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ON $\mathcal{Z}_p$-NORMS OF RANDOM VECTORS

Abstract. To any n-dimensional random vector X we may associate its L_p-centroid body $\mathcal{Z}_p(X)$ and the corresponding norm. We formulate a conjecture concerning the bound on the $\mathcal{Z}_p(X)$-norm of X and show that it holds under some additional symmetry assumptions. We also relate our conjecture with estimates of covering numbers and Sudakov-type minoration bounds.

§1. Introduction. Formulation of the Problem

Let $p \geqslant 2$ and $X = (X_1, \ldots, X_n)$ be a random vector in $\mathbb{R}^n$ such that $\mathbb{E}|X|^p < \infty$. We define the following two norms on $\mathbb{R}^n$:

$$\|t\|_{\mathcal{M}_p(X)} := (\mathbb{E}|\langle t, X\rangle|^p)^{1/p}$$

and

$$\|t\|_{\mathcal{Z}_p(X)} := \sup\{|\langle t, s\rangle|\colon\ \|s\|_{\mathcal{M}_p(X)} \leqslant 1\}.$$

By $\mathcal{M}_p(X)$ and $\mathcal{Z}_p(X)$ we will also denote unit balls in these norms, i.e.,

$$\mathcal{M}_p(X) := \{t \in \mathbb{R}^n\colon\ \|t\|_{\mathcal{M}_p(X)} \leqslant 1\}$$

and

$$\mathcal{Z}_p(X) := \{t \in \mathbb{R}^n\colon\ \|t\|_{\mathcal{Z}_p(X)} \leqslant 1\}.$$

The set $\mathcal{Z}_p(X)$ is called the L_p-centroid body of X (or rather of the distribution of X). It was introduced (under a different normalization) for uniform distributions on convex bodies in [9]. Investigation of L_p-centroid bodies played a crucial role in the Paouris proof of large deviations bounds for Euclidean norms of log-concave vectors [10]. Such bodies also appear in questions related to the optimal concentration of log-concave vectors [7].

Let us introduce a bit of useful notation. We set $|t| := \|t\|_2 = \sqrt{\langle t, t\rangle}$ and $B_2^n = \{t \in \mathbb{R}^n\colon |t| \leqslant 1\}$. By $\|Y\|_p = (\mathbb{E}|Y|^p)^{1/p}$ we denote the L_p-norm of a random variable Y. Letter C denotes universal constants (that may differ at each occurence), we write $f \sim g$ if $\frac{1}{C} f \leqslant g \leqslant Cf$.

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Let us begin with a simple case, when a random vector X is rotationally invariant. Then $X = RU$, where U has a uniform distribution on S^{n-1} and $R = |X|$ is a nonnegative random variable, independent of U. We have for any vector $t \in \mathbb{R}^n$ and $p \geqslant 2$,

$$\|\langle t, U\rangle\|_p = |t|\|U_1\|_p \sim \sqrt{\frac{p}{n+p}}|t|,$$

where U_1 is the first coordinate of U. Therefore

$$\|t\|_{\mathcal{M}_p(X)} = \|U_1\|_p\|R\|_p|t| \quad \text{and} \quad \|t\|_{\mathcal{Z}_p(X)} = \|U_1\|_p^{-1}\|R\|_p^{-1}|t|.$$

So

$$\left(\mathbb{E}\|X\|^p_{\mathcal{Z}_p(X)}\right)^{1/p} = \|U_1\|_p^{-1}\|R\|_p^{-1}(\mathbb{E}|X|^p)^{1/p} = \|U_1\|_p^{-1} \sim \sqrt{\frac{n+p}{p}}. \quad (1)$$

This motivates the following problem.

Problem 1. Is it true that for (at least a large class of) centered n-dimensional random vectors X,

$$\left(\mathbb{E}\|X\|^2_{\mathcal{Z}_p(X)}\right)^{1/2} \leqslant C\sqrt{\frac{n+p}{p}} \quad \text{for } p \geqslant 2,$$

or maybe even

$$\left(\mathbb{E}\|X\|^p_{\mathcal{Z}_p(X)}\right)^{1/p} \leqslant C\sqrt{\frac{n+p}{p}} \quad \text{for } p \geqslant 2?$$

Notice that the problem is linearly-invariant, since

$$\|AX\|_{\mathcal{Z}_p(AX)} = \|X\|_{\mathcal{Z}_p(X)} \quad \text{for any } A \in \mathrm{GL}(n). \quad (2)$$

For any centered random vector X with nondegenerate covariance matrix, random vector $Y = \mathrm{Cov}(X)^{-1/2}X$ is isotropic (i.e., centered with identity covariance matrix). We have $\mathcal{M}_2(Y) = \mathcal{Z}_2(Y) = B_2^n$, hence

$$\mathbb{E}\|X\|^2_{\mathcal{Z}_2(X)} = \mathbb{E}\|Y\|^2_{\mathcal{Z}_2(Y)} = \mathbb{E}|Y|^2 = n.$$

Next remark shows that the answer to our problem is positive in the case $p \geqslant n$.

Remark 1. For $p \geqslant n$ and any n-dimensional random vector X we have $(\mathbb{E}\|X\|^p_{\mathcal{Z}_p(X)})^{1/p} \leqslant 10$.

Proof. Let S be a $1/2$-net in the unit ball of $\mathcal{M}_p(X)$ such that $|S| \leqslant 5^n$ (such net exists by the volume-based argument, cf. [1, Corollary 4.1.15]). Then

$$(\mathbb{E}\|X\|^p_{\mathcal{Z}_p(X)})^{1/p} \leqslant 2\Big(\mathbb{E}\sup_{t\in S}|\langle t,X\rangle|^p\Big)^{1/p} \leqslant 2\Big(\mathbb{E}\sum_{t\in S}|\langle t,X\rangle|^p\Big)^{1/p}$$
$$\leqslant 2|S|^{1/p}\sup_{t\in S}(\mathbb{E}\langle t,X\rangle|^p)^{1/p} \leqslant 2\cdot 5^{n/p}. \qquad \square$$

L_p-centroid bodies play an important role in the study of vectors uniformly distributed on convex bodies and a more general class of log-concave vectors. A random vector with a nondenerate covariance matrix is called log-concave if its density has the form e^{-h}, where $h\colon\ \mathbb{R}^n \to (-\infty,\infty]$ is convex. If X is centered and log-concave then

$$\|\langle t,X\rangle\|_p \leqslant \lambda\frac{p}{q}\|\langle t,X\rangle\|_q \quad \text{ for } p \geqslant q \geqslant 2, \tag{3}$$

where $\lambda = 2$ ($\lambda = 1$ if X is symmetric and log-concave and $\lambda = 3$ for arbitrary log-concave vectors). One of open problems for log-concave vectors [7] states that for such vectors, arbitrary norm $\| \ \|$ and $q \geqslant 1$,

$$(\mathbb{E}\|X\|^q)^{1/q} \leqslant C\Big(\mathbb{E}\|X\| + \sup_{\|t\|_*\leqslant 1}\|\langle t,X\rangle\|_q\Big).$$

In particular one may expect that for log-concave vectors

$$(\mathbb{E}\|X\|^q_{\mathcal{Z}_p(X)})^{1/q} \leqslant C\Big(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)} + \sup_{t\in\mathcal{M}_p(X)}\|\langle t,X\rangle\|_q\Big)$$
$$\leqslant C\Big(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)} + \frac{\max\{p,q\}}{p}\Big).$$

As a result it is natural to state the following variant of Problem 1.

Problem 2. Let X be a centered log-concave n-dimensional random vector. Is it true that

$$(\mathbb{E}\|X\|^q_{\mathcal{Z}_p(X)})^{1/q} \leqslant C\sqrt{\frac{n}{p}} \quad \text{ for } 2 \leqslant p \leqslant n,\ 1 \leqslant q \leqslant \sqrt{pn}.$$

In Section 2 we show that Problems 1 and 2 have affirmative solutions in the class of unconditional vectors. In Section 3 we relate our problems to estimates of covering numbers. We also show that the first estimate in Problem 1 holds if the random vector X satisfies the Sudakov-type minoration bound.

§2. Bounds for unconditional random vectors

In this section we consider the class of *unconditional* random vectors in $\mathbb{R}^n$, i.e., vectors X having the same distribution as

$$(\varepsilon_1|X_1|, \varepsilon_2|X_2|, \ldots, \varepsilon_n|X_n|),$$

where (ε_i) is a sequence of independent symmetric ± 1 random variables (Rademacher sequence), independent of X.

Our first result shows that formula (1) may be extended to the unconditional case for p even. We use the standard notation – for a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$, $x \in \mathbb{R}^n$ and $m = \sum \alpha_i$, $x^\alpha := \prod_i x_i^{\alpha_i}$ and $\binom{m}{\alpha} := m!/(\prod_i \alpha_i!)$.

Proposition 2. *We have for any $k = 1, 2, \ldots$ and any n-dimensional unconditional random vector X such that $\mathbb{E}|X|^{2k} < \infty$,*

$$\Big(\mathbb{E}\|X\|^{2k}_{\mathcal{Z}_{2k}(X)}\Big)^{1/(2k)} \leqslant c_{2k} := \left(\sum_{\|\alpha\|_1 = k} \frac{\binom{k}{\alpha}^2}{\binom{2k}{2\alpha}}\right)^{1/(2k)} \sim \sqrt{\frac{n+k}{k}},$$

where the summation runs over all multiindices $\alpha = (\alpha_1, \ldots, \alpha_n)$ with nonnegative integer coefficients such that $\|\alpha\|_1 = \sum_{i=1}^n \alpha_i = k$.

Proof. Observe first that

$$\mathbb{E}|\langle t, X\rangle|^{2k} = \mathbb{E}\left|\sum_{i=1}^n t_i \varepsilon_i X_i\right|^{2k} = \sum_{\|\alpha\|_1 = k} \binom{2k}{2\alpha} t^{2\alpha} \mathbb{E}X^{2\alpha}.$$

For any $t, s \in \mathbb{R}^n$ we have

$$|\langle t, s\rangle|^k = \sum_{\|\alpha\|_1 = k} \binom{k}{\alpha} t^\alpha s^\alpha.$$

So by the Caushy–Schwarz inequality,

$$\|s\|^k_{\mathcal{Z}_{2k}(X)} = \sup\{|\langle t, s\rangle|^k : \mathbb{E}|\langle t, X\rangle|^{2k} \leqslant 1\} \leqslant \left(\sum_{\|\alpha\|_1 = k} \frac{\binom{k}{\alpha}^2}{\binom{2k}{2\alpha}} \frac{s^{2\alpha}}{\mathbb{E}X^{2\alpha}}\right)^{1/2}.$$

To see that $c_{2k} \sim \sqrt{(n+k)/k}$ observe that

$$\frac{\binom{k}{\alpha}^2}{\binom{2k}{2\alpha}} = \binom{2k}{k}^{-1} \prod_{i=1}^n \binom{2\alpha_i}{\alpha_i}.$$

Therefore, since $1 \leqslant \binom{2l}{l} \leqslant 2^{2l}$, we get

$$4^{-k}\binom{n+k-1}{k} \leqslant c_{2k}^{2k} \leqslant 4^{k}\binom{n+k-1}{k}. \quad \square$$

Corollary 3. *Let X be an unconditional n-dimensional random vector. Then*

$$\Big(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}^{2k}\Big)^{1/2k} \leqslant C\sqrt{\frac{n+p}{p}} \quad \textit{for any positive integer } k \leqslant \frac{p}{2}.$$

Proof. By the monotonicity of L_{2k}-norms we may and will assume that $k = \lfloor p/2 \rfloor$. Then by Proposition 2,

$$\Big(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}^{2k}\Big)^{1/2k} \leqslant \Big(\mathbb{E}\|X\|_{\mathcal{Z}_{2k}(X)}^{2k}\Big)^{1/2k} \leqslant C\sqrt{\frac{n+k}{k}} \leqslant C\sqrt{\frac{n+p}{p}}. \quad \square$$

In the unconditional log-concave case we may bound higher moments of $\|X\|_{\mathcal{Z}_p(X)}$.

Theorem 4. *Let X be an unconditional log-concave n-dimensional random vector. Then for $p, q \geqslant 2$,*

$$(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}^{q})^{1/q} \leqslant C\bigg(\sqrt{\frac{n+p}{p}} + \sup_{t\in\mathcal{M}_p(X)} \|\langle t, X\rangle\|_q\bigg) \leqslant C\bigg(\sqrt{\frac{n+p}{p}} + \frac{q}{p}\bigg).$$

In order to show this result we will need the following lemma.

Lemma 5. *Let $2 \leqslant p \leqslant n$, X be an unconditional random vector in $\mathbb{R}^n$ such that $\mathbb{E}|X|^p < \infty$ and $\mathbb{E}|X_i| = 1$. Then*

$$\|s\|_{\mathcal{Z}_p(X)} \leqslant \sup_{\substack{I\subset[n],\\ |I|\leqslant p}} \ \sup_{\substack{\|t\|_{\mathcal{M}_p(X)}\\ \leqslant 1}} \Big|\sum_{i\in I} t_i s_i\Big| + C_1 \sup_{\substack{\|t\|_{\mathcal{M}_p(X)}\leqslant 1,\\ \|t\|_2 \leqslant p^{-1/2}}} \Big|\sum_{i=1}^{n} t_i s_i\Big|. \qquad (4)$$

Proof. We have by the unconditionality of X and Jensen's inequality,

$$\|t\|_{\mathcal{M}_p(X)} = \Big\|\sum_{i=1}^{n} t_i \varepsilon_i |X_i|\Big\|_p \geqslant \Big\|\sum_{i=1}^{n} t_i \varepsilon_i \mathbb{E}|X_i|\Big\|_p.$$

By the result of Hitczenko [5], for numbers $a_1, \ldots, a_n$,

$$\Big\|\sum_{i=1}^{n} a_i \varepsilon_i\Big\|_p \sim \sum_{i\leqslant p} a_i^* + \sqrt{p}\Big(\sum_{i>p} |a_i^*|^2\Big)^{1/2}, \qquad (5)$$

where $(a_i^*)_{i\leqslant n}$ denotes the nonincreasing rearrangement of $(|a_i|)_{i\leqslant n}$. Thus

$$\sqrt{p}\Big(\sum_{i>p}|t_i^*|^2\Big)^{1/2}\leqslant C_1\|t\|_{\mathcal{M}_p(X)}$$

and (4) easily follows. □

Proof of Theorem 4. The last bound in the assertion follows by (3). It is easy to see that (increasing q if necessary) it is enough to consider the case $q\geqslant\sqrt{np}$.

If $q\geqslant n$ then the similar argument as in the proof of Remark 1 shows that

$$\Big(\mathbb{E}\|X\|^q_{\mathcal{Z}_p(X)}\Big)^{1/q}\leqslant 2\cdot 5^{n/q}\sup_{t\in\mathcal{M}_p(X)}\|\langle t,X\rangle\|_q\leqslant 10\sup_{t\in\mathcal{M}_p(X)}\|\langle t,X\rangle\|_q.$$

Finally, consider the remaining case $\sqrt{pn}\leqslant q\leqslant n$. By (2) we may assume that $\mathbb{E}|X_i|=1$ for all i. By the log-concavity

$$\|\langle t,X\rangle\|_{q_1}\leqslant C\frac{q_1}{q_2}\|\langle t,X\rangle\|_{q_2}$$

for $q_1\geqslant q_2\geqslant 1$, in particular $\sigma_i:=\|X_i\|_2\leqslant C$.

Let $\mathcal{E}_1,\ldots,\mathcal{E}_n$ be i.i.d. symmetric exponential random variables with variance 1. By [6, Theorem 3.1] we have

$$\Big\|\sup_{\substack{\|t\|_{\mathcal{M}_p(X)}\leqslant 1,\\ \|t\|_2\leqslant p^{-1/2}}}\Big|\sum_{i=1}^n t_iX_i\Big|\Big\|_q \leqslant C\bigg(\Big\|\sup_{\substack{\|t\|_{\mathcal{M}_p(X)}\leqslant 1,\\ \|t\|_2\leqslant p^{-1/2}}}\Big|\sum_{i=1}^n t_i\sigma_i\mathcal{E}_i\Big|\Big\|_1+\sup_{\substack{\|t\|_{\mathcal{M}_p(X)}\leqslant 1,\\ \|t\|_2\leqslant p^{-1/2}}}\|\langle t,X\rangle\|_q\bigg).$$

We have

$$\sup_{\substack{\|t\|_{\mathcal{M}_p(X)}\leqslant 1,\\ \|t\|_2\leqslant p^{-1/2}}}\|\langle t,X\rangle\|_q\leqslant\sup_{\|t\|_{\mathcal{M}_p(X)}\leqslant 1}\|\langle t,X\rangle\|_q$$

and

$$\Big\|\sup_{\substack{\|t\|_{\mathcal{M}_p(X)}\leqslant 1,\\ \|t\|_2\leqslant p^{-1/2}}}\Big|\sum_{i=1}^n t_i\sigma_i\mathcal{E}_i\Big|\Big\|_1\leqslant\frac{1}{\sqrt{p}}\Big\|\sqrt{\sum_{i=1}^n\sigma_i^2\mathcal{E}_i^2}\Big\|_1\leqslant\frac{1}{\sqrt{p}}\sqrt{\sum_{i=1}^n\sigma_i^2}\leqslant C\sqrt{\frac{n}{p}}.$$

Thus

$$\Big\| \sup_{\substack{\|t\|_{\mathcal{M}_p(X)}\leqslant 1,\\ \|t\|_2\leqslant p^{-1/2}}} \Big|\sum_{i=1}^n t_iX_i\Big|\Big\|_q \leqslant C\Big(\sqrt{\frac{n}{p}}+\sup_{\|t\|_{\mathcal{M}_p(X)}\leqslant 1}\|\langle t,X\rangle\|_q\Big).$$

Let for each $I\subset[n]$, $P_IX=(X_i)_{i\in I}$ and S_I be a 1/2-net in $\mathcal{M}_p(P_IX)$ of cardinality at most $5^{|I|}$. We have

$$\begin{aligned}\Big\|\sup_{\substack{I\subset[n],\\|I|\leqslant p}}\ \sup_{\substack{\|t\|_{\mathcal{M}_p(X)}\\ \leqslant 1}}\Big|\sum_{i\in I}t_iX_i\Big|\Big\|_q &\leqslant 2\Big\|\sup_{\substack{I\subset[n],\\|I|\leqslant p}}\ \sup_{t\in S_I}\Big|\sum_{i\in I}t_iX_i\Big|\Big\|_q\\ &\leqslant 2\Big(\sum_{\substack{I\subset[n],\\|I|\leqslant p}}\sum_{t\in S_I}\mathbb{E}\Big|\sum_{i\in I}t_iX_i\Big|^q\Big)^{1/q}\\ &\leqslant 2\cdot 5^{p/q}|\{I\subset[n],|I|\leqslant p\}|^{1/q}\sup_I\sup_{t\in S_I}\Big\|\sum_{i\in I}t_iX_i\Big\|_q\\ &\leqslant 10\left(\frac{en}{p}\right)^{p/q}\sup_{t\in\mathcal{M}_p(X)}\Big\|\sum_{i\in I}t_iX_i\Big\|_q\\ &\leqslant C\sup_{t\in\mathcal{M}_p(X)}\Big\|\sum_{i\in I}t_iX_i\Big\|_q,\end{aligned}$$

where the last estimate follows from $q\geqslant\sqrt{np}$.

Hence the assertion follows by Lemma 5. □

Corollary 6. *Let X be an unconditional log-concave n-dimensional random vector and $2\leqslant p\leqslant n$. Then*

$$\frac{1}{C}\sqrt{\frac{n}{p}}\leqslant\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}\leqslant\Big(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}^{\sqrt{np}}\Big)^{1/\sqrt{np}}\leqslant C\sqrt{\frac{n}{p}} \tag{6}$$

and

$$\mathbb{P}\left(\|X\|_{\mathcal{Z}_p(X)}\geqslant\frac{1}{C}\sqrt{\frac{n}{p}}\right)\geqslant\frac{1}{C},$$
$$\mathbb{P}\left(\|X\|_{\mathcal{Z}_p(X)}\geqslant Ct\sqrt{\frac{n}{p}}\right)\leqslant e^{-t\sqrt{np}}\quad\text{for}\quad t\geqslant 1.$$

Proof. The upper bound in (6) easily follows by Theorem 4. In fact we have for $t \geqslant 1$,

$$\Big(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}^{t\sqrt{np}}\Big)^{1/(t\sqrt{np})} \leqslant Ct\sqrt{\frac{n}{p}},$$

hence the Chebyshev inequality yields the upper tail bound for $\|X\|_{\mathcal{Z}_p(X)}$.

To establish lower bounds we may assume that X is additionally isotropic. Then by the result of Bobkov and Nazarov [3] we have

$$\|\langle t, X\rangle\|_p \leqslant C(\sqrt{p}\|t\|_2 + p\|t\|_\infty).$$

This easily gives

$$\mathbb{E}\|X\|_{\mathcal{Z}_p(X)} \geqslant \frac{1}{C}\sqrt{\frac{n}{p}}\mathbb{E}X^*_{\lceil n/2\rceil} \geqslant \frac{1}{C}\sqrt{\frac{n}{p}},$$

where the last inequality follows by Lemma 7 below.

By the Paley–Zygmund inequality we get

$$\mathbb{P}\bigg(\|X\|_{\mathcal{Z}_p(X)} \geqslant \frac{1}{C}\sqrt{\frac{n}{p}}\bigg) \geqslant \mathbb{P}\Big(\|X\|_{\mathcal{Z}_p(X)} \geqslant \frac{1}{2}\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}\Big) \geqslant \frac{(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)})^2}{4\mathbb{E}\|X\|^2_{\mathcal{Z}_p(X)}} \geqslant c. \qquad \square$$

Lemma 7. *Let X by a symmetric isotropic n-dimensional log-concave vector. Then $\mathbb{E}X^*_{\lceil n/2\rceil} \geqslant \frac{1}{C}$.*

Proof. Let $a_i > 0$ be such that $\mathbb{P}(X_i \geqslant a_i) = 3/8$. Then by the log-concavity of X_i, $\mathbb{P}(|X_i| \geqslant ta_i) = 2\mathbb{P}(X_i \geqslant ta_i) \leqslant (3/4)^t$ for $t \geqslant 1$ and integration by parts yields $\|X_i\|_2 \leqslant Ca_i$. Thus $a_i \geqslant c_1$ for a constant $c_1 > 0$.

Let $S = \sum_{i=1}^n I_{\{|X_i| \geqslant c_1\}}$. Then $\mathbb{E}S = \sum_{i=1}^n \mathbb{P}(|X_i| \geqslant c_1) \geqslant 3n/4$. On the other hand $\mathbb{E}S \leqslant \frac{n}{2} + n\mathbb{P}(X^*_{\lceil n/2\rceil} \geqslant c_1)$, so

$$\mathbb{E}X^*_{\lceil n/2\rceil} \geqslant c_1\mathbb{P}(X^*_{\lceil n/2\rceil} \geqslant c_1) \geqslant c_1/4. \qquad \square$$

The next example shows that the tail and moment bounds in Corollary 6 are optimal.

Example. Let $X = (X_1, \ldots, X_n)$ be an isotropic random vector with i.i.d. symmetric exponential coordinates (X is of density $2^{n/2}\exp(-\sqrt{2}\|x\|_1)$).

Then $(\mathbb{E}|X_i|^p)^{1/p} \leqslant p/2$, so $\frac{2}{p}e_i \in \mathcal{M}_p(X)$ and

$$\mathbb{P}\left(\|X\|_{\mathcal{Z}_p(X)} \geqslant t\sqrt{n/p}\right) \geqslant \mathbb{P}(|X_i| \geqslant t\sqrt{np}/2) \geqslant e^{-t\sqrt{np}/\sqrt{2}}$$

and for $q = s\sqrt{np}$, $s \geqslant 1$,

$$\left(\mathbb{E}\|X\|^q_{\mathcal{Z}_p(X)}\right)^{1/q} \geqslant \frac{2}{p}\|X_i\|_q \geqslant cq/p = cs\sqrt{n/p}.$$

§3. General case – approach via entropy numbers

In this section we propose a method of deriving estimates for $\mathcal{Z}_p$-norms via entropy estimates for $\mathcal{M}_p$-balls and Euclidean distance. We use a standard notation – for sets $T, S \subset \mathbb{R}^n$, by $N(T,S)$ we denote the minimal number of translates of S that are enough to cover T. If S is the ε-ball with respect to some translation-invariant metric d then $N(T,S)$ is also denoted as $N(T,d,\varepsilon)$ and is called the metric entropy of T with respect to d.

We are mainly interested in log-concave vectors or random vectors which satisfy moment estimates

$$\|\langle t, X\rangle\|_p \leqslant \lambda\frac{p}{q}\|\langle t, X\rangle\|_q \quad \text{for } p \geqslant q \geqslant 2. \tag{7}$$

Let us start with a simple bound.

Proposition 8. *Suppose that X is isotropic in $\mathbb{R}^n$ and (7) holds. Then for any $p \geqslant 2$ and $\varepsilon > 0$ we have*

$$\left(\mathbb{E}\|X\|^2_{\mathcal{Z}_p(X)}\right)^{1/2} \leqslant \varepsilon\sqrt{n} + \frac{e\lambda}{p}\max\{p, \log N(\mathcal{M}_p(X), \varepsilon B_2^n)\}.$$

Proof. Let $N = N(\mathcal{M}_p(X), \varepsilon B_2^n)$ and choose $t_1, \dots, t_N \in \mathcal{M}_p(X)$ such that $\mathcal{M}_p(X) \subset \bigcup_{i=1}^N (t_i + \varepsilon B_2^n)$. Then

$$\|x\|_{\mathcal{Z}_p(X)} \leqslant \varepsilon|x| + \sup_{i \leqslant N}\langle t_i, x\rangle.$$

Let $r = \max\{p, \log N\}$. We have

$$\begin{aligned}\left(\mathbb{E}\sup_{i\leqslant N}|\langle t_i, X\rangle|^2\right)^{1/2} &\leqslant \left(\mathbb{E}\sup_{i\leqslant N}|\langle t_i, X\rangle|^r\right)^{1/r} \leqslant \left(\sum_{i=1}^N \mathbb{E}|\langle t_i, X\rangle|^r\right)^{1/r} \\ &\leqslant N^{1/r}\sup_i \|\langle t_i, X\rangle\|_r \\ &\leqslant e\lambda\frac{r}{p}\sup_i\|\langle t_i, X\rangle\|_p \leqslant e\lambda\frac{r}{p}.\end{aligned}$$

□

Remark 9. The Paouris inequality [10] states that for isotropic log-concave vectors and $q \geqslant 2$, $(\mathbb{E}|X|^q)^{1/q} \leqslant C(\sqrt{n}+q)$, so for such vectors and $q \geqslant 2$,

$$\Big(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}^q\Big)^{1/q} \leqslant C\varepsilon(\sqrt{n}+q) + \frac{2e}{p}\max\{p,q,\log N(\mathcal{M}_p(X),\varepsilon B_2^n)\}.$$

Unfortunately, the known estimates for entropy numbers of $\mathcal{M}_p$-balls are rather weak.

Theorem 10 ([4, Proposition 9.2.8]). *Assume that X is isotropic log-concave and $2 \leqslant p \leqslant \sqrt{n}$. Then*

$$\log N\Big(\mathcal{M}_p(X), \frac{t}{\sqrt{p}}B_2^n\Big) \leqslant C\frac{n\log^2 p \log t}{t} \qquad \textit{for } 1 \leqslant t \leqslant \min\Big\{\sqrt{p}, \frac{1}{C}\frac{n\log p}{p^2}\Big\}.$$

Corollary 11. *Let X be isotropic log-concave, then*

$$\Big(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}^p\Big)^{1/p} \leqslant C\left(\frac{n}{p}\right)^{3/4}\log p\sqrt{\log n} \quad \textit{for } 2 \leqslant p \leqslant \frac{1}{C}n^{3/7}\log^{-2/7} n.$$

Proof. We apply Theorem 10 with $t = (n/p)^{1/4}\log p \log^{1/2} n$ and Proposition 8 with $\varepsilon = tp^{-1/2}$. □

Remark 12. Suppose that X is centered and the following stronger bound than (7) (satisfied for example for Gaussian vectors) holds

$$\|\langle t, X\rangle\|_p \leqslant \lambda\sqrt{\frac{p}{q}}\|\langle t,X\rangle\|_q \qquad \text{for } p \geqslant q \geqslant 2. \tag{8}$$

Then for any $2 \leqslant p \leqslant n$,

$$\frac{1}{\lambda}\sqrt{\frac{2n}{p}} \leqslant \Big(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}^2\Big)^{1/2} \leqslant \Big(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}^n\Big)^{1/n} \leqslant 10\lambda\sqrt{\frac{n}{p}}.$$

Proof. Without loss of generality we may assume that X is isotropic. We have

$$\|\langle t,X\rangle\|_p \leqslant \lambda\sqrt{p/2}\|\langle t,X\rangle\|_2 = \lambda\sqrt{p/2}|t|,$$

so $\mathcal{M}_p(X) \supset \lambda^{-1}\sqrt{2/p}B_2^n$ and

$$\Big(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}^2\Big)^{1/2} \geqslant \frac{1}{\lambda}\sqrt{\frac{2}{p}}\left(\mathbb{E}|X|^2\right)^{1/2} = \frac{1}{\lambda}\sqrt{\frac{2n}{p}}.$$

On the other hand let S be a 1/2-net in $\mathcal{M}_p(X)$ of cardinality at most 5^n. Then

$$\Big(\mathbb{E}\|X\|^n_{\mathcal{Z}_p(X)}\Big)^{1/n} \leqslant 2\left(\mathbb{E}\sup_{t\in S}|\langle t,X\rangle|^n\right)^{1/n}$$
$$\leqslant 2\left(\sum_{t\in S}\mathbb{E}|\langle t,X\rangle|^n\right)^{1/n} \leqslant 2|S|^{1/n}\sup_{t\in S}\|\langle t,X\rangle\|_n$$
$$\leqslant 10\lambda\sqrt{\frac{n}{p}}\sup_{t\in S}\|\langle t,X\rangle\|_p \leqslant 10\lambda\sqrt{\frac{n}{p}}. \qquad \square$$

Recall that the Sudakov minoration principle [11] states that if G is an isotropic Gaussian vector in $\mathbb{R}^n$ then for any bounded $T\subset\mathbb{R}^n$ and $\varepsilon>0$,

$$\mathbb{E}\sup_{t\in T}\langle t,G\rangle \geqslant \frac{1}{C}\varepsilon\sqrt{\log N(T,\varepsilon B_2^n)}.$$

So we can say that a random vector X in $\mathbb{R}^n$ satisfies *the L_2-Sudakov minoration with a constant C_X* if for any bounded $T\subset\mathbb{R}^n$ and $\varepsilon>0$,

$$\mathbb{E}\sup_{t\in T}\langle t,X\rangle \geqslant \frac{1}{C_X}\varepsilon\sqrt{\log N(T,\varepsilon B_2^n)}.$$

Example. Any unconditional n-dimensional random vector satisfies the L_2-Sudakov minoration with constant $C\sqrt{\log(n+1)}/(\min_{i\leqslant n}\mathbb{E}|X_i|)$.

Indeed, we have by the unconditionality, Jensen's inequality and the contraction principle,

$$\mathbb{E}\sup_{t\in T}\sum_{i=1}^n t_iX_i = \mathbb{E}\sup_{t\in T}\sum_{i=1}^n t_i\varepsilon_i|X_i| \geqslant \mathbb{E}\sup_{t\in T}\sum_{i=1}^n t_i\varepsilon_i\mathbb{E}|X_i|$$
$$\geqslant \min_{i\leqslant n}\mathbb{E}|X_i|\mathbb{E}\sup_{t\in T}\sum_{i=1}^n t_i\varepsilon_i.$$

On the other hand, the classical Sudakov minoration and the contraction principle yields

$$\frac{1}{C}\varepsilon\sqrt{\log N(T,\varepsilon B_2^n)} \leqslant \mathbb{E}\sup_{t\in T}\sum_{i=1}^n t_ig_i \leqslant \mathbb{E}\max_{i\leqslant n}|g_i|\mathbb{E}\sup_{t\in T}\sum_{i=1}^n t_i\varepsilon_i$$
$$\leqslant C\sqrt{\log(n+1)}\mathbb{E}\sup_{t\in T}\sum_{i=1}^n t_i\varepsilon_i.$$

However the L_2-Sudakov minoration constant may be quite large in the isotropic case even for unconditional vectors if we do not assume that L_1 and L_2 norms of X_i are comparable. Indeed, let $\mathbb{P}(X = \pm n^{1/2}e_i) = \frac{1}{2n}$ for $i = 1, \ldots, n$, where $e_1, \ldots, e_n$ is the canonical basis of $\mathbb{R}^n$. Then X is isotropic and unconditional. Let $T = \{t \in \mathbb{R}^n \colon \ \|t\|_\infty \leqslant n^{-1/2}\}$. Then

$$\mathbb{E} \sup_{t \in T} |\langle t, X\rangle| \leqslant 1.$$

However, by the volume-based estimate,

$$N(T, \varepsilon B_2^n) \geqslant \frac{\mathrm{vol}(T)}{\mathrm{vol}(\varepsilon B_2^n)} \geqslant \left(\frac{1}{\varepsilon C}\right)^n,$$

hence

$$\sup_{\varepsilon > 0} \varepsilon \sqrt{\log N((T, \varepsilon B_2^n)} \geqslant \frac{1}{C}\sqrt{n}.$$

Thus the L_2-Sudakov constant $C_X \geqslant \sqrt{n}/C$ in this case.

Next proposition shows that random vectors with uniformly log-convex density satisfy the L_2-Sudakov minoration.

Proposition 13. *Suppose that a symmetric random vector X in $\mathbb{R}^n$ has the density of the form e^h such that* $\mathrm{Hess}(h) \geqslant -\alpha \mathrm{Id}$ *for some $\alpha > 0$. Then X satisfies the L_2-Sudakov minoration with constant $C_X \leqslant C\sqrt{\alpha}$.*

Proof. We will follow the method of the proof of the (dual) classical Sudakov inequality (cf. (3.15) and its proof in [8]).

Let T be a bounded symmetric set and

$$A := \mathbb{E} \sup_{t \in T} |\langle t, X\rangle|.$$

By the duality of entropy numbers [2] we need to show that

$$\log^{1/2} N(\varepsilon^{-1} B_2^n, T^o) \leqslant C \varepsilon^{-1} \alpha^{1/2} A$$

for $\varepsilon > 0$ or equivalently that

$$N(\delta B_2^n, 6AT^o) \leqslant \exp(C\alpha\delta^2) \quad \text{ for } \delta > 0. \tag{9}$$

To this end let $N = N(\delta B_2^n, 6AT^o)$. If $N = 1$ there is nothing to show, so assume that $N \geqslant 2$. Then we may choose $t_1, \ldots, t_N \in \delta B_2^n$ such that the balls $t_i + 3AT^0$ are disjoint. Let $\mu = \mu_X$ be the distribution of X. By the Chebyshev inequality,

$$\mu(3AT^0) = 1 - \mathbb{P}\Big(\sup_{t \in T} |\langle t, X\rangle| > 3A\Big) \geqslant \frac{2}{3}.$$

Observe also that for any symmetric set K and $t \in \mathbb{R}^n$,

$$\mu(t+K) = \int_K e^{h(x-t)}dx = \int_K e^{h(x+t)}dx = \int_K \frac{1}{2}(e^{h(x-t)} + e^{h(x+t)})dx \geqslant \int_K e^{(h(x-t)+h(x+t))/2}dx.$$

By Taylor's expansion we have for some $\theta \in [0,1]$,

$$\frac{h(x-t)+h(x+t)}{2} = h(x) + \frac{1}{4}(\langle \mathrm{Hess} h(x+\theta t)t,t\rangle + \langle \mathrm{Hess} h(x-\theta t)t,t\rangle) \geqslant h(x) - \frac{1}{2}\alpha|t|^2.$$

Thus

$$\mu(t+K) \geqslant \int_K e^{h(x)-\alpha|t|^2/2} = e^{-\alpha|t|^2/2}\mu(K)$$

and

$$1 \geqslant \sum_{i=1}^N \mu(t_i + 3AT^0) \geqslant \sum_{i=1}^N e^{-\alpha|t_i|^2/2}\mu(3AT^0) \geqslant \frac{2N}{3}e^{-\alpha\delta^2/2} \geqslant N^{1/3}e^{-\alpha\delta^2/2}$$

and (9) easily follows. □

Proposition 14. *Suppose that X satisfies the L_2-Sudakov minoration with constant C_X. Then for any $p \geqslant 2$*

$$N\Big(\mathcal{M}_p(X), \frac{eC_X}{\sqrt{p}}B_2^n\Big) \leqslant e^p.$$

In particular if X is isotropic we have for $2 \leqslant p \leqslant n$,

$$\Big(\mathbb{E}\|X\|^2_{\mathcal{Z}_p(X)}\Big)^{1/2} \leqslant e\Big(C_X\sqrt{\frac{n}{p}} + 1\Big).$$

Proof. Suppose that $N = N(\mathcal{M}_p(X), eC_X p^{-1/2}B_2^n) \geqslant e^p$. We can choose $t_1, \ldots, t_N \in \mathcal{M}_p(X)$, such that $\|t_i - t_j\|_2 \geqslant eC_X p^{-1/2}$. We have

$$\mathbb{E}\sup_{i\geqslant N}\langle t_i, X\rangle \geqslant \frac{1}{C_X}eC_X p^{-1/2}\sqrt{\log N} > e.$$

However on the other hand,

$$\mathbb{E}\sup_{i\geqslant N}\langle t_i,X\rangle\leqslant\Big(\mathbb{E}\sup_{i\geqslant N}|\langle t_i,X\rangle|^p\Big)^{1/p}\leqslant\Big(\sum_{i\geqslant N}\mathbb{E}|\langle t_i,X\rangle|^p\Big)^{1/p}$$
$$\leqslant N^{1/p}\max_i\|\langle t_i,X\rangle\|_p\leqslant e.$$

To show the second estimate we proceed in a similar way as in the proof of Proposition 8. We choose $T\subset\mathcal{M}_p(X)$ such that $|T|\leqslant e^p$ and $\mathcal{M}_p(X)\subset T+eC_Xp^{-1/2}B_2^n$. We have

$$\|X\|_{\mathcal{Z}_p(X)}\leqslant eC_Xp^{-1/2}|X|+\sup_{t\in T}|\langle t,X\rangle|$$

so that

$$\Big(\mathbb{E}\|X\|^2_{\mathcal{Z}_p(X)}\Big)^{1/2}\leqslant eC_Xp^{-1/2}(\mathbb{E}|X|^2)^{1/2}+\Big(\mathbb{E}\sup_{t\in T}|\langle t,X\rangle|^2\Big)^{1/2}.$$

Vector X is isotropic, so $\mathbb{E}|X|^2=n$ and since $T\subset\mathcal{M}_p(X)$ and $p\geqslant 2$ we get

$$\Big(\mathbb{E}\sup_{t\in T}|\langle t,X\rangle|^2\Big)^{1/2}\leqslant\Big(\mathbb{E}\sup_{t\in T}|\langle t,X\rangle|^p\Big)^{1/p}\leqslant\Big(\sum_{t\in T}\mathbb{E}|\langle t,X\rangle|^p\Big)^{1/p}$$
$$\leqslant|T|^{1/p}\max_{t\in T}\||\langle t,X\rangle\|_p\leqslant e.\qquad\square$$

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