_{\mathbb{R}^4} |q|^2 \psi^2 dz \\ &\leq c(l)(\Gamma_1^5 + 1) \int_{C'_1} |\nabla v|^2 dz + c \int_{C'_1} |q|^2 dz < \infty, \end{aligned}$$

hence Lemma 4.2 yields

$$\int_{C_1} |\nabla v| \log \left( e + \frac{|\nabla v|}{(|\nabla v|)_{C_1}} \right) dz \leq 2^9 c(l)(\Gamma_1^5 + 1) \int_{C'_1} |\nabla v|^2 dz + 2^9 c \int_{C'_1} |q|^2 dz.$$

Returning to parabolic cylinders gives Theorem 2.6.

## §5. APPENDIX I

Here, we are going to prove (1.4). Indeed, we have

$$(D\nabla u) : \nabla v = u_{i,l} d_{jl} v_{i,j} = u_{i,l} \epsilon_{jls} v_{i,j} \omega_s.$$

Since  $\omega$  is an  $BMO$  function, it suffices to show that for any  $s = 1, 2, 3$ , the function

$$x \mapsto u_{i,l}(x) \epsilon_{jls} v_{i,j}(x)$$

belongs to the Hardy space and to find the corresponding estimates, compare e.g. §VII.3 of [20] about duality between Hardy and  $BMO$  spaces. To this end, let us fix a standard mollifier  $\Phi_\varrho$  and consider the function

$$H_s(x) := \sup_{\varrho > 0} |(\Phi_\varrho \star (u_{i,l} \epsilon_{jls} v_{i,j}))(x)|.$$

Taking into account properties of the Levi-Civita tensor, we have

$$H_s(x) = \sup_{\varrho > 0} |(\Phi_\varrho \star (\bar{u}_i \epsilon_{jls} v_{i,j}), l)(x)|,$$

where  $\bar{u} = u - [u]_{B(x,\varrho)}$ . After integration by parts and applying the estimate  $|\nabla \Phi_\varrho| \leq c\varrho^{-4}$ , we find

$$H_s(x) \leq \sup_{\varrho > 0} \frac{c}{\varrho} \frac{1}{|B(\varrho)|} \int_{B(x,\varrho)} |\bar{u}| |\nabla v| dy$$

$$\leq \frac{c}{\varrho} \left( \frac{1}{|B(\varrho)|} \int_{B(x, \varrho)} |\bar{u}|^3 dy \right)^{\frac{1}{3}} \times \left( \frac{1}{|B(\varrho)|} \int_{B(x, \varrho)} |\nabla v|^{\frac{3}{2}} dy \right)^{\frac{2}{3}}.$$

Now, we can use Poincaré-Sobolev inequality and pass to the standard centred (Hardy-Littlewood) maximal functions, denoted by  $M$ , thus obtaining

$$\begin{aligned} H_s(x) &\leq c \sup_{\varrho > 0} \left( \frac{1}{|B(\varrho)|} \int_{B(x, \varrho)} |\nabla u|^{\frac{3}{2}} dy \right)^{\frac{2}{3}} \left( \frac{1}{|B(\varrho)|} \int_{B(x, \varrho)} |\nabla v|^{\frac{3}{2}} dy \right)^{\frac{2}{3}} \\ &\leq c M^{\frac{2}{3}}(|\nabla u|^{\frac{3}{2}})(x) M^{\frac{2}{3}}(|\nabla v|^{\frac{3}{2}})(x). \end{aligned}$$

Integration over  $\mathbb{R}^3$ , together with  $L_p$ -estimates for maximal functions gives us

$$\begin{aligned} \|H_s\|_1 &\leq c \left( \int_{\mathbb{R}^3} M^{\frac{4}{3}}(|\nabla u|^{\frac{3}{2}})(x) dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} M^{\frac{4}{3}}(|\nabla v|^{\frac{3}{2}})(x) dx \right)^{\frac{1}{2}} \\ &\leq c \|\nabla u\|_2 \|\nabla v\|_2, \end{aligned}$$

for any  $s = 1, 2, 3$ . Therefore, by definition, for any  $s = 1, 2, 3$ ,  $H_s$  belongs to the Hardy space. Now, estimate (1.4) follows from duality between Hardy and  $BMO$  spaces.

## §6. APPENDIX II

Here we state an existence theorem for the Cauchy problem for system (1.1), compare Remark 2.2. To this end we need to introduce certain energy spaces. First, we let

$$C_{0,0}^\infty(\Omega) = \{v \in C_0^\infty(\Omega) : \operatorname{div} v = 0\}$$

and then

$$\overset{\circ}{J}_p(\Omega) = [C_{0,0}^\infty(\Omega)]^{L_p(\Omega)},$$

$\overset{\circ}{J}_p^1(\Omega)$  is the closure of the set  $C_{0,0}^\infty(\Omega)$  with respect to the semi-norm

$$|v|_{p,1,\Omega} = \left( \int_{\Omega} |\nabla v|^p dx \right)^{\frac{1}{p}}.$$

If  $\Omega = \mathbb{R}^3$ , we shall drop  $\Omega$  in the notation of the spaces. We denote  $\mathbb{R}^3 \times \mathbb{R}_+$  briefly by  $Q_+$ .

**Theorem 6.1.** *Given a divergence free drift  $b \in L_\infty(BMO)^{-1}$  and initial velocity  $u_0 \in \mathring{J}_2$ , there exists a unique pair  $v$  and  $q$  satisfying the following properties:*

- (i)  $v \in L_\infty(0, \infty; \mathring{J}_2) \cap L_2(0, \infty; \mathring{J}_2^1)$ ,  $q \in L_2(Q_+)$ ;
- (ii)  $v$  and  $q$  satisfy the problem (1.1) in the sense of distributions;
- (iii) the function

$$t \mapsto \int_{\mathbb{R}^3} v(x, t) \cdot w(x) dx$$

is continuous at any  $t \geq 0$  for each  $w \in L_2(\mathbb{R}^3)$ ;

- (iv)  $\|v(\cdot, t) - u_0(\cdot)\|_2 \rightarrow 0$  as  $t \rightarrow 0$ ;
- (v) for all  $t \geq 0$

$$\frac{1}{2} \int_{\mathbb{R}^3} |v(x, t)|^2 dx + \int_0^t \int_{\mathbb{R}^3} |\nabla v|^2 dx dt' \leq \frac{1}{2} \int_{\mathbb{R}^3} |u_0|^2 dx;$$

- (vi) for all  $t \geq 0$

$$\begin{aligned} & \int_{\mathbb{R}^3} \varphi |v(x, t)|^2 dx + 2 \int_0^t \int_{\mathbb{R}^3} \varphi |\nabla v|^2 dx dt' \\ &= \int_0^t \int_{\mathbb{R}^3} (|v|^2 (\partial_t + \Delta) \varphi - 2 D \nabla v : v \otimes \nabla \varphi + 2 q v \cdot \nabla \varphi) dx dt' \end{aligned}$$

for any non-negative  $\varphi \in C_0^\infty(Q_+)$ .

The proof of the theorem relies essentially on the estimate (1.4). Observe that it is also applicable to the pressure equation

$$-\Delta q = \operatorname{div} (D \nabla v)$$

hence gives the estimate for the pressure

$$\|q\|_{2, Q_+} \leq c \|d\|_{L_\infty(BMO)} \|\nabla v\|_{2, Q_+}.$$

Further details are standard.

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Поступило 18 июля 2017 г.

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