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# SUBGROUPS OF THE GENERAL LINEAR GROUP CONTAINING THE ELEMENTARY SUBGROUP OVER A COMMUTATIVE RING EXTENSION OF RANK 2

Abstract. Let $R = \prod_{i\in I} F_i$ be a direct product of fields and let $S = R[\sqrt{d}] = \prod_{i\in I} F_i[\sqrt{d_i}]$ be a ring extension of rank 2 of $R$. The subgroups of the general linear group $\mathrm{GL}(2n, R), n \geqslant 3$ that contain the elementary group $E(n, S)$ are described. It is shown that for every such a subgroup $H$ there exists a unique ideal $A \trianglelefteq R$ such that

$$E(n, S)E(2n, R, A) \leqslant H \leqslant N_{\mathrm{GL}(2n,R)}(E(n, S)E(2n, R, A)).$$

## §1. Introduction

Let $R$ be an associative ring with 1 and let $S$ be a ring extension of $R$, which is a free $R$-module of rank $m$. Then $S$ is considered as a subring of the matrix ring $M(m, R)$ via the regular representation, and so $\mathrm{GL}(n, S)$ is a subgroup of the general linear group $\mathrm{GL}(mn, R)$. We are interested in intermediate subgroups between $E(n, S)$ and $\mathrm{GL}(mn, R)$. When $R = S$, the lattice of all such subgroups is described for various classes of rings such as commutative rings, regular rings, rings with condition of stable rank, etc (see, e.g. [1], [2], [3]). It is shown that every subgroup $H$ of $\mathrm{GL}(n, R)$ normalized by $E(n, R)$ has a standard description, i.e., there exists a unique ideal $A \trianglelefteq R$ such that $E(n, R, A) \leqslant H \leqslant C(n, R, A)$. When $m > 1$, ShangZhi Li [4] solved the problem for division rings $R, S$. In this case, for every such a subgroup $H$, there exists an intermediate division ring $T$ between $R$ and $S$ such that $SL(nk, T) \leqslant H \leqslant \mathrm{GL}(nk, T) \rtimes \mathrm{Aut}(T/R)$, where $k = \dim_T S$. In this paper, we consider the problem when $R = \prod_{i\in I} F_i$

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is a direct product of fields and $S \subseteq \prod_{i \in I} \overline{F_i}$ is a commutative ring extension of rank 2 of $R$. We show that under some assumptions, for every subgroup $H$ in $\mathrm{GL}(2n, R)$ containing $E(n, S)$, there exists a unique ideal $A \trianglelefteq R$ such that

$$E(n,S)E(2n,R,A) \leqslant H \leqslant N_{\mathrm{GL}(2n,R)}(E(n,S)E(2n,R,A)).$$

## §2. Principal notation

Let $G$ be a group. For two elements $x, y \in G$, we denote by $[x, y] = xyx^{-1}y^{-1}$ their commutator, by $x^y = y^{-1}xy$ and ${}^y x = yxy^{-1}$ the conjugates of $x$ by $y$ and $y^{-1}$, respectively. We write $H \leqslant G$ to mean that $H$ is a subgroup of $G$. For a subset $X \subseteq G$, we denote by $\langle X \rangle$ the subgroup of $G$ generated by $X$, and by $\langle X \rangle^H$ the smallest subgroup $G$ normalized by $H$ which contains $X$. For two subgroups $F, H \leqslant G, [F, H]$ is the corresponding relative commutator subgroup generated by all commutators $[f, h], f \in F, h \in H$. The group $G$ is called perfect if $[G, G] = G$.

Now, let $R$ be an arbitrary associative ring with 1. When $A$ is an ideal of $R$, we write $A \trianglelefteq R$. For two natural numbers $m, n, M(m, n, R)$ is the additive group of $m \times n$ matrices with entries in $R$, in particular, $M(n, R) = M(n, n, R)$ is the matrix ring of degree $n$ over $R$. For every $b = (b_{rs}) \in M(mn, R)$, considering $b \in M(n, (M(m, R))$, we write $b = (B_{hk})$, $1 \leqslant h, k \leqslant n$ with $B_{hk} = (b_{ij}) \in M(m, R), (h-1)m + 1 \leqslant i \leqslant hm$, $(k-1)m + 1 \leqslant j \leqslant km$. As always, $R^*$ is the multiplicative group of $R$ and $\mathrm{GL}(n, R) = M(n, R)^*$ is the general linear group of degree $n$ over $R$. As usual, $a_{ij}$ is the entry of a matrix $a \in \mathrm{GL}(n, R)$ at the position $(i, j)$, i.e., $a = (a_{ij}), 1 \leqslant i, j \leqslant n$. Next, $a^{-1} = (a'_{ij})$ is the inverse of $a$ and $a^t$ is its transpose. By $a_{*j} = (a_{1j}, ..., a_{nj})^t$ we denote the $j$th column of $a$ and by $a_{i*} = (a_{i1}, ..., a_{in})$ its $i$th row. As usual, we denote by $e$ the identity matrix and $e_{ij}$ a standard matrix unit, that is the matrix which has 1 in the position $(i, j)$ and zeros elsewhere. For $\xi \in R$ and $1 \leqslant i \neq j \leqslant n, t_{ij}(\xi) = e + \xi e_{ij}$ is an elementary transvection. In the sequel, without any special reference we use standard relations among elementary transvections, such as the additivity formula $t_{ij}(\xi)t_{ij}(\zeta) = t_{ij}(\xi + \zeta)$ and the Chevalley commutator formula $[t_{ij}(\xi), t_{jh}(\zeta)] = t_{ih}(\xi\zeta)$.

We denote by $R^n$ the right free $R$-module of all columns of height $n$ with components from $R$, and by ${}^nR$ the left free $R$-module of all rows of length $n$ with components from $R$. The standard bases of $R^n$ and ${}^nR$ are denoted by $e_1, \dots, e_n$ and $f_1, \dots, f_n$ respectively. In other words, $e_i$ is the $i$th column $e_{*i}$ of the identity matrix $e$ of degree $n$, while $f_i$ is the $i$th row $e_{i*}$ of $e$. A column $v = (v_1, \dots, v_n)^t \in R^n$ is said to be unimodular if the left ideal generated by $v_1, \dots, v_n$ coincides with $R$. Similarly, a row $u = (u_1, \dots, u_n)^t \in^n R$ is said to be unimodular if the right ideal generated by $u_1, \dots, u_n$ coincides with $R$.

Now, let $A$ be an ideal in $R$. We denote by $E(n, A)$ the subgroup in $\mathrm{GL}(n, R)$ generated by all elementary transvections of level $A$:

$$E(n, A) = \langle t_{ij}(\xi), \xi \in A, 1 \leqslant i \neq j \leqslant n \rangle.$$

In the most important case where $A = R$, the group $E(n, R)$ generated by all elementary transvections is called the elementary group. In the sequel a major role is played by the relative elementary group $E(n, R, A)$. Recall that the group $E(n, R, A)$ is the normal closure of $E(n, A)$ in $E(n, R)$:

$$E(n, R, A) = \langle t_{ij}(\xi), \xi \in A, 1 \leqslant i \neq j \leqslant n \rangle^{E(n,R)}.$$

The canonical projection $\rho_A : R \longrightarrow R/A$ sending any element $\lambda \in R$ to the element $\overline{\lambda} = \lambda + A$, defines the corresponding reduction homomorphism

$$\rho_A : \mathrm{GL}(n, R) \longrightarrow \mathrm{GL}(n, R/A).$$

The kernel of $\rho_A$ is denoted by $\mathrm{GL}(n, R, A)$ and is called the principal congruence subgroup in $\mathrm{GL}(n, R)$ of level $A$.

In the proofs, often without any special reference, we shall use a bunch of classical facts on elementary groups. The following fact, first proved in [5], is cited as the Suslin theorem.

**Lemma 1.** *Let $R$ be a commutative ring and $n \geqslant 3$. Then for any ideal $A \trianglelefteq R$, we have $[E(n, R), \mathrm{GL}(n, R, A)] = E(n, R, A)$. In particular, $E(n, R, A)$ is normal in $\mathrm{GL}(n, R)$.*

The following statement was proved by Vaserstein and Suslin [6] and, in the context of Chevalley groups, by Tits [11].

**Lemma 2.** *For $n \geqslant 3$, the relative elementary subgroup $E(n, R, A)$ is generated by all transvections of the form*

$$z_{ij}(\xi, \zeta) = t_{ji}(\zeta) t_{ij}(\xi) t_{ji}(-\zeta), \xi \in A, \zeta \in R, 1 \leqslant i \neq j \leqslant n.$$

## §3. The regular representation

Let $R$ be a commutative ring and let $S$ be a commutative ring extension of $R$, which is a free $R$-module of rank $m$. Suppose that $1 = w_1, \dots, w_m$ is a basis of $S/R$. For $\alpha \in S, 1 \leqslant j \leqslant m$, there exist $\alpha_{1j}, \ldots, \alpha_{mj} \in R$ such that

$$\alpha w_j = \alpha_{1j} w_1 + \cdots + \alpha_{mj} w_m.$$

Then $[\alpha] := (\alpha_{ij}) \in M(m, R)$. It is clear that the map $\alpha \mapsto [\alpha]$ is a ring monomorphism from $S$ to $M(m, R)$ and so $\mathrm{GL}(n, S)$ is considered as a subgroup of $\mathrm{GL}(mn, R)$. Then $E(n, S)$ is a subgroup of $E(mn, R)$. The symbols $e, e_1, \ldots, e_{mn}, f_1, \ldots, f_{mn}, e_{ij}, t_{ij}(\xi)(\xi \in R)$ are used for $\mathrm{GL}(mn, R)$ as in §1, now we write $E, E_i, \ldots E_n$, $F_1, \ldots, F_n, E_{ij}, T_{ij}(\alpha)(\alpha \in S)$ to denote the corresponding concepts in $\mathrm{GL}(n, S)$, namely, $E$ is the identity matrix in $\mathrm{GL}(n, S)$; $E_i$ is the $i$th column of $E$; $F_j$ is the $j$th row of $E$; $E_{ij}$ is a standard matrix unit and $T_{ij}(\alpha) = E + \alpha E_{ij}$.

Suppose that $S = R[\sqrt{d}]$ be a commutative ring extension of rank 2 of $R$ with the basis $1, \sqrt{d}$. Then

$$S = \left\{ \begin{pmatrix} x & dy \\ y & x \end{pmatrix} : x, y \in R \right\},$$

where the identity is $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Note that if $d \in R^*$ and $\alpha \in S$ has a zero row or a zero column, then $\alpha = 0$.

**Lemma 3.** *Let $S = R[\sqrt{d}]$. Suppose that $2 \in R^*$ and $d \in R^*$. Then the group of all automorphisms of $S$ that are identical on $R$ is*

$$\mathrm{Aut}(S/R) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} : r \in R, r^2 = 1 \right\}.$$

**Proof.** Let $\varphi \in \mathrm{Aut}(S/R)$. Suppose that $\varphi(\sqrt{d}) = a + b\sqrt{d}$, where $a, b \in R$. We have $d = \varphi(d) = \varphi(\sqrt{d})\varphi(\sqrt{d}) = (a^2 + b^2 d) + 2ab\sqrt{d}$, so $a^2 + b^2 d = d$ and $2ab = 0$. Moreover, there exists $x + y\sqrt{d} \in S$ such that $\varphi(x + y\sqrt{d}) = \sqrt{d}$, it follows that $yb = 1$. By invertibility of $d$ and 2, we get $a = 0, b^2 = 1$, so the matrix of $\varphi$ with respect to the basis $1, \sqrt{d}$ is $\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$. Now, let $r \in R, r^2 = 1$, then the map $\psi : S \longrightarrow S$ which maps $x + y\sqrt{d}$ to $x + yr\sqrt{d}$ belongs to $\mathrm{Aut}(S/R)$ and its matrix with respect to the basis 1, $\sqrt{d}$ is $\begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}$. □

## §4. Decomposition of transvections

**Theorem 1.** *Let $R$ be a commutative ring, and let $S = R[\sqrt{d}]$ be a commutative ring extension of rank 2 of $R$ and $n \geqslant 3$. Then for any*

*$g \in \mathrm{GL}(2n, R)$, the elementary subgroup $E(n, S)$ is generated by the transvections $E + UF_j, 1 \leqslant j \leqslant n$, where the column $U \in S^n$ has at most two nonzero components and its $j$th component is zero such that $gU \in M(2n, 2, R)$ has at least one zero row.*

**Proof.** Put $T_{*j}(U) = E + UF_j$, then $T_{*j}(U) \in E(n, S)$. We need to show that any transvection $T_{ij}(\alpha)$ is expressed as a product of $T_{*j}(U), 1 \leqslant j \leqslant n$. Let $1 \leqslant i \neq j \leqslant n$ and $\alpha \in S$. Choose $h \notin \{i, j\}$. For any $1 \leqslant k \leqslant 2n$, put

$$U(k) = \alpha g'_{(2h)k}(U_{kh}E_i - U_{ki}E_h),$$

where

$$U_{kl} = \begin{pmatrix} g_{k(2l)} & dg_{k(2l-1)} \\ g_{k(2l-1)} & g_{k(2l)} \end{pmatrix} \in S, \quad l = i, h.$$

Clearly, $U(k) \in S^n$ and $U(k)_j = 0$, moreover, $(gU(k))_{k*} = 0$.

To finish the proof, it suffices to observer that

$$\begin{aligned} \prod_{k=1}^{2n} T_{*j}(U(k)) &= \prod_{k=1}^{2n} T_{ij}(\alpha g'_{(2h)k}U_{kh})T_{hj}(-\alpha g'_{(2h)k}U_{ki}) \\ &= \prod_{k=1}^{2n} T_{ij}(\alpha g'_{(2h)k}U_{kh}) \prod_{k=1}^{2n} T_{hj}(-\alpha g'_{(2h)k}U_{ki}) \\ &= T_{ij}(\alpha(\sum_{k=1}^{2n} g'_{(2h)k}U_{kh}))T_{hj}(-\alpha(\sum_{k=1}^{2n} g'_{(2h)k}U_{ki})) = T_{ij}(\alpha). \end{aligned}$$

To justify the last equality, note that

$$\sum_{k=1}^{2n} g'_{(2h)k}U_{kh} = \begin{pmatrix} \sum_{k=1}^{2n} g'_{(2h)k}g_{k(2h)} & \sum_{k=1}^{2n} g'_{(2h)k}dg_{k(2h-1)} \\ \sum_{k=1}^{2n} g'_{(2h)k}g_{k(2h-1)} & \sum_{k=1}^{2n} g'_{(2h)k}g_{k(2h)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\sum_{k=1}^{2n} g'_{(2h)k}U_{ki} = \begin{pmatrix} \sum_{k=1}^{2n} g'_{(2h)k}g_{k(2i)} & \sum_{k=1}^{2n} g'_{(2h)k}dg_{k(2i-1)} \\ \sum_{k=1}^{2n} g'_{(2h)k}g_{k(2i-1)} & \sum_{k=1}^{2n} g'_{(2h)k}g_{k(2i)} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

□

## §5. Elementary transvections in subgroups normalized by $E(n,S)$

**Proposition 1.** *Let $R$ be a commutative ring, and let $S = R[\sqrt{d}]$ be a commutative ring extension of rank 2 of $R$ and $n \geqslant 3$. For any ideal $A$ of $R$, we have*

$$E(2n,A)^{E(n,S)} = E(2n,R,A).$$

**Proof.** Put $H = E(2n,A)^{E(n,S)}$. Since $E(n,S) \leqslant E(2n,R)$, we have $H \leqslant E(2n,R,A)$. By Lemma 2, it suffices to check that the $z_{ij}(\xi,\zeta) = t_{ji}(\zeta)t_{ij}(\xi)t_{ji}(-\zeta) \in H$ for any $\xi \in A, \zeta \in R$. Note that $H$ is normalized by $E(n,S)$, conjugating by a monomial matrix from $E(n,S)$, we can suppose that $i \in \{1,2\}, j \in \{1,2,3,4\}$.

1. $(i,j) \in \{(1,2),(2,1)\}$. We have

$$\begin{aligned} z_{12}(\xi,\zeta) &= {}^{t_{21}(\zeta)}t_{12}(\xi) = {}^{t_{21}(\zeta)}[t_{13}(1),t_{32}(\xi)] \\ &= [{}^{t_{21}(\zeta)}t_{13}(1), {}^{t_{21}(\zeta)}t_{32}(\xi)] \\ &= [[t_{21}(\zeta),t_{13}(1)]t_{13}(1),[t_{21}(\zeta),t_{32}(\xi)]t_{32}(-\xi)] \\ &= [t_{23}(\zeta)t_{13}(1),t_{31}(-\xi\zeta)t_{32}(\xi)], \end{aligned}$$

therefore

$$z_{12}(\xi,\zeta)t_{31}(-\xi\zeta)t_{32}(\xi) = t_{23}(\zeta)t_{13}(1)t_{31}(-\xi\zeta)t_{32}(\xi)t_{13}(-1)t_{23}(-\zeta)$$

$$\begin{aligned} &= T_{12}(1+\zeta\sqrt{d})t_{14}(-d\zeta)t_{24}(-1)t_{31}(-\xi\zeta)t_{32}(\xi)t_{24}(1)t_{14}(d\zeta)T_{12}(-1-\zeta\sqrt{d}) \\ &= T_{12}(1+\zeta\sqrt{d})t_{14}(-d\zeta)t_{31}(-\xi\zeta)t_{14}(d\zeta)t_{24}(-1)t_{32}(\xi)t_{24}(1)T_{12}(-1-\zeta\sqrt{d}) \\ &= T_{12}(1+\zeta\sqrt{d})t_{34}(-d\xi\zeta^2)t_{31}(-\xi\zeta)t_{34}(\xi)t_{32}(\xi)T_{12}(-1-\zeta\sqrt{d}) \in H. \end{aligned}$$

It follows that $z_{12}(\xi,\zeta) \in H$. Similarly, $z_{21}(\xi,\zeta) \in H$.

2. $(i,j) = (2,3)$. We have

$$\begin{aligned} z_{23}(\xi,\zeta) &= t_{32}(\zeta)t_{23}(\xi)t_{32}(-\zeta) = {}^{t_{32}(\zeta)}[t_{25}(1),t_{53}(\xi)] \\ &= [{}^{t_{32}(\zeta)}t_{25}(1), {}^{t_{32}(\zeta)}t_{53}(\xi)] \\ &= [t_{35}(\zeta)t_{25}(1),t_{52}(-\xi\zeta)t_{53}(\xi)] \\ &= [t_{46}(\zeta)t_{35}(\zeta)t_{25}(1)t_{16}(d),t_{52}(-\xi\zeta)t_{53}(\xi)] \\ &= [T_{23}(\zeta)T_{13}(\sqrt{d}),t_{52}(-\xi\zeta)t_{53}(\xi)] \in H. \end{aligned}$$

3. $(i, j) \in \{(1,3),(1,4),(2,4)\}$. We have

$$\begin{aligned} z_{13}(\xi,\zeta) &= t_{31}(\zeta)t_{13}(\xi)t_{31}(-\zeta) \\ &= T_{21}(\zeta)t_{42}(-\zeta)t_{13}(\xi)t_{42}(\zeta)T_{21}(-\zeta) \\ &= T_{21}(\zeta)t_{13}(\xi)T_{21}(-\zeta) \in H. \end{aligned}$$

Similarly,

$$z_{14}(\xi,\zeta) = T_{21}(\zeta\sqrt{d})t_{14}(\xi)T_{21}(-\zeta\sqrt{d}) \in H,$$

and

$$z_{24}(\xi,\zeta) = T_{21}(\zeta)t_{24}(\xi)T_{21}(-\zeta) \in H.$$

□

**Lemma 4.** *Let $R$ be a commutative ring, and let $S = R[\sqrt{d}]$ be a commutative ring extension of rank 2 of $R$ and $n \geqslant 3$. Let $a = gT_{ij}(\alpha)T_{hk}(\beta)g^{-1}$, where $g \in \mathrm{GL}(2n,R), \alpha, \beta \in S, j \neq h, i \neq k$. Suppose that $d \in R^*$ and $2 \in R^*$. If $a \in N_{\mathrm{GL}(2n,R)}(E(n,S))$ and $a_{l*} = f_l$ for some $1 \leqslant l \leqslant 2n$, then $a \in \mathrm{GL}(n,S)$.*

**Proof.** Conjugating by a monomial matrix from $E(n,S)$, we can suppose that $l = 1$ or $l = 2$. Now, $a = (A_{ij}) \in N_{\mathrm{GL}(2n,R)}(E(n,S))$, by Corollary 1 of [12], there exist $\sigma \in \mathrm{Aut}(S/R)$ and $b = (B_{ij}) \in \mathrm{GL}(n,S)$ such that $a = (B_{ij}\sigma)$. We have $B_{1j} = A_{1j}\sigma^{-1}$ for all $1 \leqslant j \leqslant n$, so the $l$th row of $B_{1j}$ is zero for any $2 \leqslant j \leqslant n$. By Lemma 3, $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}$, where $r \in R, r^2 = 1$, so the $l$th row of $B_{11}$ is $r^{l+1}f_l \in {}^2R$. From $d \in R^*$ and $B_{1j} \in S$, it follows that $B_{11} = r^{l+1}I_2$ and $B_{1j} = 0$, so $A_{11} = r^{l+1}\sigma$ and $A_{1j} = 0$ for all $2 \leqslant j \leqslant n$. Since $i \neq k, j \neq h$, we have

$$(a-e)^2 = g(T_{ij}(\alpha)T_{hk}(\beta) - e)^2 g^{-1} = 0.$$

Note that $a - e = (C_{ij})$, where $C_{1j} = 0$ for all $2 \leqslant j \leqslant n$, we have

$$(r^{l+1}\sigma - I_2)^2 = C_{11}^2 = 0,$$

so $r = 1$, hence $\sigma = I_2$. Therefore $a = b \in \mathrm{GL}(n,S)$. □

**Lemma 5.** *Under assumptions of Lemma 4, let $H$ be a subgroup in $\mathrm{GL}(2n, R)$ that contains $E(n,S)$. Suppose that $d \in R^*$ and $2 \in R^*$. If $H \not\subseteq N_{\mathrm{GL}(2n,R)}(E(n,S))$, then $H$ contains a matrix $a \notin N_{\mathrm{GL}(2n,R)}(E(n,S))$ such that $a_{l*} = f_l$ for some $1 \leqslant l \leqslant 2n$, moreover, $a = gT_{ij}(\alpha)T_{hk}(\beta)g^{-1}$ for some $g \in H \setminus N_{\mathrm{GL}(2n,R)}(E(n,S)), \alpha, \beta \in S, i \neq k, j \neq h$.*

**Proof.** Let $g \in H \setminus N_{\mathrm{GL}(2n,R)}(E(n,S))$. By Corollary 2 of [12], we have $E(n,S)^g \nleqslant \mathrm{GL}(n,S)$. By Theorem 1, there exists $u = E + UF_j$, $1 \leqslant j \leqslant n$, where the column $U \in S^n$ has at most two nonzero components and $gU \in M(2n,2,R)$ has one zero row, say $l$th row, such that $gug^{-1} \notin \mathrm{GL}(n,S)$. Clearly, $u$ has the form $T_{ij}(\alpha)T_{hk}(\beta)$ with $\alpha, \beta \in S$, $i \neq k$, $j \neq h$ and $(gug^{-1})_{l*} = f_l$. Put $a = gug^{-1}$, then $a \notin N_{\mathrm{GL}(2n,R)}$ $(E(n,S))$ by Lemma 4. □

**Proposition 2.** *Let $R$ be a commutative ring, and let $S = R[\sqrt{d}]$ be a commutative ring extension of rank 2 of $R$ and $n \geqslant 3$. Let $H$ be a subgroup in $\mathrm{GL}(2n,R)$ that contains $E(n,S)$. For $1 \leqslant i \neq j \leqslant 2n$, put*

$$I_{ij} = \{\xi \in R \mid \ t_{ij}(\xi) \in H\}.$$

*Then there exist ideals $A, B$ of $R$ such that*

*(i) for $(i,j) \in \{(2k-1,2k),(2k,2k-1), 1 \leqslant k \leqslant n\}, I_{ij} = I_{12}$ if $i$ is odd, and $I_{ij} = I_{21}$ if $i$ is even;*

*(ii) for $(i,j) \notin \{(2k-1,2k),(2k,2k-1), 1 \leqslant k \leqslant n\}, I_{ij} = A$ if $i$ is odd and $j$ is even, and $I_{ij} = [A:d]$ if $i$ is even and $j$ is odd, moreover, $I_{ij} = B$ if $i,j$ are simultaneously even or odd;*

*(iii) $A \subseteq B \subseteq [A:d], A^2 \subseteq I_{12} \cap I_{21}, I_{12} \subseteq A$ and $I_{21} \subseteq [A:d]$.*

**Proof.** We have $t_{13}(\xi)t_{24}(\xi) = T_{12}(\xi) \in H$, so $I_{13} = I_{24.}$ Conjugating by monomial matrices from $E(n,S)$, we have $(i)$, moreover, for any $(i,j) \notin \{(2k-1,2k),(2k,2k-1), 1 \leqslant k \leqslant n\}$,

– if $i$ is odd and $j$ is even, then $I_{ij} = I_{14}$;

– if $i,j$ are simultaneously even or odd, then $I_{ij} = I_{13}$;

– if $i$ is even and $j$ is odd, then $I_{ij} = I_{23}$.

Put $A = I_{14}, B = I_{13}$. Clearly, $A, B$ are additional subgroups of $R$. For any $\xi \in A, \zeta \in R$, we have,

$$t_{14}(\xi\zeta) = [t_{16}(\xi), t_{64}(\zeta)] = [t_{16}(\xi), t_{64}(\zeta)t_{53}(\zeta)] = [t_{16}(\xi), T_{32}(\zeta)] \in H,$$

therefore $A \trianglelefteq R$. Moreover, for any $\xi \in B, \zeta \in R$, we have

$$t_{13}(\xi\zeta) = [t_{15}(\zeta), t_{53}(\xi)] = [t_{15}(\zeta)t_{26}(\zeta), t_{53}(\xi)] = [T_{13}(\zeta), t_{53}(\xi)] \in H,$$

so $B \trianglelefteq R$.

Let $\xi \in A$, then

$$t_{15}(\xi) = [t_{14}(\xi), t_{45}(1)] = [t_{14}(\xi), t_{45}(1)t_{36}(d)] = [t_{14}(\xi), T_{23}(\sqrt{d})] \in H,$$

so $A \subseteq B$. Since $t_{14}(d\xi)t_{23}(\xi) = T_{12}(\xi\sqrt{d}) \in E(n,S)$, we have $t_{14}(d\xi)t_{23}(\xi)$ $\in H$ for all $\xi \in R$, so $I_{23} = [A:d]$.

Let $\xi \in B$, then

$$t_{23}(\xi) = [t_{25}(1), t_{53}(\xi)] = [t_{25}(1)t_{16}(d), t_{53}(\xi)] = [T_{13}(\sqrt{d}), t_{53}(\xi)] \in H,$$

so $\xi \in I_{23} = [A : d]$. Hence $B \subseteq [A : d]$.

Let $\xi, \xi' \in A$. It follows from $A \subseteq B \subseteq [A : d]$ that $t_{12}(\xi\xi') = [t_{14}(\xi), t_{42}(\xi')] \in H$ and $t_{21}(\xi\xi') = [t_{23}(\xi), t_{31}(\xi')] \in H$, so $A^2 \subseteq I_{12} \cap I_{21}$.

Now, since

$$t_{14}(\xi) = [t_{12}(\xi), t_{24}(1)] = [t_{12}(\xi), t_{24}(1)t_{13}(1)] = [t_{12}(\xi), T_{12}(1)]$$

and

$$t_{23}(\xi) = [t_{21}(\xi), t_{13}(1)] = [t_{21}(\xi), t_{13}(1)t_{24}(1)] = [t_{21}(\xi), T_{12}(1)],$$

we claim that $I_{12} \subseteq A$ and $I_{21} \subseteq [A : d]$. □

**Corollary 1.** *Under assumptions of Proposition* 2*, suppose that any ideal of* $R$ *is idempotent and* $d \in R^*$*. Then for every subgroup* $H$ *in* $\mathrm{GL}(2n, R)$ *that contains* $E(n, S)$*, there exists a unique ideal* $A$ *of* $R$ *such that*

$$A = \{\xi \in R : t_{ij}(\xi) \in H\}$$

*for any* $1 \leqslant i \neq j \leqslant 2n$.

## §6. Main results

In this section we always assume that $R$ is a direct product of fields, $R = \prod_{i\in I} F_i$, and $S \subseteq \prod_{i\in I} \overline{F_i}$ is a commutative ring extension of rank 2 of $R$ with the basis $1, \sqrt{d}$. Then $S = R[\sqrt{d}]$, where $d = (d_i)_{i\in I} \in \prod_{i\in I} F_i$. We have

$$S = R[\sqrt{d}] = R + R\sqrt{d} = \prod_{i\in I}(F_i + F_i\sqrt{d_i}) = \prod_{i\in I} F_i(\sqrt{d_i}).$$

Let $i \in I$ and $\alpha_i + \beta_i\sqrt{d_i} = 0, \alpha_i, \beta_i \in F_i$. Put $\alpha = (\alpha_j\delta_{ij})_{j\in I}, \beta = (\beta_j\delta_{ij})_{j\in I}$, then $\alpha + \beta\sqrt{d} = 0$. It follows that $\alpha = \beta = 0$, so $\alpha_i = \beta_i = 0$. Therefore $1, \sqrt{d_i}$ is linearly independent, i.e., $F(\sqrt{d_i})$ is a field extension of rank 2 of $F_i$. In particular, $d_i \neq 0$ for any $i \in I$. Hence $d \in R^*$.

**Lemma 6.** *Let* $A, B \in M(2, R)$*. Suppose that* $B$ *has a zero row. If* $AB \in S$*, then* $AB = 0$.

**Proof.** Let $A = \begin{pmatrix} x & y \\ z & t \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ u & v \end{pmatrix}$. It follows from $AB \in S$ that $yu = tv$ and $yv = dtu$, so $d(tu)^2 = (tv)^2$. Put $t = (t_i)_{i\in I}, u = (u_i)_{i\in I}$, $v = (v_i)_{i\in I}$. We have $d_i(t_iu_i)^2 = (t_iv_i)^2$ for any $i \in I$. Since $F_i(\sqrt{d_i})$ is a field extension of rank 2 of $F_i$, we have $t_iu_i = t_iv_i = 0$, it follows that $tu = tv = 0$ and hence $AB = 0$. Similarly, we have the above conclusion for $B = \begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix}$. □

**Lemma 7.** *Let* $A = \begin{pmatrix} x & dy \\ y & x \end{pmatrix}$, *and let* $B$ *be either* $\begin{pmatrix} y & x \\ 0 & 0 \end{pmatrix}$ *or* $\begin{pmatrix} 0 & 0 \\ y & x \end{pmatrix}$. *If* $BA = 0$, *then* $A = B = 0$.

**Proof.** Since $BA = 0$, we have $2xy = 0$ and $x^2 + dy^2 = 0$. Put $x = (x_i)_{i\in I}, y = (y_i)_{i\in I}$, we have $2x_iy_i = 0$ and $x_i^2 + d_iy_i^2 = 0$ for any $i \in I$. If $\mathrm{char}(F_i) = 2$, then $x_i^2 - d_iy_i^2 = x_i^2 + d_iy_i^2 = 0$, and hence $x_i = y_i = 0$. If $\mathrm{char}(F_i) \neq 2$, then $x_iy_i = 0 = x_i^2 + d_iy_i^2$, so $x_i = y_i = 0$. Therefore $x = y = 0$. □

**Lemma 8.** *Suppose that* $2 \in R^*$ *and* $n \geqslant 3$. *Let* $H$ *be a subgroup in* $\mathrm{GL}(2n, R)$ *that contains* $E(n, S)$ *and* $H \not\subseteq N_{\mathrm{GL}(2n,R)}(E(n, S))$. *Then* $H$ *contains a matrix* $g = (G_{ij}) \in \mathrm{GL}(n, M_2(R)) \setminus N_{\mathrm{GL}(2n,R)}(E(n, S))$ *such that there exists an index* $i$, $G_{ij} = 0$ *for all* $1 \leqslant j \neq i \leqslant n$.

**Proof.** By Lemma 5, there exist $a \in H \setminus N_{\mathrm{GL}(2n,R)}(E(n, S))$ and $u = T_{ij}(\alpha)T_{hk}(\beta) \in E(n, S)$ such that $aua^{-1} \notin \mathrm{GL}(n, S)$ and $(aua^{-1})_{l*} = f_l$ for some $1 \leqslant l \leqslant 2n$. Put $g = aua^{-1} \in H$. Conjugating by monomial matrices from $E(n, S)$, we can suppose that $l \in \{1, 2\}$. For $k \in \{1, 2\} \setminus \{l\}$ and $2 \leqslant s \leqslant n$, we put

$$U_s = \begin{pmatrix} g_{k(2s)} & dg_{k(2s-1)} \\ g_{k(2s-1)} & g_{k(2s)} \end{pmatrix}.$$

For $3 \leqslant s \leqslant n$, put

$$h = (H_{ij}) = gT_{(s-1)1}(U_s)T_{s1}(-U_{s-1})g^{-1},$$

then $G_{1(s-1)}U_s - G_{1s}U_{s-1} = 0$, so $H_{1*} = F_1$. If $h \notin N_{\mathrm{GL}(2n,R)}(E(n, S))$, then we can finish here. Suppose that $h \in N_{\mathrm{GL}(2n,R)}(E(n, S))$. By Lemma 4, $h \in \mathrm{GL}(n, S)$. We have

$$H_{ij} = (G_{i(s-1)}U_s - G_{is}U_{s-1})G'_{1j} + \delta_{ij}I_2 \in S, 1 \leqslant i, j \leqslant n.$$

Note that $(g^{-1})_{l*} = f_l$, by Lemma 6, $(G_{i(s-1)}U_s - G_{is}U_{s-1})G'_{1j} = 0$ for all $2 \leqslant i, j \leqslant n$. Therefore $(G_{i(s-1)}U_s - G_{is}U_{s-1})G'_{1j} = 0$ for all $1 \leqslant i \leqslant n, 2 \leqslant j \leqslant n$, so

$$\begin{aligned} U_s G'_{1j} &= \sum_{i=1}^{n} G'_{(s-1)i} G_{i(s-1)} U_s G'_{1j} - \sum_{i=1}^{n} G'_{(s-1)i} G_{is} U_{s-1} G'_{1j} \\ &= \sum_{i=1}^{n} G'_{(s-1)i} (G_{i(s-1)} U_s - G_{is} U_{s-1}) G'_{1j} = 0. \end{aligned}$$

It follows that $G_{is}U_{s-1}G'_{1j} = 0$ for all $1 \leqslant i \leqslant n$. Since $(G_{1s}, \ldots, G_{ns})^t$ is unimodular, we have $U_{s-1}G'_{1j} = 0$.

We have proved $U_s G'_{1j} = 0$ for all $2 \leqslant s, j \leqslant n$. Now, for any $2 \leqslant s \leqslant n$, put $b = (B_{ij}) = gT_{s1}(U_s)g^{-1}$. Obviously, $b \in H, b_{l*} = f_l$ and $B_{1j} = 0$ for all $2 \leqslant j \leqslant n$. If there exists $s$ such that $b \notin \mathrm{GL}(n, S)$, then by Lemma 4, the matrix $b$ satisfies the conditions of the lemma. Therefore we can suppose that $b \in \mathrm{GL}(n, S)$ for all $2 \leqslant s \leqslant n$, then $G_{1s}U_sG'_{11} = B_{11} - I_2 \in S$. Note that the $l$th row of the matrix $G_{1s}U_sG'_{11}$ is zero, so $G_{1s}U_sG'_{11} = 0$. We have

$$G_{1s}U_s = G_{1s}U_s(\sum_{j=1}^{n} G'_{1j}G_{j1}) = \sum_{j=1}^{n} (G_{1s}U_sG'_{1j})G_{j1} = 0.$$

By Lemma 7, $G_{1s} = 0$ and the matrix $g$ itself satisfies the conditions of the lemma. □

**Lemma 9.** *Suppose that $2 \in R^*$ and $n \geqslant 3$. Let $H$ be a subgroup in $\mathrm{GL}(2n, R)$ that contains $E(n, S)$. If there exists $g = (G_{ij}) \in \mathrm{GL}(n, M(2, R)) \setminus N_{\mathrm{GL}(2n,R)}(E(n, S))$ such that $G_{1j} = 0$ for all $2 \leqslant j \leqslant n$ and $gE(n, S) g^{-1} \cup g^{-1}E(n, S)g \subseteq H$, then $H$ contains an elementary transvection $T_{ij}(C)$ with $C \notin S$.*

**Proof.** Suppose that $H$ doesn't contain any elementary transvection $T_{ij}(C)$ with $C \notin S$. For any $2 \leqslant j \leqslant n$ and $\alpha \in S$, put $g_1 = gT_{j1}(\alpha)g^{-1} \in H$. We have

$$g_1 = \begin{pmatrix} I_2 & 0 & \ldots & 0 \\ \beta_{2j} & I_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \\ \beta_{nj} & 0 & \ldots & I_2 \end{pmatrix}, \quad \beta_{ij} = G_{ij}\alpha G_{11}^{-1}, 2 \leqslant i \leqslant n.$$

Then

$T_{n1}(\beta_{ij}) = [T_{ni}(I_2), T_{i1}(\beta_{ij})] = [T_{ni}(I_2), g_1] \in H$, for all $2 \leqslant i \leqslant n-1$ and

$$T_{21}(\beta_{nj}) = [T_{2n}(I_2), T_{n1}(\beta_{nj})] = [T_{2n}(I_2), g_1] \in H,$$

therefore $\beta_{ij} \in S$ for all $2 \leqslant i, j \leqslant n, \alpha \in S$. In particular, $G_{ij}G_{11}^{-1} \in S$, i.e., there exist $\alpha_{ij} \in S, 2 \leqslant i, j \leqslant n$, such that $G_{ij} = \alpha_{ij}G_{11}$. Note that $G_{11} \in \mathrm{GL}(2, R)$ and $(G_{ij})_{2\leqslant i,j\leqslant n} \in \mathrm{GL}(2n-2, R)$, so $(\alpha_{ij})_{2\leqslant i,j\leqslant n} \in \mathrm{GL}(2n-2, R)$. We have

$$G_{11}\begin{pmatrix} 0 & d \\ 1 & 0 \end{pmatrix} G_{11}^{-1} = \begin{pmatrix} a & r \\ s & -a \end{pmatrix}$$

for some $a, r, s \in R$. Let $\alpha_{i2} = \begin{pmatrix} u_i & dv_i \\ v_i & u_i \end{pmatrix}$ with $u_i, v_i \in R$, we have

$$\beta_{i2} = \alpha_{i2}G_{11}\begin{pmatrix} 0 & d \\ 1 & 0 \end{pmatrix} G_{11}^{-1} = \begin{pmatrix} u_i a + dsv_i & u_i r - dav_i \\ v_i a + su_i & v_i r - au_i \end{pmatrix}.$$

Since $\beta_{i2} \in S$, we get the system

$$\begin{cases} u_i a + dsv_i = v_i r - au_i; \\ u_i r - dav_i = d(v_i a + su_i). \end{cases}$$

Hence $(r-sd)(u_i^2 - dv_i^2) = 0$, so $(r-sd)u_i = (r-sd)v_i = 0$ for all $2 \leqslant i \leqslant n$. Since $(u_2, v_2, \ldots, u_n, v_n)^t$ is unimodular, $r = sd$. Using invertibility of 2 and $d$, we have $au_i = av_i = 0$, so $a = 0$. Therefore

$$G_{11}\begin{pmatrix} 0 & d \\ 1 & 0 \end{pmatrix} G_{11}^{-1} = \begin{pmatrix} 0 & ds \\ s & 0 \end{pmatrix} \in S.$$

It follows that $G_{11}\alpha G_{11}^{-1} \in S$ for all $\alpha \in S$, i.e., $G_{11}SG_{11}^{-1} \subseteq S$. Replacing $g$ by $g^{-1}$,we have $G_{11}^{-1}SG_{11} \subseteq S$, therefore $G_{11}SG_{11}^{-1} = S$.
Put $h = gG_{11}^{-1} = (G_{ij}G_{11}^{-1})$, then $hE(n,S)h^{-1} = gG_{11}^{-1}E(n,S)$ $G_{11}g^{-1} = gE(n,S)g^{-1} \subseteq H$, so $g^{-1}h \in N_{\mathrm{GL}(2n,R)}(E(n,S))$.
Since $g \notin N_{\mathrm{GL}(2n,R)}(E(n,S)), h \notin N_{\mathrm{GL}(2n,R)}(E(n,S))$. Moreover, $H_{1*} = F_1$ and $H_{ij} = G_{ij}G_{11}^{-1} \in S$ for all $2 \leqslant i, j \leqslant n$, i.e., $h = \begin{pmatrix} I_2 & 0 \\ U & D \end{pmatrix}$ with $U = (\gamma_2, \ldots, \gamma_n)^t \in M_2(R)^{n-1}$, $D \in M(n-1, S)$. Replacing $g$ by $g^{-1}$, we have $g^{-1}G_{11} = \begin{pmatrix} I_2 & 0 \\ U' & D' \end{pmatrix}$ with $U' \in M_2(R)^{n-1}, D' \in M(n-1, S)$. Then

$$h^{-1} = G_{11}(g^{-1}G_{11})G_{11}^{-1} = \begin{pmatrix} I_2 & 0 \\ G_{11}U'G_{11}^{-1} & G_{11}D'G_{11}^{-1} \end{pmatrix},$$

therefore $D^{-1} = G_{11}D'G_{11}^{-1} \in M(n-1,S)$ due to $G_{11}SG_{11}^{-1} = S$. If $U \in S^{n-1}$, then $h = \begin{pmatrix} I_2 & 0 \\ U & D \end{pmatrix} \in M(n,S)$, and $h^{-1} = \begin{pmatrix} I_2 & 0 \\ -D^{-1}U & D^{-1} \end{pmatrix}$ $\in M(n,S)$, so $h \in \mathrm{GL}(n,S) \subseteq N_{\mathrm{GL}(2n,R)}(E(n,S))$, contradicting the hypothesis. Hence there exists $\gamma_k \notin S$ for some $2 \leqslant k \leqslant n$. Take $1 \leqslant j \leqslant n-1, j \neq k-1$, put

$$a = \begin{pmatrix} I_2 & 0 \\ 0 & D \end{pmatrix}^{-1} \begin{pmatrix} I_2 & 0 \\ 0 & T_{j(k-1)}(-I_2) \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ 0 & D \end{pmatrix} \in E(n,S),$$

then

$$T_{j+1,1}(\gamma_k) = \begin{pmatrix} I_2 & 0 \\ 0 & T_{j(k-1)}(I_2) \end{pmatrix} hah^{-1} \in E(n,S)h(E(n,S)h^{-1} \subseteq H,$$

contradicting the hypothesis.

□

**Lemma 10.** *Suppose that $2 \in R^*$ and $n \geqslant 3$. Let $H$ be a subgroup in $\mathrm{GL}(2n,R)$ that contains $E(n,S)$. If $H \nsubseteq N_{\mathrm{GL}(2n,R)}(E(n,S))$, then $H$ contains an elementary transvection $T_{ij}(C)$ with $C \notin S$.*

**Proof.** By Lemma 8, there exist a matrix $g = (G_{ij}) \in H \setminus N_{\mathrm{GL}(2n,R)}(E(n,S))$ and an index $i$ such that $G_{ij} = 0$ for all $1 \leqslant j \neq i \leqslant n$. Clearly, $gE(n,S)g^{-1} \cup g^{-1}E(n,S)g \subseteq H$, moreover, conjugating by monomial matrices from $E(n,S)$, we can suppose that $G_{1j} = 0$ for all $2 \leqslant j \leqslant n$. By Lemma 9, $H$ contains an elementary transvection $T_{ij}(C)$ with $C \notin S$. □

**Lemma 11.** *Suppose that $2 \in R^*$ and $n \geqslant 3$. Let $H$ be a subgroup in $\mathrm{GL}(2n,R)$ that contains $E(n,S)$. Then either $H \leqslant N_{\mathrm{GL}(2n,R)}(E(n,S))$, or $H$ contains a nontrivial elementary transvection of the form $t_{(2i)(2j)}(\xi)$.*

**Proof.** Suppose that $H \nsubseteq N_{\mathrm{GL}(2n,R)}(E(n,S))$, by Lemma 10, $H$ contains an elementary transvection $T_{hj}(C)$ with $C \notin S$. Choose $\alpha \in S$ such that its first column is the same as in $C$, then $C - \alpha = \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}$, where $x, y \in R, (x,y) \neq (0,0)$. Now, put $\gamma = -dy + x\sqrt{d} \in S$, we have

$$\gamma(C-\alpha) = \begin{pmatrix} -dy & dx \\ x & -dy \end{pmatrix} \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & x^2 - dy^2 \end{pmatrix} \neq 0.$$

Take $i \in \{1, \dots, n\} \setminus \{h, j\}$, then

$$t_{(2i)(2j)}(x^2 - dy^2) = T_{ij}(\gamma(C - \alpha)) = [T_{ih}(\gamma), T_{hj}(C - \alpha)]$$
$$= [T_{ih}(\gamma), T_{hj}(C)T_{hj}(-\alpha)] \in H.$$

□

Now, we are ready to prove the following main result of the paper.

**Theorem 2.** *Let $R = \prod_{i \in I} F_i$ be a direct product of fields, and let $S = R[\sqrt{d}] \subseteq \prod_{i \in I} \overline{F_i}$ be a commutative ring extension of rank 2 of $R$. Suppose that $2 \in R^*$ and $n \geqslant 3$. Then for every such a subgroup $H$ in $G = \mathrm{GL}(2n, R)$ that contains $E(n, S)$, there exists a unique ideal $A$ of $R$ such that*

$$E(n, S)E(2n, R, A) \leqslant H \leqslant N_G(E(n, S)E(2n, R, A)).$$

**Proof.** Let $A$ be the largest ideal such that $E(2n, A) \leqslant H$, the existence of such an ideal was established in Corollary 1. By Proposition 1, we have

$$E(2n, R, A) = E(2n, A)^{E(n,S)} \leqslant H.$$

Let $\overline{H} = \rho_A(H)$, clearly, $\overline{H}$ contains $E(n, S/SA)$. We have $A = \prod_{i \in I} A_i$, where $A_i \trianglelefteq F_i$, therefore $R/A \cong \prod_{j \in J} F_j$ and $S/SA \cong \prod_{j \in J} F_j(\sqrt{d_j})$, where $J = \{j : A_j = 0\}$. By Lemma 11, we have the following alternative: either $\overline{H} \leqslant N_{\mathrm{GL}(2n,R/A)}(E(n, S/SA))$, or $\overline{H}$ contains a nontrivial elementary transvection $t_{(2i)(2j)}(\overline{\xi})$ for some $\xi \in R \backslash A$. We show that the second possibility cannot occur. Indeed, presenting $t_{(2i)(2j)}(\xi) \in H\mathrm{GL}(2n, R, A)$ in the form

$$t_{(2i)(2j)}(\xi) = ab, a \in H, b \in \mathrm{GL}(2n, R, A).$$

Take $k \in \{1, \dots, n\} \setminus \{i, j\}$, we have

$$t_{(2i)(2k)}(\xi) = [t_{(2i)(2j)}(\xi), T_{jk}(I_2)] = {}^a[b, T_{jk}(I_2)][a, T_{jk}(I_2)].$$

The first of the commutators on the right-hand side belongs $E(2n, R, A)$, while the second lies in $H$. This means that $t_{(2i)(2k)}(\xi) \in H$, where $\xi \notin A$, which contradicts the maximality of $A$. Therefore $\overline{H} \leqslant N_{\mathrm{GL}(2n,R/A)}(E(n, S/SA))$, by Theorem 2 of [12], we have the desired inclusion

$$H \leqslant N_{\mathrm{GL}(2n,R)}(E(n, S)E(2n, R, A)).$$

□

## §7. Counterexamples

In this section, we establish counterexamples to show that the result in Theorem 2 does not hold for some rings.

**Lemma 12.** *Let $R$ be a commutative ring, and let $S = R[\sqrt{d}]$ be a commutative ring extension of rank 2 of $R$ and $n \geqslant 3$. Suppose that $d \in R^*$ and $a \in R \setminus \{0\}, ad = a$. Then the subgroup $H = E(n, S[M])$ with $M = \begin{pmatrix} 0 & 0 \\ a & a \end{pmatrix}$ does not contain any nontrivial elementary transvection in $E(2n, R)$.*

**Proof.** By Proposition 2, it suffices to prove that if $B = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \in S[M]$, then $b = 0$. Indeed, since $M \begin{pmatrix} 0 & d \\ 1 & 0 \end{pmatrix} = M$, $B$ is written in the form

$$B = \sum_{i=0}^{k} \alpha_i M^i, \text{ where } \alpha_i = \begin{pmatrix} x_i & dy_i \\ y_i & x_i \end{pmatrix}, \; x_i, y_i \in R,$$

that is

$$B = \begin{pmatrix} x_0 + d\sum_{i=1}^k y_i a^i & d(y_0 + \sum_{i=1}^k y_i a^i) \\ y_0 + \sum_{i=1}^k x_i a^i & x_0 + \sum_{i=1}^k x_i a^i \end{pmatrix}.$$

Therefore $x_0 + d\sum_{i=1}^k y_i a^i = d(y_0 + \sum_{i=1}^k y_i a^i) = x_0 + \sum_{i=1}^k x_i a^i = 0$ and $b = y_0 + \sum_{i=1}^k x_i a^i$. Now, it follows from $ad = a$ and $d \in R^*$ that $b = y_0 - x_0 = 0$ as required. □

**Theorem 3.** *Let $R = \mathbb{Z}$ or $R = \mathbb{Z}_m$ with $m \in \{4, 15, 50, 63\}$, and $n \geqslant 3$. Then there exist a commutative ring extension $S = R[\sqrt{d}]$ of rank 2 of $R$ and an intermediate subgroup $H, E(n, S) \leqslant H \leqslant \mathrm{GL}(2n, R)$ such that there is no ideal $A$ of $R$ so that*

$$E(n, S)E(2n, R, A) \leqslant H \leqslant N_{\mathrm{GL}(2n,R)}(E(n, S)E(2n, R, A)).$$

**Proof.** 1) Case $R = \mathbb{Z}_m$ with $m \in \{4, 15, 50, 63\}$. Put

$$(d, a) = \begin{cases} (-1, 2), \text{ if } m = 4; \\ (7, 5), \text{ if } m = 15; \\ (3, 25), \text{ if } m = 50; \\ (8, 9), \text{ if } m = 63. \end{cases}$$

Then $S = R[\sqrt{d}]$ is a commutative ring extension of rank 2 of $R$ and $d \in R^*$, $a \in R \setminus \{0\}, ad = a$. By Lemma 12, the subgroup $H = E(n, S[M])$ with

$M = \begin{pmatrix} 0 & 0 \\ a & a \end{pmatrix}$ does not contain any nontrivial elementary transvection in $E(2n, R)$. Suppose that there exists an ideal $A$ satisfying the property in the theorem. Then $A = 0$ and $H \leqslant N_G(E(n, S))$. By Corollary 1, [12] there exists $\sigma = \begin{pmatrix} 1 & b \\ 0 & c \end{pmatrix} \in \mathrm{Aut}(R/S)$ such that $M\sigma \in S$, so $a = 0$ and we have a contradiction.

2) Case $R = \mathbb{Z}$. Let $m, d, a, M$ and $H$ be as above. Suppose that there exists an ideal $A$ such that

$$E(n, \mathbb{Z}[\sqrt{d}])E(2n, \mathbb{Z}, A) \leqslant H \leqslant N_{\mathrm{GL}(2n,\mathbb{Z})}(E(n, \mathbb{Z}[\sqrt{d}])E(2n, \mathbb{Z}, A)).$$

Let $I = m\mathbb{Z}$ and let $\pi : \mathbb{Z} \longrightarrow \mathbb{Z}_m$ be the canonical epimorphism. We have

$$\begin{aligned} E(n, \mathbb{Z}_m[\sqrt{d}])E(2n, \mathbb{Z}_m, \pi(A)) &\leqslant \rho_I(H) \\ &\leqslant \rho_I(N_{\mathrm{GL}(2n,\mathbb{Z})}(E(n, \mathbb{Z}[\sqrt{d}])E(2n, \mathbb{Z}, A)). \end{aligned}$$

Note that

$$\begin{aligned} &\rho_I(N_{\mathrm{GL}(2n,\mathbb{Z})}(E(n, \mathbb{Z}[\sqrt{d}])E(2n, \mathbb{Z}, A)) \\ &\quad\subseteq N_{\mathrm{GL}(2n,\mathbb{Z}_m)}(E(n, \mathbb{Z}_m[\sqrt{d}])E(2n, \mathbb{Z}_m, \pi(A)) \end{aligned}$$

and we have a contradiction. □

**Remark**. By Theorem 3 with $R = \mathbb{Z}_{15}$, we see that the condition $S \subseteq \overline{R}$ in Theorem 2 can not be omitted. It is also shown that the result in Theorem 2 does not hold for $\mathbb{Z}$ and $\mathbb{Z}_m$ if $m$ is not square-free.

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