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TAMASCHKE'S RESULTS ON SCHUR RINGS AND A GENERALIZATION OF ASSOCIATION SCHEMES

Abstract. The concepts of an association scheme and a coherent configuration are generalized by analyzing a relationship between S-rings (Schur rings) and association schemes. In this connection, Tamaschke's results on S-rings and other generalizations of association schemes are discussed.

Dedicated to the memory of Sergei Evdokimov

§1. Introduction

In Wielandt's book [25], an S-ring (Schur ring) is defined as follows. Let G be a finite group and $G = T_1 \cup \cdots \cup T_\ell$ be a partition of G. Suppose that $T_1 = \{1_G\}$ and, for every $i \in \{1, \ldots, \ell\}$, there exists $i^* \in \{1, \ldots, \ell\}$ such that $T_{i^*} = \{g^{-1} \mid g \in T_i\}$. Put $\tau_i = \sum_{g \in T_i} g \in \mathbb{Z}G$. If $\mathscr{T} = \bigoplus_{i=1}^{\ell} \mathbb{Z}\tau_i$ is a subring of $\mathbb{Z}G$, then we say that $\mathscr{T}$ is an S-ring. A typical example of an S-ring is obtained by orbits of a subgroup of the automorphism group of G on G. Theory of S-rings were studied by many authors, for example, Klin–Pöschel [18], Leung–Man [19, 20], and Evdokimov–Ponomarenko [8]. A recent survey on S-rings was given in Muzychuk–Ponomarenko [21].

In [22, 23], Tamaschke used a weak definition for S-rings. Tamaschke did not assume $T_1 = \{1_G\}$. To avoid confusion, we call Tamaschke's S-rings quasi-S-rings (quasi-Schur rings). If the underlying group G is abelian, then quasi-S-rings are S-rings by a suitable identification. However they are different, in general. Tamaschke considered structure theory and representation theory of quasi-S-rings.

It is well known that an S-ring defines an association scheme in the sense of Zieschang [28], a fusion of a thin association scheme. Motivated by Tamaschke's S-rings, we generalize association schemes and coherent configurations (Definition 2.1). However, we show that they are essentially association schemes or coherent configurations (Theorem 2.9). Thus we can get no new object, but the author believe that the arguments are still

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valuable because the definitions are weaker than the usual ones. As we wrote, an S-ring is characterized as a fusion of a thin association scheme. Similarly, we can characterize a quasi-S-ring as a correspondent of a fusion of a schurian association scheme (Theorem 3.4).

In the section §4, we summarize how Tamaschke's results were generalized to association schemes. We also give answers to two Tamaschke's questions in §4.4 and §5.

§2. Generalizations of coherent configurations and association schemes

Let X be a finite set. We denote by $\mathrm{M}_X(R)$ the full matrix algebra over a commutative ring R both rows and columns of whose matrices are index by the set X. For a subset s of $X \times X$, we denote by σ_s the adjacency matrix of s, namely $\sigma_s \in \mathrm{M}_X(\mathbb{Z})$ with the (x,y)-entry is 1 if $(x,y) \in s$ and 0 otherwise. We set $1_X = \{(x,x) \mid x \in X\}$.

Definition 2.1. Let X be a finite set, and let S be a collection of non-empty subsets of $X \times X$. The pair (X,S) is called a *quasi-coherent configuration* if the following conditions hold:

(1) $X \times X = \bigcup_{s \in S} s$ is a partition,

(2) for every $s \in S$, there exists $s^* \in S$ such that

$$s^* = \{(y,x) \mid (x,y) \in s\} \in S,$$

and

(3) $\bigoplus_{s \in S} \mathbb{Z}\sigma_s$ is a subring of $\mathrm{M}_X(\mathbb{Z})$ (possibly without units).

The pair (X,S) is called a *coherent configuration* [17] if the conditions (1), (2), (3) and

(4) there is a subset S_0 of S such that $\bigcup_{s \in S_0} s = 1_X$

hold. The pair (X,S) is called a *quasi-association scheme* if the conditions (1), (2), (3) and

(5) there is an $s \in S$ such that $s \supset 1_X$

hold. The pair (X,S) is called an *association scheme* [3, 28] if the conditions (1), (2), (3) and

(6) $1_X \in S$

hold.

An association scheme is also called a *homogeneous coherent configuration* [17]. We remark that some authors use another definitions of association schemes. For example, it is assumed that $s^* = s$ for all $s \in S$ (a *symmetric association scheme*) in Bailey [1], Cameron [4], Cameron-van Lint [5], Godsil [10], and it is assumed that the ring $\bigoplus_{s\in S} \mathbb{Z}\sigma_s$ is commutative (a *commutative association scheme*) in Delsarte [7]. It is easy to see that symmetric association schemes are commutative. By definition, we have

- an association scheme is a coherent configuration,
- an association scheme is a quasi-association scheme,
- a coherent configuration is a quasi-coherent configuration, and
- a quasi-association scheme is a quasi-coherent configuration.

We will see that a quasi-coherent configuration defines a coherent configuration and a quasi-association scheme defines an association scheme in a natural way (Theorem 2.9). Thus the generalizations yield no new objects, but the author believe that they are still useful. For example, S-rings by Tamaschke's definition [22, 23] (we call them quasi-S-rings) are quasi-association schemes but not association schemes.

Let (X, S) be a quasi-coherent configuration. We set $S_0 = \{s \in S \mid s \cap 1_X \neq \emptyset\}$. We use letters $s, t, u, \dots$ for elements in S, and $a, b, c, \dots$ for elements in S_0. Note that $a^* = a$ for $a \in S_0$, since $(x, x) \in a$ for some $x \in X$. A quasi-coherent configuration (X, S) is a quasi-association scheme if and only if $|S_0| = 1$. By the condition (3) in Definition 2.1, there are non-negative integers p_{st}^u $(s, t, u \in S)$ such that $\sigma_s\sigma_t = \sum_{u\in S} p_{st}^u \sigma_u$. In other words, for $(x, y) \in u$, it holds that $\sharp\{z \in X \mid (x, z) \in s,\ (z, y) \in t\} = p_{st}^u$. We call p_{st}^u the *intersection numbers.* Clearly $p_{aa}^a \neq 0$ for $a \in S_0$. For a commutative ring R with unity, we can define an R-algebra $RS = R \otimes_{\mathbb{Z}} \left(\bigoplus_{s\in S} \mathbb{Z}\sigma_s\right)$. We call RS the *adjacency algebra* of (X, S) over R. We define the *complex product* by $st = \{u \in S \mid p_{st}^u \neq 0\}$ for $s, t \in S$. For $T, U \subset S$, we also define the complex product by $TU = \bigcup_{t\in T} \bigcup_{u\in U} tu$. It is easy to see that the complex product is associative.

Lemma 2.2. *For $s, t, u \in S$, the following conditions are equivalent:*

(1) $p_{st}^{u^*} \neq 0$,
(2) $p_{tu}^{s^*} \neq 0$,
(3) $p_{us}^{t^*} \neq 0$, *and*

(4) *there exists* $x, y, z \in X$ *such that* $(x, y) \in s$, $(y, z) \in t$, *and* $(z, x) \in u$.

Proof. This is clear by definition. □

Lemma 2.3. *For* $s, t \in S$, $s \neq t$ *and* $a \in S_0$, *we have* $p^a_{st^*} = 0$.

Proof. By definition, all diagonal entries of $\sigma_s \sigma_{t^*}$ are 0 if $s \neq t$. □

For $a \in S_0$, we set $X_a = \{x \in X \mid (x, x) \in a\}$. Then $X = \bigcup_{a \in S_0} X_a$ is a partition of X.

Lemma 2.4. *For* $a \in S_0$, *we have* $a \subset X_a \times X_a$ *and* ${\sigma_a}^2 = p^a_{aa}\sigma_a$. *Moreover,* $\sigma_a \sigma_b = 0$ *for* $a, b \in S_0$, $a \neq b$.

Proof. Suppose $s \in S$ and $s \neq a$. We have $p^a_{as^*} = 0$ by Lemma 2.3 and $a^* = a$. By Lemma 2.2, we have $p^s_{aa} = 0$. Thus ${\sigma_a}^2 = p^a_{aa}\sigma_a$ holds. Suppose that $(x, y) \in a$. By ${\sigma_a}^2 = p^a_{aa}\sigma_a$, $(x, x) \in a$, and similarly we have $(y, y) \in a$. Thus $a \subset X_a \times X_a$.

Now it is clear that $\sigma_a \sigma_b = 0$ for $a, b \in S_0$, $a \neq b$. □

Proposition 2.5. *For every* $s \in S$, *there exists a unique pair* $(a, b) \in S_0 \times S_0$ *such that* $\sigma_a \sigma_s \sigma_b \neq 0$. *Moreover, in this case,* $s \subset X_a \times X_b$, $\sigma_a \sigma_s \sigma_b = p^a_{aa} p^b_{bb} \sigma_s$ *and* $asb = as = sb = \{s\}$ *for complex products.*

Proof. Since $s \neq \emptyset$, we choose $(x, y) \in s$. There are $a, b \in S_0$ such that $(x, x) \in a$ and $(y, y) \in b$. Thus $\sigma_a \sigma_s \sigma_b \neq 0$.

Suppose $t \in S$ and $t \neq s$. Then $p^a_{st^*} = 0$ by Lemma 2.3, and thus $p^t_{as} = 0$. This means that we can write $\sigma_a \sigma_s = p^s_{as}\sigma_s$ for $p^s_{as} > 0$. If $\sigma_{a'}\sigma_s \neq 0$ for $a' \in S_0$, $a' \neq a$, then $0 \neq p^s_{as}\sigma_{a'}\sigma_s = \sigma_{a'}\sigma_a\sigma_s$ and this is impossible by Lemma 2.4. Therefore $a \in S_0$ is unique, and similarly $b \in S_0$ is unique.

By the above arguments, we can write $\sigma_a \sigma_s \sigma_b = \beta \sigma_s$ for some positive integer β. On the other hand, we have

$$\begin{aligned} {\sigma_a}^2 \sigma_s {\sigma_b}^2 &= (p^a_{aa}\sigma_a)\sigma_s(p^b_{bb}\sigma_b) = \beta p^a_{aa} p^b_{bb} \sigma_s, \\ {\sigma_a}^2 \sigma_s {\sigma_b}^2 &= \sigma_a(\sigma_a \sigma_s \sigma_b)\sigma_b = \beta \sigma_a \sigma_s \sigma_b = \beta^2 \sigma_s. \end{aligned}$$

Thus $\beta = p^a_{aa} p^b_{bb}$. □

Proposition 2.6. *The relation* $\Delta = \bigcup_{a \in S_0} a$ *is an equivalence relation on* X.

Proof. We claim that $a \in S_0$ is an equivalence relation on X_a. By definition, a is a reflexive relation. By $a^* = a$, a is a symmetric relation. Since $(x,y) \in a$ and $(y,z) \in a$ imply $(x,z) \in a$ by Lemma 2.4, a is an associative relation. Therefore a is an equivalence relation on X_a.

Now, since $X = \bigcup_{a \in S_0} X_a$ is a partition of X, the assertion holds. □

We consider the set of equivalence classes X/Δ. We denote by $x\Delta$ the equivalence class containing $x \in X$.

Lemma 2.7. *For $x \in X_a$, we have $x\Delta = \{y \in X \mid (x,y) \in a\}$ and $|x\Delta| = p_{aa}^a$.*

Proof. Since $\sigma_a \sigma_b = 0$ for $b \in S_0$, $b \neq a$, the assertions are clear. □

Lemma 2.8. *Suppose $x' \in x\Delta$, $y' \in y\Delta$, and $(x,y) \in s$. Then we have $(x',y') \in s$.*

Proof. Suppose that $(x',y') \in t$. By Proposition 2.5, there is a unique pair $(a,b) \in S_0 \times S_0$ such that $asb = as = sb = \{s\}$. Thus $x \in X_a$, $(x,x') \in a$, $y \in X_b$, and $(y,y') \in b$. Therefore $t \in asb = \{s\}$ and $t = s$. □

By Lemma 2.8, for $s \in S$, we can define $s^\Delta = \{(x\Delta, y\Delta) \mid (x,y) \in s\} \subset X/\Delta \times X/\Delta$, and we have a partition $X/\Delta \times X/\Delta = \bigcup_{s \in S} s^\Delta$. We remark that $s^\Delta \neq \emptyset$ for every $s \in S$ and $s^\Delta \cap t^\Delta = \emptyset$ for $s \neq t$. We put $S^\Delta = \{s^\Delta \mid s \in S\}$.

Theorem 2.9. *For a quasi-coherent configuration (X,S), the pair $(X/\Delta, S^\Delta)$ is a coherent configuration. If (X,S) is a quasi-association scheme, then $(X/\Delta, S^\Delta)$ is an association scheme.*

Proof. The conditions (1) and (2) in Definition 2.1 are clearly satisfied. We prove (3) in Definition 2.1. Suppose $s,t,u \in S$ and $(x\Delta, y\Delta) \in u^\Delta$. We put $I = \{z \in X \mid (x,z) \in s,\ (z,y) \in t\}$ and $I' = \{z\Delta \in X/\Delta \mid (x\Delta, z\Delta) \in s^\Delta,\ (z\Delta, y\Delta) \in t^\Delta\}$. There are $a,b \in S_0$ such that $sa = \{s\}$ and $bt = \{t\}$ by Proposition 2.5. If $a \neq b$, then $I' = \emptyset$ and so $|I'| = 0$ does not depend on $(x\Delta, y\Delta) \in u^\Delta$. Assume that $a = b$. In this case, $I \subset X_a$ and $|z\Delta| = p_{aa}^a$ for every $z\Delta \in I'$ by Lemma 2.7. Now $|I'| = |I|/p_{aa}^a = p_{st}^u/p_{aa}^a$ does not depend on $(x\Delta, y\Delta) \in u^\Delta$.

The last statement is clearly holds. □

Remark 2.10. (1) The argument here is similar to the definition of factor schemes of association schemes [28, §1.5].

(2) Let R be a commutative ring with unity. In general, the adjacency algebras of (X,S) and $(X/\Delta, S^\Delta)$ over R are non-isomorphic. However, if all p_{aa}^a $(a \in S_0)$ are invertible in R, then RS has the identity element $\sum_{a\in S_0} (p_{aa}^a)^{-1}\sigma_a$ and the adjacency algebras of (X,S) and $(X/\Delta, S^\Delta)$ over R are isomorphic by the map $(p_{aa}^a)^{-1}\sigma_s \mapsto \sigma_{s^\Delta}$, where $a \in S_0$ is determined by $as = \{s\}$.

§3. Schur rings and quasi-Schur rings

Following Wielandt's book [25], we define S-rings (Schur rings). However, Tamaschke used a weaker definition, and Wielandt also used it in [24]. To avoid confusion, we call Tamaschke's S-rings quasi-S-rings (quasi-Schur rings).

Definition 3.1. Let G be a finite group. We say that a subring (not necessarily contains units) $\mathscr{T}$ of the group ring $\mathbb{Z}G$ is a *quasi-S-ring* (quasi-Schur ring) on G if the following conditions hold:

(1) there is a partition $G = T_1 \cup \cdots \cup T_\ell$,
(2) for every $i \in \{1,\dots,\ell\}$, there exists $i^* \in \{1,\dots,\ell\}$ such that $T_{i^*} = \{g^{-1} \mid g \in T_i\}$, and
(3) we put $\tau_i = \sum_{g\in T_i} g$, then the set $\{\tau_1, \dots \tau_\ell\}$ is a $\mathbb{Z}$-basis of $\mathscr{T}$.

A quasi-S-ring is called an *S-ring* (Schur ring) if

(4) $T_1 = \{1_G\}$.

The set T_i is called a $\mathscr{T}$*-class* and τ_i is called a $\mathscr{T}$*-class sum*. For a quasi-S-ring, we suppose $T_1 \ni 1_G$. The next example is [22, Example 1.2].

Example 3.2. (1) Let $G = \{g_1,\dots,g_n\}$ be a finite group. The partition $G = \{g_1\} \cup \cdots \cup \{g_n\}$ defines an S-ring on G. In this case, $\mathscr{T} = \mathbb{Z}G$.

(2) Let G be a finite group with conjugacy classes $C_1,\dots,C_\ell$. The partition $G = C_1 \cup \cdots \cup C_\ell$ defines an S-ring on G. In this case, $\mathscr{T} = Z(\mathbb{Z}G)$, the center of the group ring.

(3) Let G be a finite group and H a subgroup of G. The double coset partition $G = Hg_1H \cup \cdots \cup Hg_\ell H$ defines a quasi-S-ring on G. We call this quasi-S-ring the *double coset quasi-S-ring*.

Let $\mathscr{T}$ be a quasi-S-ring on a finite group G with partition $G = T_1 \cup \cdots \cup T_\ell$, $1_G \in T_1$. We define $s_i = \{(g,h) \in G \times G \mid gh^{-1} \in T_i\}$ and put $S = \{s_i \mid i = 1, \ldots, \ell\}$. Then it is easy to see that (G, S) is a quasi-association scheme. If $\mathscr{T}$ is an S-ring then (G, S) is an association scheme. Let Φ be the right regular permutation representation of G. Then the adjacency matrix of s_i is $\sum_{g \in T_i} \Phi(g)$. By Theorem 2.9, every quasi-S-ring defines an association scheme. We will characterize what kind of association schemes can be obtained by quasi-S-rings in Theorem 3.4.

Example 3.3. We consider association schemes obtained from (quasi-) Schur-rings in Example 3.2.

(1) An association scheme obtained from an S-ring in Example 3.2 (1) is called a *thin association scheme*. The adjacency matrices are permutation matrices $\{\Phi(g) \mid g \in G\}$, where Φ is the right regular permutation representation of G.

(2) An association scheme obtained from an S-ring in Example 3.2 (2) is called a *group association scheme*.

(3) An association scheme obtained from a quasi-S-ring in Example 3.2 (3) is called a *schurian association scheme*. Usually, a schurian association scheme is defined by a transitive permutation group, but it is equivalent to our definition. A schurian association scheme is defined by a finite group G and its subgroup H.

Let $\mathscr{T}$ and $\mathscr{T}'$ be quasi-S-rings on G with partitions $G = T_1 \cup \cdots \cup T_\ell$ and $G = T'_1 \cup \cdots \cup T'_m$, respectively. We say that $\mathscr{T}'$ is a *fission* of $\mathscr{T}$ if every T'_i is contained in some T_j. Is this case, we also say that $\mathscr{T}$ is a *fusion* of $\mathscr{T}'$. For (quasi-) coherent configurations and (quasi-) association schemes, we also define fissions and fusions similarly.

Theorem 3.4. *The association scheme obtained from an S-ring is a fusion of thin an association scheme. The association scheme obtained from a quasi-S-ring is a fusion of a schurian association scheme.*

Proof. The first statement is clear by definition. We show the second statement. Let $\mathscr{T}$ be a quasi-S-ring on G with partition $G = T_1 \cup \cdots \cup T_\ell$. Suppose $1_G \in T_1$. Put $H = T_1$. Then H is a subgroup of G by Proposition 2.6 or [22, §1]. By Proposition 2.5, every T_i is a union of (H, H)-double cosets. Thus $\mathscr{T}$ is a fusion of the double coset quasi-S-ring, and the corresponding association scheme is a fusion of a schurian association scheme. □

The converses of the statements of Theorem 3.4 also hold. Namely, every fusion of a schurian association scheme is realized by a quasi-S-ring.

If T_1 is a normal subgroup of G, especially if G is an abelian group, then essentially the quasi-S-ring $\mathscr{T}$ can be considered as an S-ring on G/T_1 though their ring structures are different, in general (see Remark 2.10 (2)).

§4. Tamaschke's results and generalizations

In this section, we will summarize how Tamaschke's results on quasi-S-rings were generalized to association schemes. In this section, $\mathscr{T}$ is a quasi-S-ring on a finite group G with partition $G = T_1 \cup \cdots \cup T_\ell$, $H = T_1 \ni 1_G$, and (X, S) is the association scheme obtained from $\mathscr{T}$.

For a commutative ring R with unity we define an *quasi-S-algebra* $R\mathscr{T}$ of $\mathscr{T}$ over R by $R\mathscr{T} = R \otimes_{\mathbb{Z}} \mathscr{T}$. For the complex number field $\mathbb{C}$, $\mathbb{C}\mathscr{T}$ is isomorphic to the adjacency algebra $\mathbb{C}S$ of (X, S) as a $\mathbb{C}$-algebra by Remark 2.10 (2), and thus it is semisimple [28, Theorem 4.1.3]. We can identify the sets of all irreducible characters of $\mathbb{C}\mathscr{T}$ and $\mathbb{C}S$, and we will denote it by $\mathrm{Irr}(\mathscr{T})$ or $\mathrm{Irr}(S)$.

4.1. Ordinary representations. Ordinary representations, representations over the complex number field $\mathbb{C}$, of quasi-S-rings were considered in [23, §1]. Orthogonality relations [23, Theorem 1.5] and a formula on central primitive idempotents [23, Theorem 1.7] were given and generalized to coherent configurations in Higman [17]. In particular, for double coset quasi-S-rings (Example 3.2 (3)), we can understand all irreducible representations by [6, Theorem 11.25], and this fact was generalized to more general cases in [16]. Since every quasi-S-ring is a fusion of a double coset quasi-S-ring, we are interested in representations of a fusion. For commutative association schemes, fusions were considered in Bannai [2] (he call a fusion association scheme a *subscheme*). An another result for representations of fusions were given in [15].

4.2. $\mathscr{T}$-Subgroups and closed subsets. A subgroup K of G is called a *$\mathscr{T}$-subgroup* if it is a union of some $\mathscr{T}$-classes [23, §2]. A $\mathscr{T}$-subgroup corresponds to a *closed subset* of (X, S) (see [28, §1.3]).

4.3. $\mathscr{T}$-Conjugacy classes, CS-rings, and group-like association schemes. Two $\mathscr{T}$-classes T_i and T_j are said to be *$\mathscr{T}$-conjugate* if $|T_i|^{-1}\chi(\tau_i) = |T_j|^{-1}\chi(\tau_j)$, where $\tau_i = \sum_{g \in T_i} g$ is the $\mathscr{T}$-class sum, for all $\chi \in \mathrm{Irr}(\mathscr{T})$ (this is different from Tamaschke's definition [23, Definition 2.1], but essentially they are same). This was generalized to association schemes in [11, §4].

Note that the cardinality of $\mathscr{T}$-conjugacy classes is greater than or equal to $|\operatorname{Irr}(\mathscr{T})|$. We say that $\mathscr{T}$ is a *CS-ring* if the equality holds [23, Definition 3.2]. In this case, the center of $\mathscr{T}$ is also a quasi-S-ring with the partition given by $\mathscr{T}$-conjugacy classes. If $\mathscr{T}$ is commutative, then clearly $\mathscr{T}$ is a CS-ring. The definition of CS-rings was generalized to association schemes in [11, §4]. We call such an association scheme a *group-like association scheme*.

When the double coset S-ring $\mathscr{T}$ is a CS-ring, the subgroup $H = T_1$ is called a *CS-subgroup* [23, §5]. It seems that CS-subgroups are not studied so well. It is natural to define a *CS-closed subset* of an association scheme, that is a closed subset and the factor scheme is group-like.

4.4. $\mathscr{T}$-Normal subgroups and normal closed subsets. We start with Tamaschke's ambiguous definitions [22, §4]. Let K be a $\mathscr{T}$-subgroup of G.

(1) K is said to be T-*normal* if $T_iK = KT_i$ for all $i \in \{1, \dots, \ell\}$.
(2) K is said to be *T-normal* if $\sum_{g\in K} g$ is in the center of $\mathscr{T}$.

In [22, §4], Tamaschke wrote "This definition of T-normality is not sufficient for T-normality. At least we do not know yet whether it is sufficient or not", see also [22, Problem 4.12]. However, we know that they are equivalent. A closed subset U of an association scheme (X, S) is said to be *normal* if $sU = Us$ for all $s \in S$, and thus this corresponds to "T-normality". In [12, Proposition 3.3], it was shown that U is normal if and only if $\sum_{u\in U} \sigma_u$ is in the center of the adjacency algebra $\mathbb{C}S$. Thus T-normality and T-normality are equivalent for a $\mathscr{T}$-subgroup.

It is known that the intersection of two T-normal subgroups is not necessarily T-normal [26]. Thus the intersection of two normal closed subset of an association scheme is not necessarily normal [27]. If $\mathscr{T}$ is a CS-ring, then the intersection of two T-normal subgroups is T-normal [23, Theorem 4.2]. Similarly, if (X, S) is a group-like association scheme, then the intersection of two normal closed subsets is normal. We also remark that the "kernel" of a character of an association scheme is not necessarily normal [11, Example 3.3], in general. However the "kernel" of a character of an association scheme is normal, if the association scheme is group-like [11, Theorem 4.3].

The factor ring of a CS-ring $\mathscr{T}$ by a $\mathscr{T}$-normal subgroup is also a CS-ring [23, Theorem 4.3]. This fact was generalized to association schemes in [13, Proposition 4.2].

4.5. Tensor products of quasi-S-ring. Tensor products of quasi-S-rings were defined in [23, §6] and were generalized to coherent configurations or association schemes in a natural way.

4.6. Categories of S-rings and association schemes. In [22, §1], Tamaschke defined some categories. His definition of a category of quasi-S-algebras was generalized to a category of association schemes in French [9]. Their morphisms induce algebra homomorphisms. An another definition of a category of association schemes was considered in [14].

4.7. The homomorphism theorem, isomorphism theorems, and the Jordan–Hölder theorem. In [22], the homomorphism theorem, isomorphism theorems, and the Jordan–Hölder theorem for quasi-S-rings were given. We can find them for association schemes in Zieschang [28].

§5. An answer to a question by Tamaschke

When a partition of a finite group, closed by taking inverse, defines a semigroup by complex products $ab = \{gh \mid g \in a,\ h \in b\}$, we call the semigroup an *S-semigroup* [22, Definition 1.9]. Let $\mathscr{T}$ be a quasi-S-ring with a partition $G = T_1 \cup \cdots \cup T_\ell$. Then the semigroup generated by $\{T_1, \ldots, T_\ell\}$ is an S-semigroup. Thus a quasi-S-ring defines an S-semigroup.

A Tamaschke's question [22, Question 1.19] is whether all S-semigroups are obtained by S-rings or not. We will give an example of an S-semigroup which does not come from an S-ring. Therefore S-semigroups are not necessarily obtained by S-rings.

Example 5.1. Let G be the symmetric group of degree 4. Put

$$\begin{aligned} T_1 &= \{()\}, \\ T_2 &= \{(1,2),\ (1,3),\ (1,4),\ (2,3),\ (1,2,3),\ (1,3,2), \\ &\qquad (2,3,4),\ (2,4,3),\ (1,2,3,4),\ (1,4,3,2)\}, \\ T_3 &= G \setminus (T_1 \cup T_2). \end{aligned}$$

Then $G = T_1 \cup T_2 \cup T_3$, $T_i^{-1} = T_i$ $(i = 1, 2, 3)$ and

$$T_2T_2 = T_3T_3 = G = T_1 \cup T_2 \cup T_3, \quad T_2T_3 = T_3T_2 = G \setminus \{()\} = T_2 \cup T_3.$$

Thus this partition defines an S-semigroup $\{T_1,\ T_2,\ T_3,\ T_2 \cup T_3,\ T_1 \cup T_2 \cup T_3\}$, but easily we can see that this partition does not define an S-ring.

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