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# ON AN EXPONENTIAL FUNCTIONAL FOR GAUSSIAN PROCESSES AND ITS GEOMETRIC FOUNDATIONS

Abstract. After setting geometric notions, we revisit an exponential functional that has arisen in several contexts, with special attention to a set of geometric parameters and associated inequalities.

## §1. Introduction

It is an honor and a pleasure to contribute to this volume. V.N. Sudakov's work has had a great influence on my own interests. In that spirit, what follows is a note on an exponential functional that bears on the structure of bounded Gaussian processes. The content is largely expository and begins with a review of relevant notions from classical convex geometry and their extension to infinite dimensions. We then recall the exponential functional, including a basic inequality, and a set of geometric parameters. The latter are re-examined for an alternate representation and then related inequalities are discussed.

## §2. Background

In what follows, aspects of geometric convexity not otherwise referenced can be found in the excellent monograph [19]. As stated there, the key feature of *Brunn–Minkowski theory* is the interaction of volume evaluation and vector addition of convex bodies (non-empty, compact, convex subsets): for convex bodies $K_1, K_2, \ldots, K_n$ in $\mathbb{R}^d$ and positive coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$,

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$$\mathrm{vol}_d\,(\lambda_1 K_1 + \lambda_2 K_2 + \cdots + \lambda_n K_n) = \sum_{i_1, i_2, \cdots, i_d = 1}^{n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_d} V(K_{i_1}, K_{i_2}, \ldots, K_{i_d}), \quad (1)$$

where, without loss of generality, the "mixed volumes" $V(\cdots)$ are taken to be symmetric in their arguments. For the special case of a parallel body $K + \lambda B_d$ ($B_d$, the unit ball in $I\!R^d$), (1) is the classical Steiner formula

$$\mathrm{vol}_d(K + \lambda B_d) = \sum_{j=0}^{d} \lambda^i \binom{d}{j} W_j(K), \quad (2)$$

where

$$W_j(K) = V(\underbrace{K, K, \cdots, K}_{k-j}, \underbrace{B_d, B_d, \cdots, B_d}_{j}), \quad 0 \leqslant j \leqslant d$$

are the *quermassintegrals* or *Minkowski functionals* (one should note that the latter term also refers to a different object in the literature). Unfortunately, they have the inconvenient property of depending on $d$, the dimension of the specific ambient space. A modified collection is free of this property: the *intrinsic volumes* [2, 16] are given by

$$V_j(K) = \frac{\binom{d}{j}}{\kappa_j} W_{d-j}(K), \quad 0 \leqslant j \leqslant d. \quad (3)$$

Here $\kappa_j$ is the volume of $B_j$, and one can extend (3) by taking $V_j(K) = 0$ for $d < j$ (by contrast, infinite–dimensional $K$ will have $V_j(K) > 0$ for all $j$). We note $V_0(K) = 1$ and three other specific cases: $V_d(K) = \mathrm{vol}_d(K)$, $V_{d-1}(K) = (1/2)S_{d-1}(K)$ (i.e., 1/2 the surface area of $K$), and $V_1(K)$, which is a mean–width type functional normalized so that if $K$ is a line segment, then $V_1(K)$ is its length.

The corresponding version of the Steiner formula reads

$$\mathrm{vol}_d(K + \lambda B_d) = \sum_{i=0}^{d} \lambda^j \kappa_j V_{d-j}(K). \quad (4)$$

The *Alexandrov–Fenchel inequality* asserts that for convex bodies $K_1$, $K_2, \dots, K_d$ in $\mathbb{R}^d$

$$V^2(K_1, K_2, K_3, \dots, K_d) \geqslant V(K_1, K_1, K_3, \dots, K_d)\, V(K_2, K_2, K_3, \dots, K_d). \quad (5)$$

Specifying to intrinsic volumes and making an appropriate adjustment of constants, (5) can be shown to imply logconcavity of the sequence $\{j!\, V_j(K)\}_{j=0}^{\infty}$ :

$$(j!\, V_j(K))^2 \geqslant (j-1)! V_{j-1}(K) \cdot (j+1)! V_{j+1}(K) \qquad j = 1, 2, \dots \quad (6)$$

and a direct consequence

$$V_j(K) \leqslant \frac{V_1^j(K)}{j!} \quad j = 1, 2, \dots \quad (7)$$

[2, 17].

## §3. Extension of Intrinsic Volumes to Infinite–Dimensional Bodies

It was the celebrated insight of Sudakov ([21–23]; Theorem 1 below) that connected the geometric structure just described and Gaussian processes. This was subsequently elaborated by Chevet and Tsirelson. We give a brief review.

For a convex body $K$ in Hilbert space ( $\Longleftrightarrow$ $\ell_2$), consider a Gaussian process $\{X_t, t \in K\}$[1] that is *isonormal*:

$$t \longmapsto X_t \sim N(0, \sigma_t^2),$$

where $\sigma_t^2 = \operatorname{Var} X_t = \|t\|^2$ and $\operatorname{Cov}(X_t, X_{\hat{t}}) = \langle t, \hat{t}\rangle$ (scalar product). An important question is whether there is a version that is a.s. bounded, formulated by Dudley [3] as to whether $K$ is a *GB–set*.

On the geometric side, and making use of the monotonicity of $V_1(\cdot)$, set

$$V_1(K) = \sup\left\{V_1(\widehat{K}) \,:\, \widehat{K} \subseteq K,\ \ \widehat{K} \text{ finite-dimensional}\right\}. \quad (8)$$

Then Sudakov established

**Theorem 1.** *$K$ is a GB–set if and only if $V_1(K)$ is finite.*

[1]Here, and below, $t \in K$ means by convention that $t$ ranges over a (resp., any) countable dense subset of $K$.

In what follows, we assume that all relevant $K$ are GB.

Chevet [2] similarly extended by monotonicity the other intrinsic volumes $V_j, j = 2, 3, \ldots$, established (7), and thereby concluded that

$$V_1(K) < \infty \implies V_j(K) < \infty, \quad j = 2, 3, \ldots .$$

Sudakov showed specifically that

$$V_1(K) = \sqrt{2\pi}\ \mathrm{E} \sup_{t \in K} X_t \,. \tag{9}$$

In an important step, Tsirelson [25] placed (9) within a family of representations for all of the intrinsic volumes. Accommodating technical issues somewhat differently, a sketch is as follows: for given $j$, consider

$$X_t^{j*} = \left( X_t^{(1)}, X_t^{(2)}, \ldots, X_t^{(j)} \right),$$

where the components are independent copies of $X_t$, together with the vector process

$$X_K^{j*} = \{X_t^{j*},\, t \in K\}.$$

The closed convex hull

$$Y_{j,K} = \overline{\mathrm{conv}} \left( X_K^{j*} \right)$$

is a candidate for a random convex body in $\mathbb{R}^j$, and, accordingly, its measurability must be established. To do this, we make use of its support function $h_{Y_{j,K}} : S^{j-1} \to \mathbb{R}^1$, given by

$$\begin{aligned} h_{Y_{j,K}}(u) &= \sup \{ < y, u > \,|\, y \in Y_{j,K} \} \\ &= \sup \left\{ < x, u > \,|\, x \in X_K^{j*} \right\} \\ &= \sup \left\{ \sum_{i=1}^{j} X_t^{(i)} u_i \,|\, t \in K \right\}, \end{aligned}$$

which is evidently a random variable for each $u$. Now measurability of $Y_{j,K}$ coincides with measurability of the quantity $\delta_H\left(Y_{j,K}, L\right)$ for every convex body $L$ in $\mathbb{R}^j$, where $\delta_H$ is the Hausdorff metric. This is confirmed by recalling that

$$\begin{aligned} &\delta_H\left(Y_{j,K}, L\right) \\ &\quad = \sup \left\{ |h_{Y_{j,K}}(u) - h_L(u)| \,|\, u \in \text{a countable, dense subset of } S^{j-1} \right\}. \end{aligned}$$

With the foregoing in place, Tsirelson's representation [25, Theorem 6] is

$$V_j(K) = \frac{(2\pi)^{j/2}}{j!\,\kappa_j}\,\mathrm{E}\,\mathrm{vol}_j\,(Y_{j,K}) \qquad j = 1, 2, \dots . \tag{10}$$

For what follows, and in view of the standard isonormal map $t \mapsto X_t = \langle t, Z\rangle = \sum_1^\infty t_i Z_i$, where $\{Z_i\}_1^\infty$ is a sequence of standard normal random variables, we introduce the suggestive notation

$$Z_{[\,j,\infty]}K = Y_{j,K}, \tag{11}$$

where $Z_{[\,j,\infty]}$ is a $j \times \infty$ matrix of independent standard normal random variables. Finally we mention that an alternate proof of the representation was given by the author [31] based on a theorem of Hadwiger characterizing intrinsic volumes ( [6]; see also [10]).

## §4. The Wills Functional

In various forms, the functional of the title has arisen independently in (i) geometry [7, 8, 32] (from where we take its name), (ii) maximum likelihood estimation of location [24–26], and (iii) financial mathematics [1]; see also [27–29]. For a convex body $K$ in $\mathbb{R}^d$, the *Wills functional* is given by

$$W(K) = \sum_{j=0}^{d} V_j((1/\sqrt{2\pi})K) = \sum_{j=0}^{d}(1/(2\pi)^{j/2})V_j(K) \tag{12}$$

[32].[2] A different expression for $W(K)$ also obtains:

$$\int_{\mathbb{R}^d} e^{-\pi\,\mathrm{dist}^2(x,(1/\sqrt{2\pi})K)}\mathrm{d}x. \tag{13}$$

where $\mathrm{dist}(x, (1/\sqrt{2\pi})\,K) = \inf_{t\in(1/\sqrt{2\pi})K}\|x-t\|$. Following [7], the equivalence of the two expressions was shown in [27], and we repeat that here for the reader's convenience. Consider

$$W(K) = \mathrm{E}\,\mathrm{vol}_d\left((1/\sqrt{2\pi})K + \Lambda B_d\right), \tag{14}$$

[2]We note that the scaling of $K$ by $1/\sqrt{2\pi}$ does not appear in the original formulation of Wills as followed in the geometry literature and also in [27]. The present normalization was adopted by the author in [29] as somewhat better fitted to Gaussian contexts; see also [25].

where $\Lambda$ is a random variable with density $f(\lambda) = 1(\lambda \geqslant 0) 2\pi\, \lambda\, e^{-\pi \lambda^2} x$. Expanding the volume expression, taking expectations, and making note of $\mathrm{E}\,\Lambda^j = \frac{1}{\kappa_j}, j = 0, 1, 2, \ldots$ yields (12). For the second representation, again start with (14), but now set

$$\mathrm{vol}_d((1/\sqrt{2\pi})K + \Lambda B_d) = \int\limits_{\mathbb{R}^d} 1 \left[ \mathrm{dist}\left(x, (1/\sqrt{2\pi})K\right) \leqslant \Lambda \right] \mathrm{d}x.$$

Taking expectations and invoking Fubini gives (13).

Now we make a change of variables $z = \sqrt{2\pi}\, x$ in (13) to get equivalently

$$\left(\frac{1}{2\pi}\right)^{d/2} \int\limits_{\mathbb{R}^d} e^{-(1/2)\,\mathrm{dist}^2(z,K)} \mathrm{d}z = \left(\frac{1}{2\pi}\right)^{d/2} \int\limits_{\mathbb{R}^d} e^{\sup_{t \in K} [\langle t, z\rangle - (1/2)\|t\|^2]} e^{-(1/2)\|z\|^2} \mathrm{d}z.$$

For an isonormal Gaussian process $\{X_t, t \in K\}$ given by $X_t = \langle t, Z \rangle$, $Z$ $d$–dimensional standard normal, we have thus shown that

$$W(K) = \mathrm{E}\, e^{\sup_{t \in K} \left[X_t - (1/2)\sigma_t^2\right]}. \tag{15}$$

Extension of the domain of $W$ to infinite–dimensional $K$ is naturally done via finite–dimensional approximation as in (8). Representation (15), and also (12) in the form

$$W(K) = \sum_{j=0}^{\infty} (1/(2\pi)^{j/2}) V_j(K), \tag{16}$$

are maintained. Tsirelson [25] gave a proof of this using specifically polytopal approximants and a result of Chevet [2]. He further showed, by inserting the domination (7) into (16), the inequality

$$W(K) \leqslant e^{(1/\sqrt{2\pi}) V_1(K)}, \tag{17}$$

equivalently,

$$\mathrm{E}\, e^{\sup_{t \in K} \left\{X_t - (1/2)\sigma_t^2\right\}} \leqslant e^{\mathrm{E} \sup_{t \in K} X_t}. \tag{18}$$

( [25]; see also [17,27,28] and Remark 1 below). The latter guarantee that (15) and (16) are in fact finite for any GB $K$ and are interesting in their own right as well. In section 6, we discuss variants.

The asymptotic form of $W(rK)$, $r \to \infty$, was studied in [29]. The context there (see also [11]) was a geometric treatment of the Itô–Nisio phenomenon [9] which showed that, in a weak sense, a local neighborhood of a discontinuity of $\{X_t, t \in K\}$ generically resembles a ball of small radius and high dimension. Relevant here is the following: for $t \in K$, let $B(t,\varepsilon)$ be the $t$-centered ball of radius $\varepsilon$ and set

$$\delta(t) = \lim_{\varepsilon \to 0} \left[ \sup_{s \in K \cap B(t,\varepsilon)} X_s - \inf_{s \in K \cap B(t,\varepsilon)} X_s \right].$$

Each of these limits is an almost sure constant. Considering them as numbers, set $\Delta(K) = \sup_{t \in K} \delta(t)$ (departing from convention, we regard this as over *all* $t \in K$). Then

$$W(rK) = e^{(\Delta(K)/2)r + o(r)}. \tag{19}$$

An important tool in [29] was a class of geometric parameters $\{m_j(K)\}_1^\infty$ such that

$$\mathrm{E} \sup_{t \in K} X_t = m_1(K) \geqslant \cdots \geqslant m_{j-1}(K) \geqslant m_j(K) \geqslant \cdots \to \Delta(K)/2. \tag{20}$$

In what follows we examine their structure further and discuss related inequalities.

## §5. Quasi–widths

Following [29, 30], we set

$$m_j(K) = \frac{j V_j(K)}{\sqrt{2\pi} V_{j-1}(K)} \quad j = 1, 2, \ldots \tag{21}$$

For each $j$, $m_j(rK)$ is homogeneous of degree 1 in $r$ and accordingly we call it the *quasi–width of order* $j$. One has

$$m_1(K) = (1/\sqrt{2\pi}) V_1(K) = \mathrm{E} \sup_{t \in K} X_t, \tag{22}$$

and that, as a consequence of (6), the quasi–widths form a decreasing sequence. For a further understanding, we derive an alternate expression to (21). In the numerator, recall that

$$V_j(K) = \frac{(2\pi)^{j/2}}{j!\,\kappa_j} \mathrm{E}\, \mathrm{vol}_j \left( Z_{[j,\infty]} K \right). \tag{23}$$

Similarly, in the denominator there is

$$V_{j-1}(K) = \frac{(2\pi)^{(j-1)/2}}{(j-1)!\,\kappa_{j-1}}\,\mathrm{E}\,\mathrm{vol}_{j-1}(Z_{[\,j-1,\infty]}K), \tag{24}$$

which we re–express by noting that in distribution

$$Z_{[\,(j-1),\infty]} = \Pi_{j-1} Z_{[\,j,\infty]},$$

where the independent matrix $\Pi_{j-1}$ consists of the first $j-1$ rows of a random $j \times j$ orthogonal matrix. Then

$$\begin{aligned}\mathrm{E}\,\mathrm{vol}_{j-1}&(Z_{[\,j-1,\infty]}K)\\ &= \mathrm{E}\,\mathrm{vol}_{j-1}(\Pi_{j-1} Z_{[\,j,\infty]}K)\\ &= \mathrm{E}\{\mathrm{E}\left[\mathrm{vol}_{j-1}(\Pi_{j-1} Z_{[\,j,\infty]}K) \mid Z_{[\,j,\infty]}K\right]\}.\end{aligned} \tag{25}$$

Now with *Kubota's integral recursion* [2, 19, 25] in the special case of *Cauchy's surface area formula,* one has for $j$–dimensional $K_0$

$$\mathrm{E}\,\mathrm{vol}_{j-1}(\Pi_{j-1}K_0) = \frac{2\kappa_{j-1}}{j\,\kappa_j} V_{j-1}(K_0) = \frac{\kappa_{j-1}}{j\kappa_j} S_{j-1}(K_0)$$

Applying this to the inner expectation in the final expression in (25), we have

$$\mathrm{E}\left[\mathrm{vol}_{j-1}(\Pi_{j-1} Z_{[\,j,\infty]}K) \mid Z_{[\,j,\infty]}K\right] = \frac{\kappa_{j-1}}{j\,\kappa_j} S_{j-1}(Z_{[\,j,\infty]}K).$$

It follows that

$$\mathrm{E}\,\mathrm{vol}_{j-1}(Z_{[\,j-1,\infty]}K) = \frac{\kappa_{j-1}}{j\,\kappa_j}\,\mathrm{E}\,S_{j-1}(Z_{[\,j,\infty]}K).$$

Inserting this into (24) gives

$$V_{j-1}(K) = \frac{(2\pi)^{(j-1)/2}}{j!\,\kappa_j}\,\mathrm{E}\left[S_{j-1}(Z_{[\,j,\infty]}K)\right]. \tag{26}$$

Substituting this and (23) into (21), we finally get

$$m_j(K) = \frac{j \cdot \mathrm{E}\,\mathrm{vol}_j(Z_{[\,j,\infty]}K)}{\mathrm{E}\,S_{j-1}(Z_{[\,j,\infty]}K)}\quad, \tag{27}$$

which was our goal. It expresses $m_j(K)$ in terms of mean behavior of the single $j$–dimensional random convex body $Z_{[\,j,\infty]}K$.

## §6. A Class of Inequalities

We turn now to a generalization of (17). Specifically, following [30], we show that a class of bounds in terms of quasi–widths comes about by varying the domination (7): recall from (6) that $a_j = j!\,V_j(K)$, $j = 0, 1, 2, \ldots$ is a log-concave sequence:

$$\log a_j \leqslant \log a_i + (\log a_{i+1} - \log a_i)(j - i),$$

for all $i, j = 0, 1, 2, \ldots$. Equivalently, for any fixed $i \in \{0, 1, 2, \ldots\}$, this can be read as

$$V_j(K) \leqslant \frac{i!\,V_i(K)}{j!} \left( \frac{(i+1)V_{i+1}(K)}{V_i(K)} \right)^{j-i} \qquad j = 0, 1, 2, \ldots. \tag{28}$$

It is of interest to re–express this. From (21), one has

$$\frac{(i+1)V_{i+1}(K)}{V_i(K)} = (2\pi)^{1/2} m_{i+1}(K) \tag{29}$$

and taking the product of (21) for $j = 1, 2, \ldots, i$ provides

$$i!V_i(K) = (2\pi)^{i/2} \prod_{j=1}^{i} m_j(K). \tag{30}$$

Substituting (29) and (30) into (28) and re–arranging gives

$$V_j(K) \leqslant c_i(K) \cdot \frac{(2\pi)^{j/2}\, m_{i+1}^{j}(K)}{j!}, \tag{31}$$

where

$$c_i(K) = \frac{\prod_{j=1}^{i} m_j(K)}{m_{i+1}^{i}(K)} = \prod_{j=1}^{i} \frac{m_j(K)}{m_{i+1}(K)} \tag{32}$$

(taking $c_0(K) = 1$). Finally, substituting the domination (31) into (16) yields

$$W(K) \leqslant c_i(K) e^{m_{i+1}(K)}, \quad i = 0, 1, 2, \ldots \tag{33}$$

thus generalizing (17) (i.e., $i = 0$) to the other quasi–widths (we note there is a minor typo in the corresponding expression in [30]).

A class of deviation bounds can also be deduced. First note that, from (33) with $r \geqslant 0$,

$$W(rK) \leqslant c_i(K)e^{m_{i+1}(K)r}, \quad i = 0, 1, 2, \dots, \tag{34}$$

using the fact that $c_i(rK)$ is homogeneous of degree 0 in $r$. Then following [27], one can re–express (34) as

$$\mathrm{E}\, e^{\sup_{t\in K}\left\{rX_t - r^2(1/2)\sigma_t^2\right\}} \leqslant c_i(K)e^{m_{i+1}(K)r}. \tag{35}$$

Setting $\sigma^2 = \sup_{t\in K}\sigma_t^2$ and re-arranging then provides

$$\mathrm{E}\, e^{r\left[\sup_{t\in K} X_t - m_{i+1}(K)\right]} \leqslant c_i(K)e^{(1/2)\sigma^2 r^2}. \tag{36}$$

Applying Markov's inequality gives for $a > 0$

$$\mathrm{P}(\sup_{t\in K} X_t - m_{i+1}(K) \geqslant a) \leqslant c_i(K)e^{(1/2)\sigma^2 r^2 - ar}$$

and minimizing the bound at $r = a/\sigma^2$ finally yields

$$\mathrm{P}(\sup_{t\in K} X_t - m_{i+1}(K) \geqslant a) \leqslant c_i(K)e^{-a^2/(2\sigma^2)}. \quad i = 0, 1, 2, \dots \tag{37}$$

The case $i = 0$, well–known in the probability literature in the form

$$\mathrm{P}(\sup_{t\in K} X_t - \mathrm{E}\sup_{t\in K} X_t \geqslant a) \leqslant e^{-a^2/(2\sigma^2)}$$

(e.g., [12, 14]), was similarly shown in [27].

In a different vein, one can think of looking for bounds sharper than those in (33). One option is to express (31) for both $i \geqslant 1$ and $i-1$. Then, for a given $j$, choose the domination that is tighter. This amounts to using the first domination for $j \geqslant i$ and the second for $j \leqslant i-1$ (the two being the same at $j = i$). That is,

$$V_j(K) \leqslant \begin{cases} c_{i-1}(K)\cdot \frac{(2\pi)^{j/2}\, m_i^j(K))}{j!} & j = 0, 1, 2, \dots, i-1 \\ c_i(K)\cdot \frac{(2\pi)^{j/2}\, m_{i+1}^j(K))}{j!} & j = i, i+1, i+2, \dots \end{cases},$$

and consequently

$$W(K) \leqslant c_{i-1}(K)\sum_{j=0}^{i-1}\frac{m_i^j(K)}{j!} + c_i(K)\sum_{j=i}^{\infty}\frac{m_{i+1}^j(K)}{j!}.$$

Finally, echoing a comment in [30], we note that the natural way in which quasi–widths emerge in the derivation of (33), as well as their appearance in (20), suggests that they bear further examination as functionals of interest for both $K$ and the process $\{X_t, t \in K\}$. In this regard, we mention as well the functionals $c_i(K), i = 1, 2, \ldots$, which, as noted, are homogeneous of degree 0 in $r$ and thus can be regarded as "shape" parameters for $K$.

## §7. Final Remarks

(1) A significant generalization of (18), including a left–tail probability bound, was shown by Borell [1].

(2) Following the above discussion, it is not possible to let $i \to \infty$ in (37), make use of (20), and produce the analogous statement with $m_i(K)$ replaced by $\Delta/2$, this because of no established control over the $c_i(K)$. However, a result of this type was shown in [29] using other means, in which the explicit intermediate estimates (35), (37) are bypassed (note there that the statement of Theorem 4 has a typographical error ("=" should be "$\leqslant$") and, in any case, does not always reflect the exact asymptotics as claimed (e.g., [13, 14]); the reader is also cautioned that in [29] the definition of "oscillation" carries a factor of 1/2 compared to the conventional definition).

(3) For additional geometric understanding of $m_2(K)$ (via $V_2(K)$), see [2] and [4].

(4) In view of the key role that (6) played in the discussion above, we note that it appeared in [18] as *ultra–logconcavity of order* $\infty$ of the sequence $\{V_j(K)\}_1^\infty$. In that study (relating to negative dependence of random variables), the closure of the class of such sequences under convolution was conjectured. This was verified in [15] with a later, geometrically–based, proof in [5] using a theorem of Shephard [20] involving mixed volumes and a special case of (1).

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