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T. Kühn, W. Linde

GAUSSIAN APPROXIMATION NUMBERS AND METRIC ENTROPY

Abstract. The aim of this paper is to survey properties of Gaussian approximation numbers. We state the basic relations between these numbers and and other s-numbers as e.g. entropy, approximation or Kolmogorov numbers. Furthermore, we fill a gap and prove new two-sided estimates in the case of operators with values in a K-convex Banach space. In a final section we apply the relations between Gaussian and other s-numbers to the d-dimensional integration operator defined on $L_2[0,1]^d$.

Dedicated to the memory of Vladimir Nikolaevich Sudakov

§1. Introduction

Basic results of R. M. Dudley in 1967 and of V. N. Sudakov in 1969 relate regularity properties of a Gaussian random process with compactness properties of its reproducing kernel Hilbert space (RKHS). These results show that certain metric entropy conditions for the RKHS are either sufficient (Dudley) or necessary (Sudakov) for the boundedness of a Gaussian random process.

To get more refined results one has to analyze the degree of compactness of the RKHS more thoroughly. A basic tool for those investigations is the behavior of certain s-numbers of operators related to Gaussian processes. Among these numbers the Gaussian approximation numbers turn out to be very useful. For example, they have been used to study approximation and small deviation properties of fractional Brownian motions or Riemann-Liouville processes, or to verify general properties of Gaussian processes (cf. [16, 18, 23] or [11]).

The aim of the present paper is to survey the properties of Gaussian approximation numbers. In particular, we state the basic relations between

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these numbers and other s-numbers as e.g. entropy, approximation or Kolmogorov numbers. Hereby, we fill a gap and prove new two-sided estimates in the case of so-called K-convex Banach spaces.

Throughout the paper, all spaces are assumed to be real, where H always denotes a separable Hilbert space and E a Banach space.

For sequences of non-negative real numbers we write $a_n \preceq b_n$, if there is a constant $C > 0$ such that $a_n \leqslant C \cdot b_n$ for all $n \in \mathbb{N}$, while $a_n \approx b_n$ means $a_n \preceq b_n \preceq a_n$.

A decreasing (resp. increasing) sequence (a_n) is said to satisfy the *doubling condition*, if $a_n \approx a_{2n}$. Clearly, in this case we have $a_n \approx a_{mn}$ for all $m \in \mathbb{N}$.

§2. The Dudley–Sudakov Theorem for Operators

In order to formulate the above mentioned theorems of Dudley and Sudakov in a functional analytical language, we need the following definitions.

Let E and F be Banach spaces and let $T : E \to F$ be a (bounded linear) operator. Its (dyadic) entropy numbers $e_n(T)$ are then defined by

$$e_n(T) := \inf \left\{ \varepsilon > 0 : T(B_E) \subseteq \bigcup_{j=1}^{2^{n-1}} B(y_j; \varepsilon) \text{ for some } y_1, \ldots, y_{2^{n-1}} \in F \right\}.$$

Here the set B_E denotes the closed unit ball in E while $B(y;\varepsilon)$ is the (open) ε-ball in F with center $y \in F$. For properties of the entropy numbers we refer to [12, 22] and [5].

Suppose we are given an operator T from a separable Hilbert space H into a Banach space E. Then its l-norm is defined by

$$l(T) := \sup_{H_0 \subseteq H} \left\{ \left(\int_{H_0} \|Th\|^2 \, d\gamma_{H_0}(h) \right)^{1/2} \right\},$$

where the supremum is taken over all finite dimensional subspaces $H_0 \subseteq H$ and γ_{H_0} denotes the (unique) standard Gaussian measure on H_0. This norm was introduced in the context of cylinder measures and operator ideals in [19] under the name π_γ-norm.

Another way to look at $l(T)$ is as follows. Let $(\xi_k)_{k=1}^\infty$ be an i.i.d. sequence of $\mathcal{N}(0,1)$-distributed random variables. Then $l(T) < \infty$ if and only if for one (or, equivalently, each) orthonormal basis (ONB) $(e_k)_{k=1}^{\infty}$ in H

$$\sup_{n\geqslant 1}\left\|\sum_{k=1}^{n}\xi_k Te_k\right\| < \infty \quad \text{a.s.}.$$

In particular, if the series

$$\sum_{k=1}^{\infty}\xi_k Te_k \quad \text{converges a.s. in } E$$

for some ONB $(e_k)_{k=1}^{\infty}$, then $l(T) < \infty$ and, moreover,

$$l(T) = \left(\mathbb{E}\left\|\sum_{k=1}^{\infty}\xi_k Te_k\right\|^2\right)^{1/2},$$

which is independent of the special choice of the ONB $(e_k)_{k\geqslant 1}$ in H.

Now we can state the above mentioned functional-analytic version of the Dudley–Sudakov Theorem (cf. [7] and [25]). This reformulation was given in [13].

Theorem 2.1. *There are universal constants $c, C > 0$ such that for all operators $T : H \to E$ from a Hilbert space H into a Banach space E the inequalities*

$$c\sup_{n\geqslant 1} n^{1/2}e_n(T^*) \leqslant l(T) \leqslant C\sum_{n=1}^{\infty} n^{-1/2}e_n(T^*) \tag{2.1}$$

hold, where $T^* : E^* \to H$ denotes the dual operator of T.

Remarks:

(1) In view of the basic result in [1] about the duality of entropy numbers, inequalities (2.1) are also valid for T, i.e., one has

$$c\sup_{n\geqslant 1} n^{1/2}e_n(T) \leqslant l(T) \leqslant C\sum_{n=1}^{\infty} n^{-1/2}e_n(T) \tag{2.2}$$

with suitable constants c, $C > 0$.

A direct proof of the first inequality in (2.2), without using the duality of entropy numbers, was given in [9].

(2) A slightly stronger version of Theorem 2.1 is as follows: If

$$\sum_{n=1}^{\infty} n^{-1/2}e_n(T^*) < \infty,$$

then

$$\sum_{k=1}^{\infty} \xi_k T e_k \quad \text{exists a.s. in } E\,, \tag{2.3}$$

and, moreover, $l(T)$ may be estimated as stated in (2.1). Recall that $l(T) < \infty$ does in general not imply the a.s. convergence of the random series in E.

(3) The estimates in (2.1) are optimal and cannot be improved, as can be seen, e.g., from the examples given in [13]. In order to get sharp two-sided estimates, more refined tools like majorizing measures are needed (cf. [26]).

§3. Gaussian Approximation Numbers

The Dudley–Sudakov Theorem asserts the following: If the operator $T : H \to E$ is sufficiently compact, then $\sum_{k=1}^{\infty} \xi_k T e_k$ converges almost surely in E for one or, equivalently, for all orthonormal bases $(e_k)_{k\geqslant 1}$ in H. Conversely, if this series converges almost surely, then necessarily the operator satisfies $e_n(T) \leqslant c\,n^{-1/2}$.

In this context, the following natural question arises: Which additional properties does the series $\sum_{k=1}^{\infty} \xi_k T e_k$ possess, if the entropy numbers $e_n(T)$ tend to zero much faster than of order $n^{-1/2}$? An answer in the language of the corresponding Gaussian process was given in [17] and [18]: The behavior of $e_n(T)$ as $n \to \infty$ is directly connected to small deviation properties of the Gaussian process, that is, with the behavior of

$$-\log\left(\mathbb{P}\left\{\Big\|\sum_{k=1}^{\infty} \xi_k T e_k\Big\| < \varepsilon\right\}\right) \to \infty$$

as $\varepsilon \to 0$.

Another possibility is to ask for the speed of convergence of the series $\sum_{k=1}^{\infty} \xi_k T e_k$. Is this related to the behavior of $e_n(T)$ as $n \to \infty$? In other words, does a faster convergence of $e_n(T) \to 0$ imply a faster convergence of $\sum_{k=1}^{n} \xi_k T e_k$ as $n \to \infty$?

To make this more precise, we have to introduce so-called Gaussian approximation numbers, which, to the best of our knowledge, appeared for the first time in [20] (see also [23]).

Let $T : H \to E$ be an operator for which the series in (2.3) converges a.s. in E. Then its Gaussian approximation numbers (sometimes also called l-numbers) are defined by

$$l_n(T) := \inf \left\{ \left(\mathbb{E} \left\| \sum_{k=n}^{\infty} \xi_k T e_k \right\|^2 \right)^{1/2} \ : \ (e_k)_{k=1}^{\infty} \text{ ONB in } H \right\} .$$

It is well-known and easy to see (cf. [22] or [23]) that these numbers may also be defined via

(i) $l_n(T) = \inf \{ l(T-S) : S \text{ operator from } H \text{ to } E,\ \operatorname{rank}(S) < n \}$

or

(ii) $l_n(T) = \inf \{ l(T-TP) : P \text{ orthogonal projection in } H, \operatorname{rank}(P) < n \}$

or

(iii) $l_n(T) = \inf \left\{ l(T\big|_{H_0^{\perp}}) : H_0 \subseteq H,\ \dim(H_0) < n \right\}$.

§4. Relations to Other Approximation Quantities

The aim of this section is to relate the numbers $l_n(T)$ with entropy and approximation numbers of T or T^*, the dual of T, respectively. Given an operator T between two Banach spaces E and F, its approximation numbers $a_n(T)$ and its Kolmogorov numbers $d_n(T)$ are defined as usual:

$$\begin{aligned} a_n(T) &:= \inf \{ \|T - S\| \ : S \text{ operator from } E \text{ to } F, \operatorname{rank}(S) < n \} , \\ d_n(T) &:= \inf \left\{ \left\| Q_N^F T \right\| : N \subseteq F,\ \dim(N) < n \right\} . \end{aligned}$$

Here the operator Q_N^F appearing in the definition of $d_n(T)$ is the canonical quotient map from F onto F/N.

Note that all numbers a_n, d_n, e_n and l_n are additive in the sense of [5], p.21. While the first three numbers are also multiplicative (cf. [5], same place), the Gaussian approximation numbers satisfy the following modified multiplicativity property. Let $T : H \to E$ and $S : H \to H$. Then for all $n, m \in \mathbb{N}$

$$\begin{aligned} l_{n+m-1}(T \circ S) &\leqslant l_n(T) \cdot a_m(S) \quad \text{and} \\ l_{n+m-1}(T \circ S) &\leqslant a_n(T) \cdot l_m(S) . \end{aligned}$$

Moreover, let us mention that $a_n(T) = d_n(T)$ whenever T has values in a Hilbert space (cf. [22]). Since $a_n(T) = a_n(T^*)$ for compact operators, we also have

$$a_n(T) = d_n(T^*) \quad \text{for} \quad T : H \to E \quad \text{compact.}$$

To become acquainted with these numbers, we first consider a special case. If T is a compact operator from a Hilbert space H into itself (or into another Hilbert space), its singular numbers are defined by $s_n(T) = \sqrt{\lambda_n}$, where $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant 0$ are the eigenvalues of the (compact and non-negative self-adjoint) operator $T^*T : H \to H$, counted according to their multiplicities.

Assume the singular numbers of T are known, say $s_n(T) = \sigma_n$. Then for the usual and for the Gaussian approximation numbers of T the exact formulae (cf. [22] or [5], 4.4.12 and 1.5.11)

$$a_n(T) = \sigma_n \qquad \text{and} \qquad l_n(T) = \left(\sum_{k=n}^{\infty} \sigma_k^2 \right)^{1/2} \tag{4.1}$$

are valid, while for its entropy numbers a nice two-sided estimate holds, which is due to [10],

$$\sup_{k \in \mathbb{N}} 2^{-n/k} \left(\prod_{j=1}^{k} \sigma_j \right)^{1/k} \leqslant e_{n+1}(T) \leqslant 6 \cdot \sup_{k \in \mathbb{N}} 2^{-n/k} \left(\prod_{j=1}^{k} \sigma_j \right)^{1/k} . \tag{4.2}$$

Our aim is to prove sharp two-sided estimates between entropy numbers and Gaussian approximation numbers, not only for a single operator $T : H \to E$ but for whole classes of operators. To this end, we have to impose certain regularity conditions on the sequences $(l_n(T))$ or $(e_n(T))$. We will show that, under fairly general conditions on the sequence (σ_n), one has

$$l_n(T) \approx \sigma_n \quad \Longleftrightarrow \quad e_n(T) \approx n^{-1/2} \sigma_n \, . \tag{4.3}$$

We will also need an additional assumption on the Banach space E, namely that it is K-convex. For this notion we refer to [23]. Note that L_p -spaces are K-convex provided that $1 < p < \infty$.

Next we list the relevant regularity conditions in our context. It is easy to verify that a decreasing sequence (σ_n) of positive real numbers satisfies the *doubling condition* $\sigma_n \approx \sigma_{2n}$ if and only if

$$\inf_{n \geqslant k} \frac{n^\alpha \sigma_n}{k^\alpha \sigma_k} > 0 \quad \text{for some } \alpha > 0 \, . \tag{4.4}$$

In other words, the sequence $(n^\alpha \sigma_n)$ is *almost increasing.* Obviously, (4.4) implies $\sigma_n \succeq n^{-\alpha}$, and therefore (σ_n) cannot decay faster than polynomially, which excludes exponential decay.

A similar condition is

$$\sup_{n \geqslant k} \frac{n^\beta \sigma_n}{k^\beta \sigma_k} < \infty \quad \text{for some } \beta > 0\,, \tag{4.5}$$

i.e. $(n^\beta \sigma_n)$ is *almost decreasing.* In particular, (4.5) implies $\sigma_n \preceq n^{-\beta}$, and therefore (σ_n) cannot decay slower than polynomially, which excludes logarithmic decay.

These regularity conditions are not new, in fact they have been widely used in the literature. For instance, the doubling condition plays an important role in the monograph [5], and the conditions (4.4) and (4.5) have been used in the paper [14] to investigate entropy numbers of general diagonal operators $D : \ell_p \to \ell_q$. A common feature of the three conditions is that they are easy to check in concrete cases.

Note that (4.4) and (4.5) are indeed very mild regularity conditions and hold for large classes of sequences. Typical examples are $\sigma_n \approx n^{-\alpha}(1 + \log n)^\beta$ with $\alpha > 0$ and $\beta \in \mathbb{R}$, but also, more generally, $\sigma_n \approx n^{-\alpha}\varphi_n$ with φ_n slowly varying. Even more, given any $0 < \alpha \neq \beta < \infty$, one can construct (σ_n) satisfying (4.4) and (4.5) such that there are subsequences (n_k) and (m_k) with $\sigma_{m_k} \approx m_k^{-\alpha}$ and $\sigma_{n_k} \approx n_k^{-\beta}$ as $k \to \infty$. This construction can be done similarly as the examples of weights with different indices in subsection 4.4 of [15].

Now we state some concrete examples which illustrate how different the relations between $l_n(T)$ and $e_n(T)$ can be. Moreover, these examples will provide a motivation why conditions (4.4) and (4.5) are very useful and quite natural in our context.

Examples.

(1) Let $D : \ell_2 \to \ell_2$ be a diagonal operator, defined by $D(x_n) = (n^{-\alpha}x_n)$. Then $l(D) < \infty$ if and only if $\alpha > 1/2$. In this case, (4.1) and (4.2) give

$$l_n(D) \approx n^{1/2-\alpha} \quad \text{and} \quad e_n(D) \approx n^{-\alpha}\,,$$

i.e. the desired relation (4.3) between Gaussian approximation numbers and entropy numbers is fulfilled with $\sigma_n := n^{1/2-\alpha}$. Clearly (σ_n) satisfies both regularity conditions (4.4) and (4.5).

This remains true, if we replace $n^{-\alpha}$ in the definition of D by $n^{-\alpha}(1+\log n)^{\beta}$ with $\alpha > 1/2$ and $\beta \in \mathbb{R}$.

(2) If $D : \ell_2 \to \ell_2$ is defined by $D(x_n) = (n^{-1/2}(1+\log n)^{-1/2-\alpha} x_n)$, with $\alpha > 0$, then (4.1) and (4.2) imply

$$l_n(D) \approx (1+\log n)^{-\alpha} \text{ and } e_n(D) \approx n^{-1/2}(1+\log n)^{-1/2-\alpha}\,.$$

This shows that the desired relation (4.3) is not true if $\sigma_n = (1+\log n)^{-\alpha}$. Obviously, since (σ_n) decays too slowly, it fails (4.5) for any $\beta > 0$.

(3) Now let $D : \ell_2 \to \ell_2$ be defined by $D(x_n) = (2^{-n} x_n)$. Then, again by (4.1) and (4.2), we have

$$l_n(D) \approx 2^{-n} \quad \text{and} \quad e_n(D) \approx 2^{-\sqrt{n}}\,.$$

Hence the desired relation (4.3) does not hold if $\sigma_n = 2^{-n}$. Clearly, (σ_n) does not satisfy the doubling condition, neither the regularity condition (4.4) for any $\alpha > 0$.

(4) Finally we consider the Volterra integration operator $V : L_2[0,1] \to C[0,1]$, defined by $Vf(x) = \int\limits_0^x f(t)\,dt$. It is well known (see also Sec. 5) that

$$l_n(V) \approx n^{-1/2}(1+\log n)^{1/2} \quad \text{and} \quad e_n(V) \approx n^{-1}\,.$$

Clearly $\sigma_n = n^{-1/2}(1+\log n)^{1/2}$ satisfies both regularity conditions (4.4) and (4.5) for appropriate $\alpha > 0$ resp. $\beta > 0$, but nevertheless (4.3) does not hold. Note that the target space $C[0,1]$ of the operator V is not K-convex.

These examples show that the relation (4.3) between Gaussian approximation numbers and entropy numbers $l_n(T) \approx \sigma_n \iff e_n(T) \approx n^{-1/2}\sigma_n$ can only hold in general if

- the sequence (σ_n) is regular in the sense of (4.4) and (4.5), and
- the operator $T : H \to E$ has values in a K-convex Banach space E.

Before proceeding further we need some more preparations. First let us state the following improved version of Carl's inequality (cf. [4], Thm. 1.3).

Proposition 4.1. *Let $(b_k)_{k=1}^{\infty}$ be an increasing sequence of real numbers such that*

$$b_{2k} \leqslant \kappa \cdot b_k \qquad \textit{for all} \;\; k \in \mathbb{N}$$

and some $\kappa \geqslant 1$. Then there is a constant $c > 0$ only depending on κ such that for all operators T between any Banach spaces and for all $n \in \mathbb{N}$

$$\max_{1 \leqslant k \leqslant n} b_k \, e_k(T) \leqslant c \cdot \max_{1 \leqslant k \leqslant n} b_k \, d_k(T) \, .$$

Let us point out that the same result holds for Gelfand numbers $c_k(T)$ instead of Kolmogorov numbers (cf. [5], Sec. 3.1), and using $c_k(T^*) \leqslant d_k(T)$ (cf. [5], Prop. 2.5.5) we obtain

$$\max_{1 \leqslant k \leqslant n} b_k \, e_k(T^*) \leqslant c \cdot \max_{1 \leqslant k \leqslant n} b_k \, d_k(T) \, .$$

Later on we have to relate quite often the entropy numbers of an operator with those of its dual. A deep result about duality of entropy numbers for operators between arbitrary Banach spaces was proved in [1]. Note that here we only investigate operators T which are either defined on a Hilbert space or map into a Hilbert space. In this situation Tomczak-Jaegermann proved much earlier (cf. [28], Thm. 1), that for all $\alpha > 0$ and all $n \in \mathbb{N}$ the estimates

$$\frac{1}{32} \cdot \max_{1 \leqslant k \leqslant n} k^\alpha e_n(T^*) \leqslant \max_{1 \leqslant k \leqslant n} k^\alpha e_n(T^*) \leqslant 32 \cdot \max_{1 \leqslant k \leqslant n} k^\alpha e_n(T) \qquad (4.6)$$

hold. This would have been sufficient for our purposes.

The next result is due to Pajor and Tomczak-Jaegermann (cf. [21] or [23], Thm. 5.8). It estimates the Kolmogorov numbers of an operator $T : H \to E$ satisfying $l(T) < \infty$. In view of Proposition 4.1, under some mild regularity conditions, it improves the left-hand estimate in Theorem 2.1.

Proposition 4.2. *There is a universal constant $c > 0$ such that for all operators T from a Hilbert space H into a Banach space E*

$$\sup_{n \in \mathbb{N}} n^{1/2} \, d_n(T) \leqslant c \cdot l(T) \, . \qquad (4.7)$$

As a direct consequence of Proposition 4.2 we obtain the following lower estimate for Gaussian approximation numbers.

Proposition 4.3. *If T is an operator from a Hilbert space H into a Banach space E, then for all $n \in \mathbb{N}$*

$$n^{1/2} \, d_{2n-1}(T) \leqslant c \cdot l_n(T)$$

with some universal constant $c > 0$.

Proof. Let S be an operator from H into E of rank less than n such that

$$l(T-S) \leqslant 2 \cdot l_n(T)\,. \tag{4.8}$$

Then (4.7) yields

$$\sup_{k\geqslant 1} k^{1/2}\, d_k(T-S) \leqslant c \cdot l(T-S)\,. \tag{4.9}$$

Since S is of rank less than n, we have $d_n(S) = 0$, and estimates (4.8) and (4.9) imply

$$\begin{aligned} n^{1/2}\, d_{2n-1}(T) &\leqslant n^{1/2}\, d_n(T-S) + n^{1/2}\, d_n(S) \\ &\leqslant c \cdot l(T-S) \leqslant 2c \cdot l_n(T) \end{aligned}$$

as asserted. □

Remark. An essential argument in the previous proof was that $d_n(S) = 0$. Note that for every operator $S \neq 0$ one has $e_n(S) > 0$ for all $n \in \mathbb{N}$, while $l_n(S) = 0$ if $\operatorname{rank}(S) < n$. Hence, a similar direct estimate between $e_{2n-1}(T)$ and $l_n(T)$ cannot hold.

As a consequence of Propositions 4.1 and 4.3 we obtain the following.

Corollary 4.4. *Let $(b_k)_{k=1}^{\infty}$ be an increasing sequence of positive real numbers satisfying the doubling condition $b_k \approx b_{2k}$. Then there is a constant $c > 0$ such that for all operators T from a Hilbert space into an arbitrary Banach space and for all $n \in \mathbb{N}$ it follows that*

$$\max_{1\leqslant k\leqslant n} b_k\, k^{1/2} \max\{e_k(T), e_k(T^*)\} \leqslant c \cdot \max_{1\leqslant k\leqslant n} b_k\, l_k(T)\,. \tag{4.10}$$

Our next objective is to estimate $l_n(T)$ from above suitably. Here the following deep result due to G. Pisier turns out to be very useful (cf. [23], Theorem 9.1).

Proposition 4.5. *There exist universal constants $c_1, c_2 > 0$ such that for all operators T from H into E and all $n \in \mathbb{N}$ the estimate*

$$l_n(T) \leqslant c_1 \sum_{k\geqslant c_2 n} e_k(T^*) k^{-1/2}(1+\log k) \tag{4.11}$$

holds. Moreover, if E is K-convex (e.g. $E = L_p$, $1 < p < \infty$), i.e. E does not contain l_1^n's uniformly (cf. [23], Thm. 2.4*), then* (4.11) *is valid without the* log*-term on the right hand side.*

Remark. Note that Proposition 4.5 was originally stated with so-called volume numbers $v_n(T^*)$ on the right hand side of (4.11). The above weaker form follows by $v_n(T^*) \leqslant 2 \cdot e_n(T^*)$.

Proposition 4.6. *Let $(\sigma_n)_{n=1}^{\infty}$ be a decreasing sequence satisfying*

$$\sigma_{2n} \approx \sigma_n \qquad \text{and} \qquad \sup_{n \geqslant k} \frac{n^\alpha \sigma_n}{k^\alpha \sigma_k} < \infty \quad \text{for some } \ \alpha > 0\,.$$

Then one has for operators T from a Hilbert space H into any Banach space E

$$l_n(T) \preceq \sigma_n \implies e_n(T) \preceq n^{-1/2}\sigma_n \implies l_n(T) \preceq \sigma_n\,(1+\log n). \qquad (4.12)$$

If E is K-convex, then one even has the equivalence

$$l_n(T) \preceq \sigma_n \quad \iff \quad e_n(T) \preceq n^{-1/2}\sigma_n\,.$$

Proof. Let $l_k(T) \preceq \sigma_k$. By our assumption $\sigma_{2n} \approx \sigma_n$ it is possible to apply Corollary 4.4 with $b_k = 1/\sigma_k$, and this immediately gives $e_k(T) \preceq k^{-1/2}\sigma_k$.

Let now $e_k(T) \preceq k^{-1/2}\sigma_k$. By (4.6) this implies

$$e_k(T^*) \leqslant c_3\, k^{-1/2}\sigma_k\,.$$

Moreover, due to the second assumption on (σ_n), we have

$$\sigma_k \leqslant c_4 \left(\frac{c_2 n}{k}\right)^\alpha \sigma_{[c_2 n]} \quad \text{for } k \geqslant c_2 n\,.$$

Inserting this in formula (4.11) from Proposition 4.5 we obtain

$$\begin{aligned} l_n(T) &\leqslant c_1 \sum_{k \geqslant c_2 n} k^{-1/2} e_k(T^*)(1+\log k) \\ &\leqslant c_1\, c_3\, c_4\, (c_2 n)^\alpha\, \sigma_{[c_2 n]} \sum_{k \geqslant c_2 n} k^{-1-\alpha}(1+\log k) \\ &\approx \sigma_{[c_2 n]}\,(1+\log n) \approx \sigma_n\,(1+\log n)\,. \end{aligned}$$

If E is K-convex, then this estimate holds without the log-term, and the proof is finished. □

Conjecture. It is very likely that, under the assumptions on (σ_n) in Proposition 4.6, we even have the following stronger version of the second implication in (4.12). At least this is suggested by all known examples.

$$e_n(T) \preceq n^{-1/2}\sigma_n \quad \implies \quad l_n(T) \preceq \sigma_n(1+\log n)^{1/2}$$

Remark. However, if we only assume the doubling condition $\sigma_{2n} \approx \sigma_n$, then the order of the gap between $l_n(T)$ and $e_n(T)$ can be strictly larger than $(n \log n)^{1/2}$, even for operators in Hilbert spaces. For example, if we slightly modify example 2 from above and define $D : \ell_2 \to \ell_2$ by

$$D(x_n) = (n^{-1/2}(1+\log n)^{-1/2}(\log(2+\log n))^{-1/2-\alpha} x_n)$$

for some $\alpha > 0$, then

$$l_n(D) \approx (\log\log n)^{-\alpha} \text{ while } e_n(D) \approx n^{-1/2}(\log n)^{-1/2}(\log\log n)^{-1/2-\alpha} .$$

Hence in this case the gap between $l_n(D)$ and $e_n(D)$ is of order

$$(n \log n \log\log n)^{1/2}.$$

Let us state an important special case of Proposition 4.6.

Proposition 4.7. *Suppose $\alpha > 0$ and $\beta \in \mathbb{R}$. Then for general Banach spaces E and operators T from H into E we have*

$$l_n(T) \preceq \frac{(1+\log n)^\beta}{n^\alpha} \Rightarrow e_n(T) \preceq \frac{(1+\log n)^\beta}{n^{\alpha+1/2}} \Rightarrow l_n(T) \preceq \frac{(1+\log n)^{\beta+1}}{n^\alpha}. \tag{4.13}$$

Note that it is not clear at all whether or not relations similar to (4.12) and (4.13) are valid for lower estimates. But for equivalences we have the following result.

Theorem 4.8. *Let T be an operator from a Hilbert space H into a K-convex Banach space E (e.g. $E = L_p$ for $1 < p < \infty$). If $(\sigma_n)_{n=1}^\infty$ satisfies the assumptions of Proposition* 4.6*, then the following are equivalent:*

(1) $l_n(T) \approx \sigma_n$
(2) $e_n(T) \approx n^{-1/2}\sigma_n$
(3) $e_n(T^*) \approx n^{-1/2}\sigma_n$.

Proof. As mentioned above, the equivalence of (2) and (3) follows either by the results in [1] or in [28]. So it remains to verify (1) $\Longleftrightarrow$ (3).

Let us first assume (1). From Proposition 4.6 we derive then

$$e_n(T^*) \preceq n^{-1/2}\sigma_n .$$

Next we prove that (1) implies

$$e_n(T^*) \succeq n^{-1/2}\sigma_n .$$

By assumption and Proposition 4.5 we have

$$c_0 \cdot \sigma_n \leqslant l_n(T) \leqslant c_1 \cdot \sum_{k \geqslant c_2 n} k^{-1/2}\, e_k(T^*)\,. \tag{4.14}$$

Recall that E is assumed to be K-convex. For some (large integer) $\rho > c_2$, to be specified later, we split the sum on the right hand side of (4.14) as

$$\sum_{c_2 n \leqslant k \leqslant \rho n} \dots + \sum_{k > \rho n} \dots \tag{4.15}$$

and estimate the two parts separately. For the first sum we have

$$\begin{aligned} \sum_{c_2 n \leqslant k \leqslant \rho n} k^{-1/2}\, e_k(T^*) &\leqslant e_{[c_2 n]}(T^*) \sum_{c_2 n \leqslant k \leqslant \rho n} k^{-1/2} \\ &\leqslant 2\,(\rho n)^{1/2} e_{[c_2 n]}(T^*)\,. \end{aligned} \tag{4.16}$$

Now we pass to the second sum in (4.15). We have already shown that (1) implies

$$e_k(T^*) \leqslant c_3 \cdot k^{-1/2} \sigma_k \tag{4.17}$$

for some $c_3 > 0$. Moreover, by the second assumption on (σ_n) there is an $\alpha > 0$ such that

$$\sigma_k \leqslant c_4 \left(\frac{n}{k}\right)^{\alpha} \sigma_n \quad \text{for all} \quad k \geqslant n\,, \tag{4.18}$$

and some constant $c_4 > 0$. Inserting (4.17) and (4.18) into the second sum in (4.15) we obtain

$$\begin{aligned} \sum_{k > \rho n} k^{-1/2} e_k(T^*) \quad &\leqslant \quad c_3 \sum_{k > \rho n} k^{-1} \sigma_k \;\leqslant\; c_3\, c_4\, n^{\alpha}\, \sigma_n \sum_{k > \rho n} k^{-1-\alpha} \\ &\leqslant \quad c_3\, c_4\, n^{\alpha}\, \sigma_n \cdot \frac{1}{\alpha\,(\rho n)^{\alpha}} = \frac{c_3\, c_4}{\alpha\, \rho^{\alpha}} \cdot \sigma_n. \end{aligned}$$

Combining this with (4.14), (4.15) and (4.16) we get

$$\begin{aligned} c_0 \cdot \sigma_n \quad &\leqslant \quad c_1 \cdot \sum_{c_2 n \leqslant k \leqslant \rho n} k^{-1/2} e_k(T^*) + c_1 \cdot \sum_{k > \rho n} k^{-1/2} e_k(T^*) \\ &\leqslant \quad c_1 \cdot (2\rho n)^{1/2} \cdot e_{[c_2 n]}(T^*) \;+\; \frac{c_1\, c_3\, c_4}{\alpha\, \rho^{\alpha}} \cdot \sigma_n\,. \end{aligned}$$

Now we choose $\rho > c_2$ so large that

$$\frac{c_1\, c_3\, c_4}{\alpha\, \rho^{\alpha}} \leqslant \frac{c_0}{2}\,.$$

Note that the constants $c_0, ..., c_4$ are independent on ρ. This gives

$$e_{[c_2 n]}(T^*) \geqslant \frac{c_0}{2c_1(2\rho)^{1/2}} \cdot n^{-1/2}\sigma_n \,,$$

and by $\sigma_{2n} \approx \sigma_n$ we conclude that $e_n(T^*) \succeq n^{-1/2}\sigma_n$. Thus we have shown (1) $\Rightarrow$ (3).

Let us verify now (3) $\Rightarrow$ (1). The upper estimate for $l_n(T)$ follows directly from (3) via Proposition 4.6.

For the lower estimate we use Corollary 4.4, formula (4.10), with

$$b_k = \frac{k}{\sigma_k} \,.$$

This is possible, since (b_k) is increasing and satisfies the doubling condition $b_k \approx b_{2k}$. Hence there are constants $c_1, c_2, c_3 > 0$ such that

$$e_k(T^*) \geqslant c_1\, k^{-1/2}\sigma_k \quad , \quad l_k(T) \leqslant c_2\sigma_k \;\text{ for all } \quad k \in \mathbb{N} \tag{4.19}$$

and

$$\max_{1\leqslant k\leqslant mn} b_k\, k^{1/2} e_k(T^*) \leqslant c_3 \max_{1\leqslant k\leqslant mn} b_k l_k(T) \qquad \text{for all } m, n \in \mathbb{N}\,. \tag{4.20}$$

Here $m \in \mathbb{N}$ is an auxiliary parameter that will be specified later. The simple idea, based on a clever argument in [3] which has also been used in [24] and [6], is to show that the maximum on the right hand side of (4.20) cannot be attained for $k \leqslant n$, provided that m is large enough. The proper choice is $m > c_2\, c_3/c_1$. Indeed, then (4.19) implies

$$\max_{1\leqslant k\leqslant mn} b_k\, k^{1/2} e_k(T^*) \geqslant c_1\, mn \;\text{ and }\; c_3 \max_{1\leqslant k\leqslant n} b_k\, l_k(T) \leqslant c_3\, c_2\; mn < c_1\, mn\,.$$

Hence the maximum on the right hand side of (4.20) is attained for some $k \geqslant n$ and we get

$$c_1\, mn \leqslant c_3 \max_{n\leqslant k\leqslant mn} b_k\, l_k(T) \leqslant c_3\, b_{mn}\, l_n(T) = c_3 \cdot \frac{mn}{\sigma_{mn}} \cdot l_n(T)\,.$$

This shows the desired lower bound for the Gaussian approximation numbers

$$l_n(T) \succeq \sigma_{mn} \approx \sigma_n$$

and completes the proof of the Proposition. $\square$

An important special case of Theorem 4.8 reads as follows:

Proposition 4.9. *If $\alpha > 0$ and $\beta \in \mathbb{R}$, then we have for operators T from H into a K-convex Banach space E*

$$l_n(T) \approx n^{-\alpha}(1+\log n)^{\beta} \quad \text{if and only if} \quad e_n(T) \approx n^{-\alpha-1/2}(1+\log n)^{\beta}\,.$$

Finally let us state a result in [16] which relates the Gaussian approximation numbers $l_n(T)$ with the "ordinary" approximation numbers $a_n(T)$.

Proposition 4.10. *Let T be an operator from H into E and let m, n be any natural numbers. Then it holds*

$$\sqrt{\log(m+1)}\cdot a_{n+m}(T) \leqslant c\cdot l_n(T)$$

with some universal $c > 0$. In particular, for all $n \geqslant 1$

$$\sqrt{\log(n+1)}\cdot a_{2n}(T) \leqslant c\cdot l_n(T).$$

§5. An Example

Given $d \geqslant 1$, the d-dimensional integration operator V_d is defined by

$$(V_d f)(t) = \int_0^{t_1}\cdots\int_0^{t_d} f(x_1,\ldots,x_d)\,\mathrm{d}x_d\cdots\mathrm{d}x_1\,, \quad t=(t_1,\ldots,t_d)\,.$$

It is known and easy to see that V_d is a bounded operator from $L_2[0,1]^d$ into the Banach space $C[0,1]^d$ of continuous functions on $[0,1]^d$.

The following has been proved in [16].

Proposition 5.1. *The Gaussian approximation numbers of V_d satisfy*

$$l_n(V_d) \approx n^{-1/2}(1+\log n)^{d-1} \quad \text{if} \quad V_d : L_2[0,1]^d \to L_p[0,1]^d\,,\ 1<p<\infty\,,$$

and

$$l_n(V_d) \approx n^{-1/2}(1+\log n)^{d-1/2} \quad \text{if} \quad V_d : L_2[0,1]^d \to C[0,1]^d\,.$$

From Propositions 4.7 and 4.9 together with

$$e_n\big(V_d : L_2[0,1]^d \to L_2[0,1]^d\big) \leqslant e_n\big(V_d : L_2[0,1]^d \to C[0,1]^d\big)$$

we derive the following result, which by different methods had been proved in [8].

Proposition 5.2. *The entropy numbers of V_d satisfy*

$$e_n(V_d) \approx n^{-1}(1+\log n)^{d-1} \quad \text{if} \quad V_d : L_2[0,1]^d \to L_p[0,1]^d\,,\ 1<p<\infty\,,$$

and

$$n^{-1}(1+\log n)^{d-1} \preceq e_n(V_d) \preceq n^{-1}(1+\log n)^{d-1/2} \quad \text{if} \quad V_d : L_2[0,1]^d \to C[0,1]^d\,. \tag{5.1}$$

Remark. For $d = 1$, the behavior of $e_n\big(V_d : L_2[0,1]^d \to C[0,1]^d\big)$ is well-known, and for $d = 2$ it is due to M. Talagrand in [27]. It holds

$$e_n(V_1) \approx n^{-1} \quad \text{and} \quad e_n(V_2) \approx n^{-1}(1+\log n)^{3/2}\,.$$

If $d > 2$, then the exact behavior of $e_n(V_d)$ is an open problem. There is a partial result in [2] asserting the following.

Proposition 5.3. *If $d > 2$, there is an $\eta > 0$ such that*

$$n^{-1}(1+\log n)^{d-1+\eta} \preceq e_n\big(V_d : L_2[0,1]^d \to C[0,1]^d\big)\,.$$

In particular, if $d \geqslant 2$, then the lower estimate in (5.1) *is not sharp.*

Finally, for the sake of completeness, let us also mention the behavior of $a_n(V_d)$. Here we have (see [16])

$$a_n(V_d : L_2[0,1]^d \to L_2[0,1]^d) \approx n^{-1}(1+\log n)^{d-1}$$

and

$$a_n(V_d : L_2[0,1]^d \to C[0,1]^d) \approx n^{-1/2}(1+\log n)^{d-1}\,.$$

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Universität Leipzig, Augustusplatz 10,
04109 Leipzig, Germany
E-mail: kuehn@math.uni-leipzig.de

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University of Delaware, 402 Ewing Hall,
Newark DE, 19716, USA
E-mail: lindew@udel.edu