Записки научных
семинаров ПОМИ
Том 455, 2017 г.

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ON T-AMORPHOUS ASSOCIATION SCHEMES

Abstract. A scheme is called T-amorphous if it is antisymmetric and any tournament obtained by an appropriate merging of its classes is doubly regular. The goal of this paper is to study basic properties of this class of schemes.

Dedicated to the memory of Sergei Evdokimov

§1. Introduction

Let $\mathfrak{X} := (\Omega, \mathcal{R} = \{R_i\}_{i=0}^d)$ be an association scheme with d classes. It is not assumed that that $\mathfrak{X}$ is commutative. Most of notation used in this paper follow the book [1]. As usual $i \mapsto i'$ is a transposition map, that is $R_{i'} = R_i^\top$. For a subset $I \subseteq [0,d]$ we set $I' := \{i' \,|\, i \in I\}$. The structure constants and the valencies of a scheme $\mathfrak{X}$ are denoted by p_{ij}^k and v_i, respectively. The adjacency matrix of R_i is denoted by A_i. Given a field F, a linear span of the adjacency matrices A_i, denoted by $F[\mathcal{R}]$, is a subalgebra of the full matrix algebra $M_\Omega(F)$. It is called the *adjacency* (or *Bose–Mesner*) algebra of $\mathfrak{X}$ over F. An *algebraic isomorphism* between two schemes $\mathfrak{X} = (\Omega, \mathcal{R} = \{R_i\}_{i=0}^d))$ and $\mathfrak{X}' = (\Omega', \mathcal{R}' = \{R_i'\}_{i=0}^d))$ is a bijection $\varphi : [0,d] \to [0,d]$ which preserves the structure constants, that is $p_{i^\varphi j^\varphi}^{k^\varphi} = p_{ij}^k$ holds for each triple i, j, k of indices. Notice that algebraic isomorphisms coincide with pseudo-isomorphisms [7], BM-isomorphisms [8] and weak-equivalences [14]. An algebraic isomorphism of a scheme to itself is called an algebraic automorphism of the scheme.

A partition $\mathcal{P} = \{\mathcal{P}_i \,|\, i = 0, \dots, d'\}$ of the index set $[0,d]$ into non-empty subsets is called *admissible* [6] if $\mathcal{P}_0 = \{0\}$, and for each $i, 1 \le i \le d'$ there exists $j \in [0,d']$ such that $\mathcal{P}_i' = \mathcal{P}_j$. Given an admissible partition $\mathcal{P} = \{\mathcal{P}_i \,|\, i = 0, \dots, d'\}$, we denote by $\mathcal{R}/\mathcal{P}$ a partition of Ω^2 with d' classes $R_{\mathcal{P}_i} := \bigcup_{j \in \mathcal{P}_i} R_j, 0 \le i \le d'$. We say that $\mathcal{P}$ *gives rise* to an association scheme if a pair $(\Omega, \mathcal{R}/\mathcal{P})$ is an association scheme. In this case the scheme $(\Omega, \mathcal{R}/\mathcal{P})$ is called a *fusion* of $(\Omega, \mathcal{R})$. Following [6] we call a

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Key words and phrases: assosiation scheme, adjacency algebra, fusion.

scheme *amorphous* if $(\Omega, \mathcal{R}/\mathcal{P})$ is an association scheme for any admissible partition $\mathcal{P}$ of $[0, d]$. It should be noted that the original definition given in [6] dealt only with commutative schemes. But it was shown by Ma [10] that being amorphous implies commutativity. Symmetric amorphous schemes known as *amorphic* were classified up to algebraic isomorphism by Gol'fand, Ivanov and Klin and in [9]. Amorphous association schemes with at least three symmetrized classes were classified (up to algebraic isomorphism) by Ito, Munemasa and Yamada. Recently J. Ma classified amorphous schemes with two symmetrized classes [10], up to algebraic isomorphism.

A scheme is called *anti-symmetric* (skew-symmetric in [10]) if $i' \neq i$ for each $i \neq 0$. An anti-symmetric scheme with two classes is equivalent to a *doubly regular tournament*. The parameters of a doubly regular tournament are uniquely determined by its order (the number of vertices of a tournament). A necessary condition for an existence of an anti-symmetric scheme on v points is $v \equiv 3 (\mathrm{mod}\ 4)$. Given an anti-symmetric scheme $(\Omega, \mathcal{R})$ with d classes, we say that a subset $I \subset [1, d]$ is a *$'$-transversal* if I intersects each pair $\{j, j'\}, j \in [1, d]$ by one element. In what follows we number the relations of $\mathcal{R}$ in a way that the set $[1, d/2]$ is a $'$-transversal. Every $'$-transversal I gives rise to an admissible partition with three classes $\{0\}, I, I'$. We say that an anti-symmetric scheme is *T-amorphous* if partition $\{0\}, I, I'$ gives rise to an anti-symmetric scheme for each $'$-transversal I. Of course, every amorphous anti-symmetric scheme would be an example of a T-amorphous scheme. Unfortunately, as was shown by Ma [10], they do not exist. Thus, one could ask whether T-amorphous schemes exist at all. Fortunately, the answer is affirmative. Recently Feng and Xiang [11] built an infinite series of T-amorphous cyclotomic schemes. The purpose of this note is to study the properties of this class of schemes.

§2. Some basic facts about T-amorphous schemes

First, we fix some notation. For the rest of the paper $\mathfrak{X} = (\Omega, \mathcal{R})$ will stand for an anti-symmetric scheme with $d = 2e$ classes. It is also assumed that $[1, e]$ is a $'$-transversal. We start with the following characterization of anti-symmetric schemes [15].

Proposition 2.1. *A scheme $(\Omega, \mathcal{R})$ is anti-symmetric if and only if it is odd, that is all its valencies and $|\Omega|$ are odd numbers.*

As a direct consequence we obtain the following

Corollary 2.2. *If $\mathfrak{X}$ is T-amorphous, then e is odd.*

Proof. The statement follows from the equailty

$$|\Omega| = 1 + \sum_{i=1}^{e} 2v_i \equiv 2e + 1(\mathrm{mod}\ 4)$$

and the congruence $|\Omega| \equiv 3(\mathrm{mod}\ 4)$. □

Now, we can easily show that a T-amorphous scheme with $e > 1$ cannot be amorphous[1]. Indeed, if an anti-symmetic scheme is amorphous, then its symmetrization is an amorphic scheme with $e \geq 3$ classes. Therefore, the number v of scheme points is a square, contradicting $v \equiv 3(\mathrm{mod}\ 4)$.

The main result of this section is the following characterization of anti-symmetric T-amorphous schemes.

Theorem 2.3. *An anti-symmetric scheme $(\Omega, \mathcal{R} = \{R_i\}_{i=0}^{2e})$ is T-amorphous if and only if*

$$\forall_{i \neq j}\ X_i X_j = -X_j X_i \ \textit{and} \ \sum_{i=1}^{e} X_i^2 = -|\Omega| I_\Omega + J_\Omega, \tag{1}$$

where $X_i := A_i - A_{i'}, 1 \leq i \leq e$.

To prove this statement we recall a well-known characterization of doubly regular tournaments.

Proposition 2.4. *A regular tournament (Ω, R) is doubly regular if and only if $(A - A^\top)^2 = -|\Omega| I_\Omega + J_\Omega$, where A is the adjacency matrix of R.*

Proof of Theorem 2.3. It follows from Proposition 2.4 that the scheme $\mathfrak{X}$ is T-amorphous iff for any function $\varepsilon : [1, e] \to \{\pm 1\}$ the matrix $X_\varepsilon := \sum_{i=1}^{e} \varepsilon(i) X_i$ satisfies the equation $X_\varepsilon^2 = -|\Omega| I_\Omega + J_\Omega$. We may assume that $e \geq 3$.

Pick an arbitrary pair of indices $i \neq j \in [1, e]$ and set $Y_1 = X_i$, $Y_2 = X_j$, $Y_3 = \sum_{k \neq i,j} X_k$. Then $(Y_1 + Y_2 + Y_3)^2 = -|\Omega| I_\Omega + J_\Omega$ and $(-Y_1 + Y_2 + Y_3)^2 = -|\Omega| I_\Omega + J_\Omega$, implying $Y_1 \star Y_2 + Y_1 \star Y_3 = 0$, where

[1]This argument was found by Misha Klin.

$A \star B := AB + BA$. Permutting the indices $1, 2, 3$ we obtain the following system of matrix equations

$$\begin{cases} Y_1 \star Y_2 + Y_1 \star Y_3 &= 0; \\ Y_1 \star Y_2 + Y_2 \star Y_3 &= 0; \\ Y_2 \star Y_3 + Y_1 \star Y_3 &= 0. \end{cases}$$

Solving the system, we obtain $Y_1 \star Y_2 = 0, Y_1 \star Y_3 = 0, Y_2 \star Y_3 = 0$. Thus, $X_i X_j + X_j X_i = 0$. The rest follows easily. $\square$

§3. Non-commutative case.

The main goal of this section is to show that a T-amorphous scheme should be commutative. With this end, we study irreducible complex representations of the adjacency algebra $\mathcal{A} := \mathbb{C}[\mathcal{R}]$. We write $\mathcal{A}_+$ and $\mathcal{A}_-$ for subspaces of $\mathcal{A}$ consisting of symmetric and skew-symmetric matrices, respectively. We also abbreviate $v := |\Omega|$, $J := J_\Omega, I := I_\Omega$.

Let $E_0 = v^{-1}J, E_1, \ldots, E_r$ be a complete set of minimal central idempotents of $\mathcal{A}$. It is well-known that a two-sided ideal $\mathcal{A}_i := E_i\mathcal{A}$ is isomorphic (as an algebra) to the full matrix algebra $M_{n_i}(\mathbb{C})$. This isomorphism yields an irreducible complex representation of $\mathcal{A}$ which will be denoted by Δ_i. We denote by n_i the dimension of Δ_i. The multiplicity of Δ_i in the decomposition of the standard module $\mathbb{C}^\Omega$ will be denoted by m_i. Since the scheme is odd, all multiplicities $m_i, 0 \leq i \leq r$, are odd integers. Notice that $\Delta_i : \mathcal{A} \to M_{n_i}(\mathbb{C})$ is a $\mathbb{C}$-algebra epimorphism such that $\Delta_i(E_i) = I_{n_i}$. According to [4] we may assume that $\Delta_i(X^*) = \Delta_i(X)^*$ for each $X \in \mathcal{A}$ (here and later on $X^* := \overline{X}^\top$ is the Hermitian conjugate of a matrix X).

Notice that as a matrix of $\mathcal{M}_\Omega(\mathbb{C})$, each E_i is Hermitian, that is $E_i^* = E_i$. A complex matrix $\overline{E}_i = E_i^\top$ is also a minimal central idempotent of $\mathcal{A}$. Thus, $E_i^\top = E_{i'}$ for some i'. Notice that the mapping $i \mapsto i'$ is an involution and $0' = 0$. The representation $\Delta_{i'}$ corresponding to $E_{i'}$ is equivalent to $X \mapsto \Delta_i(X^\top)^\top$.

In order to proceed further, we introduce additional notation. We write $\widetilde{X}$ for the $2n \times 2n$ real matrix which is obtained from $n \times n$ complex matrix X by replacing a complex number $X_{ij} = a + b\iota, \iota = \sqrt{-1}$, by the 2×2-matrix $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Notice that the mapping $X \mapsto \widetilde{X}$ is an $\mathbb{R}$-algebra monomorphism between the algebras $M_n(\mathbb{C})$ and $M_{2n}(\mathcal{R})$, and $\widetilde{X^*} = \widetilde{X}^\top$.

The following result of J. Putter (Theorem 1, [13]) plays a central role in this section.

Theorem 3.1. *Let $X_1, \dots, X_q$ be a set of $n \times n$ skew symmetric, pairwise anticommuting (that is $X_i X_j = -X_j X_i$ for $i \neq j$) non-zero real matrices. Then $q \leq n - 1$.*

Proposition 3.2. *If $i' \neq i$, then $n_i = 1$. If $n_i = 1$ and $i \neq 0$, then $i' \neq i$.*

Proof. First we prove a general fact that $\Delta_i(\mathcal{A}_-) = \Delta_i(\mathcal{A})$ (and, therefore, $\Delta_i(\mathcal{A}_-) = M_{n_i}(\mathbb{C})$). Since Δ_i maps $E_i\mathcal{A}$ onto $M_{n_i}(\mathbb{C})$ bijectively, it suffices to show that $E_i\mathcal{A} = E_i\mathcal{A}_-$. Let $Y \in E_i\mathcal{A}$ be an arbitrary matrix. Then $Y = YE_i$ implies $Y^\top = Y^\top E_{i'}$. Together with $E_i E_{i'} = 0$ we conclude that

$$E_i(Y - Y^\top) = E_i Y - E_i Y^\top E_{i'} = E_i Y = Y.$$

Thus, $E_i\mathcal{A} = E_i\mathcal{A}_-$. Since the matrices $X_1, \dots, X_e$ form a basis of $\mathcal{A}_-$, the matrices $E_i X_j, j = 1, \dots, e$, span $E_i\mathcal{A}$. Therefore, the matrices $Y_j := \Delta_i(X_j), j = 1, \dots, e$, span $M_{n_i}(\mathbb{C})$. In particular, the number of non-zero matrices among Y_j's is at least n_i^2. Since $X_j^* = -X_j$, we conclude that $Y_j^* = -Y_j$. Also, the matrices Y_j are pairwise anti-commuting. Now, the matrices $\widetilde{Y}_j, j = 1, \dots, e$, form a set of $2n_i \times 2n_i$ real skew symmetric matrices which anti-commute pairwise. By Theorem 3.1 the number of non-zero matrices among $\widetilde{Y}_j$ is at most $2n_i - 1$. On the other hand, among these matrices at least n_i^2 matrices are non-zero. Thus, $n_i^2 \leq 2n_i - 1$ implying $n_i = 1$.

Next, assume that $n_i = 1, i \neq 0$. Then Δ_i is one-dimensional and, therefore, $\Delta_{i'}(A_j) = \Delta_i(A_j^\top)^\top = \Delta_i(A_j^\top)$. If $i' = i$, then $\Delta_i(A_j) = \Delta_{i'}(A_j) = \Delta_i(A_j^\top)$ implying that $\Delta_i(X_j) = 0$ for each $j = 1, \dots, e$. But it follows from (1) that $\sum_j \Delta_i(X_j)^2 = -v$. A contradiction. Thus $i' \neq i$. $\square$

Finally, we consider the remaining case $i' = i$, that is $E_i^\top = \overline{E}_i = E_i$. In this case $\mathcal{A}_i^\top = \mathcal{A}_i$ and the mapping $X \mapsto X^\top$ is an order two anti-automorphism of $\mathcal{A}_i$. Since $\mathcal{A}_i$ is isomorphic to $M_{n_i}(\mathbb{C})$, the above mapping yields an order two anti-automorphism of $M_{n_i}(\mathbb{C})$. Each anti-automorphism of the full matrix algebra has a form $A \mapsto S^{-1}A^\top S$ where S is either symmetric or a skew-symmetric matrix [2]. Thus, $\Delta_i(X^\top) = S^{-1}\Delta_i(X)^\top S$.

If S is symmetric, then there exists a unitary matrix U and diagonal real matrix D with $D_{ii} > 0$ such that $S = UDU^\top$ (Lemma 4.4.4 [5]). Replacing $\Delta_i(X)$ by $\sqrt{D}U^*\Delta_i(X)U\sqrt{D}^{-1}$ we obtain an equivalent representation which satisfies $\Delta_i(X^\top) = \Delta_i(X)^\top$ and $\Delta_i(X^*) = \Delta_i(X)^*$ for each $X \in \mathcal{A}$.

In particular, $\Delta_i(A_j)$ is a real matrix for each $j \in [0, 2e]$ (that is Δ_i is irreducible over $\mathbb{R}$). In this case Δ_i is called a real representation.

If S is skew symmetric, then n_i is even and there exists a unitary matrix U such that $S = UVU^\top$ where V is a direct sum of $n_i/2$ matrices of the form $\begin{pmatrix} 0 & z_\ell \\ -z_\ell & 0 \end{pmatrix}, z_\ell \in \mathbb{C}$, $\ell = 1, \ldots, n_i/2$ [5, Section 4.4, Problem 26]. Therefore, there exists an orthonormal basis of $\mathbb{C}^{n_i}$ such that in the new basis $\Delta_i(X^\top) = V^{-1}\Delta_i(X)^\top V$. In this case we call Δ_i a representation of a *symplectic* type.

Proposition 3.3. *If Δ_i is real, then $n_i = 2$. There exists a unique $j \in \{1, \ldots, e\}$ with $\Delta_i(X_j) \neq 0$. Moreover, $\Delta_i(X_j) = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$, where $\lambda = \pm\sqrt{v}$.*

Proof. It follows from $X_j^\top = -X_j$ that $\Delta_i(X_j)^\top = -\Delta_i(X_j)$. Thus, $\Delta_i(X_j), j = 1, \ldots, e$ are skew symmetric, real and pairwise anticommuting matrices of order n_i. By Theorem 3.1 the number of non-zero matrices among them is at most $n_i - 1$.

Since X_j's form a basis of $\mathcal{A}_-$, the number of non-zero matrices among $\Delta_i(X_j)$ is at least $\dim(\Delta_i(\mathcal{A}_-))$. It follows from $\Delta_i(X^\top) = \Delta_i(X)^\top$ that $\Delta_i(\mathcal{A}_-)$ coincides with the subspace of skew symmetric matrices of $M_{n_i}(\mathbb{C})$. Therefore $\dim(\Delta_i(\mathcal{A}_-)) = (n_i - 1)n_i/2$ implying $(n_i - 1)n_i/2 \leq n_i$. Hence $n_i \leq 2$. By Proposition 3.2 $n_i \geq 2$. Therefore $n_i = 2$.

The subspace of skew symmetric matrices of $M_2(\mathbb{C})$ is one dimensional and is spanned by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Since $\Delta_i(X_j)$'s are anticommuting, we conclude that $\Delta_i(X_j)$ is non-zero for at most one matrix X_j. The rest of the statement follows from the equality $\sum_{j=1}^{e} X_j^2 = -vI_\Omega + J_\Omega$. $\square$

Proposition 3.4. *If Δ_i is symplectic, then $n_i = 2$. There exists at most three indices $j \in [1, e]$ such that $\Delta_i(X_j) \neq 0$. If there are exactly three indices $a, b, c \in [1, e]$ with non-zero images $\Delta_i(X_j), j \in \{a, b, c\}$, then, up to a conjugation by a unitary matrix, the matrices $\Delta_i(X_a), \Delta_i(X_b), \Delta_i(X_c)$ have the form*

$$\begin{pmatrix} \iota\theta_a & 0 \\ 0 & -\iota\theta_a \end{pmatrix}, \begin{pmatrix} 0 & \theta_b \\ -\theta_b & 0 \end{pmatrix}, \begin{pmatrix} 0 & \iota\theta_c \\ \iota\theta_c & 0 \end{pmatrix},$$

where $\theta_a, \theta_b, \theta_c$ are real numbers satisfying $\theta_a^2 + \theta_b^2 + \theta_c^2 = v$.

Proof. According to [14] the dimension of $\Delta_i(\mathcal{A}_-)$ is $n_i(n_i+1)/2$. Therefore, among the matrices $Y_j := \Delta_i(X_j), j = 1, \dots, e$ there are at least $n_i(n_i+1)/2$ non-zero matrices. Since these matrices are skew Hermitian and pairwise anti-commute, the matrices $\widetilde{Y}_j$ are pairwise anti-commuting skew-symmetric real matrices of order $2n_i$. By Theorem 3.1 the number of non-zero matrices among $\widetilde{Y}_j$, $1 \le j \le e$, is at most $2n_i - 1$. Thus, $n_i(n_i+1)/2 \le 2n_i - 1 \implies n_i \le 2$. Since n_i is even, we conlcude that $n_i = 2$. Thus $V = \begin{pmatrix} 0 & z \\ -z & 0 \end{pmatrix}$ for some $z \in \mathbb{C}$ and

$$\Delta_i(X^\top) = V^{-1}\Delta_i(X)^\top V = T^{-1}\Delta_i(X)^\top T \quad \text{where} \quad T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It follows from the above arguments that the number of non-zero matrices among $Y_j, j \in [1,e]$, is at most three. Next, assume that there are exactly three non-zero matrices Y_a, Y_b, Y_c with distinct $a, b, c \in [1,e]$. It follows from $X_j^\top = X_j^* = -X_j$ that $T^{-1}Y_j^\top T = Y_j^* = -Y_j$. Now a direct computation shows that each of the matrices Y_a, Y_b, Y_c has a form

$$Y_j = \begin{pmatrix} \iota r_j & z_j \\ -\overline{z_j} & -\iota r_j \end{pmatrix}, \quad \text{where} \quad r_j \in \mathbb{R}, z_j \in \mathbb{C} \quad \text{and} \quad j \in \{a,b,c\}.$$

The characteristic polynomial of Y_j is $x^2 + r_j^2 + |z_j|^2$. Since $Y_j \neq 0$, the number $r_j^2 + |z_j|^2$ is a positive real. Therefore, the eigenvalues of Y_j are $\pm\iota\theta_j$, where $\theta_j = \sqrt{r_j^2 + |z_j|^2}$. Since each of the matrices is skew Hermitian, it has an orthonormal eigenbasis. Therefore, there exists a unitary matrix U such that $U^*Y_aU = \begin{pmatrix} \iota\theta_a & 0 \\ 0 & -\iota\theta_a \end{pmatrix}$. The matrices U^*Y_bU, U^*Y_cU are skew Hermitian matrices with zero trace. Hence

$$U^*Y_bU = \begin{pmatrix} \iota s_b & w_b \\ -\overline{w_b} & -\iota s_b \end{pmatrix}, \qquad U^*Y_cU = \begin{pmatrix} \iota s_c & w_c \\ -\overline{w_c} & -\iota s_c \end{pmatrix},$$

$$\text{where } s_b, s_c \in \mathbb{R}, \quad w_b, w_c \in \mathbb{C}.$$

The conditions $Y_aY_b = -Y_bY_a, Y_aY_c = -Y_cY_a$ imply $s_b = s_c = 0$. Conjugating by a unitary matrix $D := \begin{pmatrix} e^{\iota\omega_b} & 0 \\ 0 & 1 \end{pmatrix}$ with $\omega_b = \arg(w_b)$ we obtain the following triple of matrices

$$(UD)^*Y_a(UD) = \begin{pmatrix} \iota\theta_a & 0 \\ 0 & -\iota\theta_a \end{pmatrix},$$
$$(UD)^*Y_b(UD) = \begin{pmatrix} 0 & \theta_b \\ -\theta_b & 0 \end{pmatrix},$$
$$(UD)^*Y_c(UD) = \begin{pmatrix} 0 & u \\ -\overline{u} & 0 \end{pmatrix}.$$

Now the condition $Y_bY_c = -Y_cY_b$ implies that u is an imaginary number. Thus $(UD)^*Y_c(UD) = \begin{pmatrix} 0 & \iota\theta_c \\ \iota\theta_c & 0 \end{pmatrix}$. The condition $\theta_a^2 + \theta_b^2 + \theta_c^2 = v$ follows from $\sum_{j=1}^{e} X_j^2 = -vI_\Omega + J_\Omega$. □

Notice that the matrices described in the above proposition with $\theta_a = \theta_b = \theta_c = 1$ generate a quaternion group of order 8. For this reason we call these type of representations of *quaternion type*.

It follows from Propositions 3.2, 3.3, and 3.4 that the set of all non-principal irreducible representations of $\mathcal{A}$ splits into three parts

(1) one-dimensional non-real representations $\Delta_1, \Delta_{1'}, \ldots, \Delta_k, \Delta_{k'}$;
(2) two-dimensional real representations $\Delta_{2k+1}, \ldots, \Delta_{2k+\ell}$;
(3) two-dimensional non-real representations of quaternion type $\Delta_{2k+\ell+1}, \ldots \Delta_{2k+\ell+m}$.

For the representation of the first type we have $\Delta_i(X^\top) = \Delta_{i'}(X)^\top$, $X \in \mathcal{A}$. For the representation of the second type $\Delta_i(X^\top) = \Delta_i(X)^\top$, and for the representations of the third type $\Delta_i(X^\top) = T^{-1}\Delta_i(X)^\top T$, where $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

As a complex algebra $\mathcal{A}$ is isomorphic to the direct sum

$$\mathbb{C}^{2k+1} \oplus M_2(\mathbb{C})^{\ell+m}.$$

Counting the dimension of $\mathcal{A}$ in two ways we obtain $1+2k+4\ell+4m = 1+2e$, implying that $k+2\ell+2m = e$. The anti-automorphism $^\top$ acts on the direct sum $\mathbb{C}^{2k+1} \oplus M_2(\mathbb{C})^{\ell+m}$ as follows:

(1) it interchanges the coordinates i and i' for $1 \le i \le k$;
(2) it acts as a transposition on the first ℓ summands $M_2(\mathbb{C})$;
(3) it acts as $X \mapsto T^{-1}X^\top T$ on the last m summands $M_2(\mathbb{C})$.

Also, $^\top$ acts trivially on the zero-indexed summand (which corresponds to Δ_0).

The dimension of $^\top$-invariant subspace of $\mathcal{A}$ counted in two ways yields us the equality $1+e = 1+k+3\ell+m$. Comparing this with $e = k+2\ell+2m$ we conclude that $\ell = m$.

Proposition 3.5. *For each $j \in \{1,\dots,e\}$ there exists a unique $r(j) \in \{1,\dots,k,2k+1,\dots,2k+2\ell\}$ such that $\Delta_{r(j)}(X_j) \neq 0$. The function r has the following properties*

(1) *r is surjective;*
(2) *for each $i \in \{1,\dots,k,2k+1,\dots,2k+2\ell\}$ one has*

$$|r^{-1}(i)| = \begin{cases} 1, & i \leq 2k+\ell, \\ 3, & i > 2k+\ell; \end{cases}$$

(3) *for each $i \in \{1,\dots,k,2k+1,\dots,2k+2\ell\}$ one has*

$$\sum_{j \in r^{-1}(i)} \Delta_i(X_j)^2 = -vI_{n_i}.$$

Proof. Define a bipartite graph Γ between the sets $D := \{1,\dots,k,2k+1,\dots,2k+2\ell\}$ and $E := \{1,..,e\}$ by connecting $i \in D$ and $j \in E$ iff $\Delta_i(X_j) \neq 0$. Since the intersection of $\ker(\Delta_i), i \in D \cup \{0\}$ is trivial, for each $j \in E$ there exists at least one $i \in D \cup \{0\}$ with $\Delta_i(X_j) \neq 0$. Since $\Delta_0(X_j) = 0$, one has $i \neq 0$. Thus, each $j \in E$ is connected with at least one element of D. Therefore, Γ has at least $|E| = e$ distinct edges.

It follows from $\sum_{j=1}^{e} X_j^2 = -vI + J$ that $\sum_{j=1}^{e} \Delta_i(X_j)^2 = -vI_{n_i}$. Therefore, for each $i \in D$ there exists at least one $j \in E$ with $\Delta_i(X_j) \neq 0$. In other words, each vertex $i \in D$ has at least one neighbour in E.

Since X_j's are anti-commuting, if Δ_i is one-dimensional there is at most one $j \in E$ with $\Delta_i(X_j) \neq 0$. Hence, $|\Gamma(i)| = 1$ for $i \in \{1,\dots,k\} \subseteq D$.

If Δ_i is 2-dimensional and real, then by Proposition 3.3 there exists at most one $j \in E$ with $\Delta_i(X_j) \neq 0$. Therefore, $|\Gamma(i)| = 1$ for each $i \in \{2k+1,\dots,2k+\ell\} \subset D$.

If Δ_i is 2-dimesional of quaternion type, then by Proposition 3.4 we have $|\Gamma(i)| \leq 3$. Thus,

$$k+\ell+3\ell \geq \sum_{i \in D} |\Gamma(i)| = |E(\Gamma)| = \sum_{j \in E} |\Gamma(j)| \geq |E| = e = k+4\ell.$$

Thus, we obtain that $|E(\Gamma)| = e$ and $|\Gamma(i)| = 3$ for each $i \in \{2k+\ell+1,\dots,2k+2\ell\} \subset D$. This means that each $j \in E$ is connected with exactly one $i \in D$. Thus, we obtain a function $r : E \to D$ such that $i \in D$ is connected with $j \in E$ iff $i = r(j)$.

Part (3) is an immediate consequence of $\sum\limits_{j=1}^{e} X_j^2 = -vI + J$. □

Now we are ready to prove the main result of this section

Theorem 3.6. *A T-amorphic scheme should be commutative.*

Proof. Assume, towards a contradiction, that the scheme is non-commutative, that is $\ell \geq 1$. Then $\mathcal{A}$ has at least one two-dimensional non-real representation of quaternion type, say Δ_q. Let E_q denote the minimal central idempotent correponding to Δ_q. Since $\Delta_q(X^\top) = T^{-1}\Delta_q(X)^\top T$, we conclude that $\Delta_q(X)$ is a scalar matrix whenever X is symmetric. By Proposition 3.5 $r^{-1}(q) = \{a,b,c\}$ for pairwise distinct indices a,b,c. By Proposition 3.4 we may assume that

$$\Delta_q(X_a) = \begin{pmatrix} \theta_a \iota & 0 \\ 0 & -\theta_a \iota \end{pmatrix}, \quad \Delta_q(X_b) = \begin{pmatrix} 0 & \theta_b \\ -\theta_b & 0 \end{pmatrix},$$
$$\Delta_q(X_c) = \begin{pmatrix} 0 & \iota\theta_c \\ \iota\theta_c & 0 \end{pmatrix}$$

for some $\theta_a, \theta_b, \theta_c \in \mathbb{R}$. Squaring these matrices we obtain

$$\Delta_q(X_a^2) = \begin{pmatrix} -\theta_a^2 & 0 \\ 0 & -\theta_a^2 \end{pmatrix}, \quad \Delta_q(X_b^2) = \begin{pmatrix} -\theta_b^2 & 0 \\ 0 & -\theta_b^2 \end{pmatrix},$$
$$\Delta_q(X_c^2) = \begin{pmatrix} -\theta_c^2 & 0 \\ 0 & -\theta_c^2 \end{pmatrix}.$$

Since $\Delta_p(X_a) = \Delta_p(X_b) = \Delta_p(X_c) = \Delta_p(E_q) = 0$ for each $p \neq q$, we obtain

$$X_a^2 = -\theta_a^2 E_q,\ X_b^2 = -\theta_b^2 E_q,\ X_c^2 = -\theta_c^2 E_q.$$

Taking traces we obtain

$$-2v_a v = -2m_q\theta_a^2,\ -2v_b v = -2m_q\theta_b^2,\ -2v_c v = -2m_q\theta_c^2,$$

implying that

$$\theta_a = \pm\sqrt{\frac{v_a v}{m_q}},\ \theta_b = \pm\sqrt{\frac{v_b v}{m_q}},\ \theta_c = \pm\sqrt{\frac{v_c v}{m_q}}.$$

It follows from $\Delta_q(X_a)\Delta_q(X_b) = \frac{\theta_a\theta_b}{\theta_c}\Delta_q(X_c)$ that $X_aX_b = \frac{\theta_a\theta_b}{\theta_c}X_c$. Since X_a, X_b, X_c are $\{0, 1, -1\}$ matrices, the number $\frac{\theta_a\theta_b}{\theta_c}$ is an integer. It follows from

$$\frac{\theta_a^2\theta_b^2}{\theta_c^2} = \frac{v_av_bv}{m_qv_c}$$

that $\frac{\theta_a\theta_b}{\theta_c}$ is an odd integer. We can conclude that $X_iX_j \equiv X_k(\text{mod } 2)$.

Finally, consider the product $\Delta_q(A_a + A_{a'})\Delta_q(X_b)$. Since $A_a + A_{a'}$ is symmetric, we conclude that $\Delta_q(A_a + A_{a'})$ is a scalar matrix λI_2 for some λ. Therefore,

$$\Delta_q((A_a + A_{a'})X_b) = \Delta_q(A_a + A_{a'})\Delta_q(X_b) = \lambda\Delta_q(X_b).$$

Since $\Delta_p(X_b) = 0$ for each $p \neq q$, we conclude $(A_a + A_{a'})X_b = \lambda X_b$. Clearly, λ is an integer. It follows from $X_a \equiv (A_a + A_{a'})(\text{mod } 2)$ that $X_aX_b \equiv \lambda X_b(\text{mod } 2)$. Therefore, $X_c \equiv \lambda X_b(\text{mod } 2)$, contrary to the fact that X_a, X_b, X_c are linearly independent over any field. $\square$

§4. Commutative case.

Here it is assumed that the scheme is commutative. In this case (1) reads as follows

$$\forall_{i\neq j}\ X_iX_j = 0 \ \text{ and } \ \sum_{i=1}^{e} X_i^2 = -vI + J. \tag{2}$$

In the commutative case the set of primitive idempotents has the following form $E_0, E_1, E_{1'}, \ldots, E_e, E_{e'}$, where $E_i^\top = E_{i'}, i = 1, \ldots, e$. Notice that in this case the direct sum $\mathcal{A} = \mathcal{A}_+ + \mathcal{A}_-$ is a $\mathbb{Z}_2$-graded decomposition, that is

$$\mathcal{A}_+\mathcal{A}_+ \subseteq \mathcal{A}_+,\ \mathcal{A}_+\mathcal{A}_- \subseteq \mathcal{A}_-,\ \mathcal{A}_-\mathcal{A}_- \subseteq \mathcal{A}_+.$$

In this section we use the following abbreviations

$$A_i^+ := A_i + A_{i'},\ A_i^- := A_i - A_{i'},\ E_i^+ := E_i + E_{i'},\ E_i^- := E_i - E_{i'}.$$

Notice that the matrices $A_0, A_1^+, \ldots, A_e^+$ $(E_0, E_1^+, \ldots, E_e^+)$ form the first (resp. the second) standard basis of $\mathcal{A}_+$ which is the BM-algebra of a symmetrized scheme $\mathcal{R}^+$. The character table and the structure constants of $\mathcal{A}^+$ are denoted by P^+ and $\overset{+}{p}{}_{ij}^{k}$. Also, each of the sets $\{A_i^-\}_{i\in[1,e]}$ and $\{E_i^-\}_{i\in[1,e]}$ is a basis of $\mathcal{A}_-$. It follows from Proposition 3.5 that the mapping $r : [1, e] \to [1, e]$ is a bijection. Renumbering idempotents we can always assume that $r(i) = i, i \in [1, e]$. Thus, for each i the matrix A_i^- is proportional to E_i^-, that is $A_i^- = \lambda_i E_i^-$ for some $\lambda_i \in \mathbb{C}$. It follows from $\sum_{i=1}^{e}(A_i^-)^2 = -vI + J$ that

$$-vI + J = \sum_{i=1}^{e} \lambda_i^2 (E_i^-)^2 = \sum_{i=1}^{e} \lambda_i^2 (E_i + E_{i'})$$
$$\Longrightarrow \quad -v\sum_{i=1}^{e}(E_i + E_{i'}) = \sum_{i=1}^{e} \lambda_i^2 (E_i + E_{i'}).$$

Therefore, $\lambda_i = \pm\iota\sqrt{v}$. Exchanging E_i with $E_{i'}$, if necessary, we can always assume that $A_i^- = \iota\sqrt{v}E_i^-$ for $i = 1, \ldots, e$. As a direct consequense of this fact we obtain the following statement.

Proposition 4.1. *Let P and Q be the first and the second eigenmatrices of the scheme (Ω, R). Then for any pair of indices $i \neq j \in [1, e]$ the following conditions hold*

(1) $P_{ji} = P_{j'i} = P_{ji'} = P_{j'i'} = \frac{1}{2}P_{ji}^+$;

(2) $\begin{cases} P_{ii} = P_{i'i'} = \frac{P_{ii}^+ + \iota\sqrt{v}}{2}, \\ P_{ii'} = P_{i'i} = \frac{P_{ii}^+ - \iota\sqrt{v}}{2}; \end{cases}$

(3) $A_i A_j^- = P_{ji} A_j^-$ *and* $P_{ji} = p_{ij}^j - p_{ij'}^j$;

(4) $A_i^+ A_i^- = P_{ii}^+ A_i^-$ *and* $P_{ii}^+ = p_{ii}^i - p_{i'i'}^i$ *is odd;*

(5) $Q_{ij} = Q_{i'j} = Q_{ij'} = Q_{i'j'} = p_{jj'}^i - p_{j'j'}^i$;

(6) $Q_{ii} = Q_{i'i'} = P_{ii'}, Q_{i'i} = Q_{ii'} = P_{ii}$.

Proof. Substracting $A_{i'} = v_i E_0 + \sum_{j=1}^{e}(P_{ji'}E_j + P_{j'i'}E_{j'})$ from $A_i = v_i E_0 + \sum_{j=1}^{e}(P_{ji}E_j + P_{j'i}E_{j'})$ we obtain that $P_{ji} = P_{ji'}, P_{j'i'} = P_{j'i}$ for $i \neq j$ and $P_{ii} - P_{ii'} = P_{i'i} - P_{i'i'} = \iota\sqrt{v}$. Combining this with $P_{ji} = P_{j'i'}, P_{ij'} = P_{i'j}$ and $P_{ji}^+ = P_{ji} + P_{ji'}, P_{ii}^+ = P_{ii} + P_{ii'}$ we obtain parts (1)–(2) of the statement.

Part (3). It follows from $A_j^- = \iota\sqrt{v}E_j^-$ that $A_i A_j^- = A_i \iota\sqrt{v}(E_j - E_{j'}) = \iota\sqrt{v}(P_{ji}E_i - P_{j'i}E_{j'})$. By part (1) $P_{ji} = P_{j'i}$. Thus, $A_i A_j^- = P_{ji}A_j^-$. Counting the coefficients of A_j in both sides of $A_i A_j^- = P_{ji}A_j^-$, we obtain that $P_{ji} = p_{ij}^j - p_{ij'}^j$.

PART (4). It follows from $E_i^+ A_i^- = A_i^-$ that

$$A_i^+ A_i^- = A_i^+(E_i^+ A_i^-) = (A_i^+ E_i^+) A_i^- = P_{ii}^+ E_i^+ A_i^- = P_{ii}^+ A_i^-.$$

Thus, $A_i^2 - A_{i'}^2 = P_{ii}^+(A_i - A_{i'})$. Comparing the coefficients of A_i in both sides, we conclude that $P_{ii}^+ = p_{ii}^i - p_{i'i'}^i$. Rewriting $A_i^2 - A_{i'}^2 = P_{ii}^+(A_i - A_{i'})$ as $A_i^2 - P_{ii}^+ A_i = A_{i'}^2 - P_{ii}^+ A_{i'}$, we obtain $(A_i^2 - P_{ii}^+ A_i)^\top = A_i^2 - P_{ii}^+ A_i$. Therefore, $A_i^2 - P_{ii}^+ A_i = \sum_{j=1}^{e} \lambda_j A_j^+$ for some integers λ_j. The valency of the right-hand side part is an even number. Therefore, $v_i^2 - P_{ii}^+ v_i$ is an even integer. Since v_i is odd, we conclude that P_{ii}^+ is odd.

PARTS (5) AND (6) follow from the well-known relations $Q_{ij}/m_j = \overline{P_{ji}}/v_i$. □

Let $\tau_i, i \in [1,e]$, denote a permutation of the set $\{0\} \cup [1,e] \cup [1',e']$ which interchanges i and i' and leaves the rest of the elements fixed (that is τ_i is a transposition $(i\,i')$). It follows from parts (1)–(2) of Proposition 4.1 that $P_{uv} = P_{u^{\tau_i} v^{\tau_i}}$ holds for any pair $u, v \in \{0\} \cup [1,e] \cup [1'e']$. Therefore, τ_i is an algebraic automorphism of $\mathcal{A}$. The transpositions $\tau_i, i \in [1,e]$, generate an elementary abelian 2-group T of order 2^e. This group acts regularly on the set of all rank 3 antisymmetric fusions of $\mathfrak{X}$.

Proposition 4.2. *$m_i = v_i$ for each $i \in [1,e]$.*

Proof. Squaring the equality $A_i - A_{i'} = \iota\sqrt{v}(E_i - E_{i'})$ we obtain

$$A_i^2 - 2A_i A_{i'} + A_{i'}^2 = -v(E_i + E_{i'}).$$

Now by taking the traces of both sides we obtain the result. □

It follows from the previous statement that the entries P_{ij}^+ are even if $i \neq j$ and odd otherwise.

Proposition 4.3. *If $i \neq j$, then $P_{ij}^+ + P_{ji}^+ \equiv 2 \pmod 4$ and $p_{ii}^j \equiv 0 \pmod 2$.*

Proof. Let M denote the diagonal matrix of size e such that $M_{ii} = m_i = v_i$. Since the principal part P_0^+ is congruent to I_e modulo two, we can write $P_0^+ = I_e + 2P_1$ for some integer matrix P_1. The character table P^+ of the scheme $\mathcal{R}^+$ has the form $\begin{pmatrix} 1 & 2\cdot\mathbf{1}M \\ \mathbf{1}^\top & I_e + 2P_1 \end{pmatrix}$, where $\mathbf{1}$ is the all-one row vector of length e. Let us write the orthogonality relations for P^+ in the matrix form

$$\begin{pmatrix} 1 & \mathbf{1} \\ 2M\mathbf{1}^\top & I_e + 2P_1^\top \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2M \end{pmatrix} \begin{pmatrix} 1 & 2\cdot\mathbf{1}M \\ \mathbf{1}^\top & I_e + 2P_1 \end{pmatrix} = v \begin{pmatrix} 1 & 0 \\ 0 & 2M \end{pmatrix}.$$

Comparing the entries of both sides in the position $(2,2)$, we obtain that

$$4\cdot MJ_eM + 2\cdot M + 4\cdot MP_1 + 4\cdot P_1^\top M + 8\cdot P_1^\top MP_1 = 2v\cdot M,$$

or, equivalently,

$$MJ_eM + MP_1 + P_1^\top M + 2\cdot P_1^\top MP_1 = \frac{v-1}{2}\cdot M.$$

Since $\frac{v-1}{2}$ is an odd integer and $M\equiv I_e(\mathrm{mod}\ 2)$, we obtain

$$J_e + P_1 + P_1^\top \equiv I_e(\mathrm{mod}\ 2).$$

This yields us the first congruence of the statement.

According to the well-known formula we have

$$v_jvp_{ii}^j = v_i^2v_j + \sum_{s=1}^{e}(P_{si}^2P_{sj'} + P_{s'i}^2P_{s'j'})m_s.$$

If $s\neq i,j$, then by Proposition 4.1 $P_{sj'} = P_{s'j'}$ implying that

$$v_jvp_{ii}^j \equiv v_i^2v_j + (P_{ii}^2P_{ij'} + P_{i'i}^2P_{i'j'})m_i + (P_{ji}^2P_{jj'} + P_{j'i}^2P_{jj'})m_j(\mathrm{mod}\ 2).$$

Since the numbers $m_i = v_i, m_j = v_j, v$ are odd and $P_{ij'} = P_{i'j'}, P_{ji} = P_{ji'}$, we obtain

$$p_{ii}^j \equiv 1 + (P_{ii}^2 + P_{i'i}^2)P_{ij} + P_{ji}^2(P_{jj'} + P_{jj'})(\mathrm{mod}\ 2).$$

By part (2) of Proposition 4.1, $P_{ii}^2 + P_{ii'}^2 = \frac{(P_{ii}^+)^2 - v}{2}$, $P_{jj} + P_{jj'} = P_{jj}^+$. Since P_{ii}^+, P_{jj}^+ are odd and $v\equiv 3(\mathrm{mod}\ 4)$, the numbers $P_{ii}^2 + P_{ii'}^2 = \frac{(P_{ii}^+)^2 - v}{2}$, $P_{jj} + P_{jj'} = P_{jj}^+$ are odd. Therefore,

$$p_{ii}^j \equiv 1 + P_{ij} + P_{ji}^2(\mathrm{mod}\ 2) \equiv 1 + P_{ij} + P_{ji}(\mathrm{mod}\ 2).$$

Together with $P_{ij} = \frac{P_{ij}^+}{2}, P_{ji} = \frac{P_{ji}^+}{2}$ and $P_{ij}^+ + P_{ji}^+ \equiv 2(\mathrm{mod}\ 4)$ we conclude that $P_{ij} + P_{ji}$ is an odd number. This yields us $p_{ii}^j \equiv 0(\mathrm{mod}\ 2)$. □

Corollary 4.4. *The mapping $A_i^+ \mapsto E_i^+$ is not a duality of $\mathcal{R}^+$.*

Proof. A mapping $A_i^+ \mapsto E_i^+$ is a duality if and only if $\frac{P_{ij}^+}{v_j} = \frac{P_{ji}^+}{v_i}$ holds for any pair of indices. But for $i\neq j$ both entries P_{ij}^+, P_{ji}^+ are even numbers whose sum is congruent to 2 modulo 4. Since v_i and v_j are odd numbers, this implies that the highest 2-powers which divide $\frac{P_{ij}^+}{v_j}, \frac{P_{ji}^+}{v_i}$ are distinct for $i\neq j$. □

Corollary 4.5. *If* $v \equiv 7 \pmod 8$ *then* $A_i^2 \equiv A_i \pmod 2$ *for each* $i \in [1,e]$. *If* $v \equiv 3 \pmod 8$ *then* $A_i^2 \equiv A_{i'} \pmod 2$ *for each* $i \in [1,e]$.

Proof. Pick an arbitrary $j \in [1,e], j \neq i$. By Proposition 4.3, p_{ii}^j is even. Since $p_{ii}^{j'} = p_{i^{\tau_j} i^{\tau_j}}^{j^{\tau_j}} = p_{ii}^j$, we obtain that $p_{ii}^{j'} \equiv 0 \pmod 2$. Therefore, $A_i^2 \equiv p_{ii}^i A_i + p_{ii}^{i'} A_{i'} \pmod 2$. This also implies that $v_i^2 \equiv p_{ii}^i v_i + p_{ii}^{i'} v_{i'} \pmod 2$. Since v_i is odd, we conclude that $p_{ii}^i + p_{ii}^{i'} \equiv 1 \pmod 2$. Thus, for any $i \in [1,e]$ either $A_i^2 \equiv A_i \pmod 2$ or $A_i^2 \equiv A_{i'} \pmod 2$.

The matrix $A := A_1 + \cdots + A_e$ is the matrix of a doubly regular tournament of order v. Therefore, $A^2 = \frac{v-3}{4} A + \frac{v+1}{4} A^\top$. If $v \equiv 7 \pmod 8$, then $A^2 \equiv A \pmod 2$ implying that $\sum_{i=1}^{e} A_i^2 \equiv \sum_{i=1}^{e} A_i \pmod 2$. Thus, $A_i^2 \equiv A_i \pmod 2$ for each $i \in [1,e]$. The case of $v \equiv 3 \pmod 8$ is considered similarly. □

§5. Fusions in $\mathfrak{X}$ and $\mathfrak{X}^+$

Let $\mathcal{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_f\}$ be a partition of $[1,e]$ with f non-empty classes. Let D be an $e \times f$ matrix defined as follows: $D_{ij} = 1$ if $i \in \mathcal{P}_j$ and $D_{ij} = 0$, otherwise. We define the *dual* partition $\mathcal{P}^\perp$ of $[1,e]$ by putting $i,j \in [1,e]$ at the same class iff the i-th and j-th rows of the matrix $P_0^+ D$ are equal. It is well-known that $\mathcal{P}$ gives rise to a fusion scheme iff $|\mathcal{P}^\perp| = |\mathcal{P}|$, that is $\mathcal{P}$ and $\mathcal{P}^\perp$ have the same number of classes. Since $P_0^+ \equiv I_e \pmod 2$, we obtain $(P_0^+ D) \equiv D \pmod 2$. Therefore, $\mathcal{P}^\perp$ is a refinement of $\mathcal{P}$. This immediately implies the following

Proposition 5.1. *A partition* $\mathcal{P}$ *gives rise to a fusion scheme iff* $\mathcal{P}^\perp = \mathcal{P}$.

Every partition $\mathcal{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_f\}$ of $[1,e]$ determines a partition $\mathcal{P}' := \{\mathcal{P}'_1, \ldots, \mathcal{P}'_f\}$ of $[1', e']$. The union $\mathcal{P} \cup \mathcal{P}'$ is an admissible partition of $[1,e] \cup [1', e']$.

Proposition 5.2. *A partition* $\mathcal{P}$ *of* $[1,e]$ *gives rise to a fusion of* $\mathcal{R}^+$ *iff the partition* $\mathcal{P} \cup \mathcal{P}'$ *gives rise to a fusion of* $\mathcal{R}$. *If* $\mathcal{R}/(\mathcal{P} \cup \mathcal{P}')$ *is a scheme, then it is T-amorphous.*

Proof. If $\mathcal{P} \cup \mathcal{P}'$ determines a fusion scheme of $\mathcal{R}$, then this scheme is anti-symmetric and its symmetrization is a fusion of $\mathcal{R}^+$ determined by $\mathcal{P}$.

Conversly, assume that $\mathcal{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_f\}$ is a partition of $[1,e]$ which determines a fusion scheme of $\mathcal{R}$. Since $\mathcal{P}^\perp = \mathcal{P}$, the sets $\{A_0\} \cup \{A_{\mathcal{P}_k}^+\}_{k=1}^f$ and $\{E_0\} \cup \{E_{\mathcal{P}_k}^+\}_{k=1}^f$ form the first and the second standard bases of the fusion scheme $\mathcal{R}^+/\mathcal{P}$. Let $\mathcal{B}$ denote the linear span of the matrices $A_0, A_{\mathcal{P}_k}, A_{\mathcal{P}'_k}, k \in [1, f]$. It suffices to show that the linear spans $\langle A_0, \{A_{\mathcal{P}_k}, A_{\mathcal{P}'_k}\}_{k\in[1,f]}\rangle$ and $\langle E_0, \{E_{\mathcal{P}_k}, E_{\mathcal{P}'_k}\}_{k\in[1,f]}\rangle$ are equal. It follows from

$$A_{\mathcal{P}_k} = \frac{1}{2}\left(A^+_{\mathcal{P}_k} + A^-_{\mathcal{P}_k}\right), A_{\mathcal{P}'_k} = \frac{1}{2}\left(A^+_{\mathcal{P}_k} - A^-_{\mathcal{P}_k}\right)$$

and

$$E_{\mathcal{P}_k} = \frac{1}{2}\left(E^+_{\mathcal{P}_k} + E^-_{\mathcal{P}_k}\right), E_{\mathcal{P}'_k} = \frac{1}{2}\left(E^+_{\mathcal{P}_k} - E^-_{\mathcal{P}_k}\right),$$

that

$$\langle A_0, \{A_{\mathcal{P}_k}, A_{\mathcal{P}'_k}\}_{k\in[1,f]}\rangle = \langle A_0, \{A^+_{\mathcal{P}_k}, A^-_{\mathcal{P}_k}\}_{k\in[1,f]}\rangle$$

and

$$\langle E_0, \{E_{\mathcal{P}_k}, E_{\mathcal{P}'_k}\}_{k\in[1,f]}\rangle = \langle E_0, \{E^+_{\mathcal{P}_k}, E^-_{\mathcal{P}_k}\}_{k\in[1,f]}\rangle.$$

Now the claim follows from

$$\langle A_0, \{A^+_{\mathcal{P}_k}\}_{k\in[1,f]}\rangle = \langle E_0, \{E^+_{\mathcal{P}_k}\}_{k\in[1,f]}\rangle$$

and

$$A^-_{\mathcal{P}_k} = \iota\sqrt{v}E^-_{\mathcal{P}_k}, k = 1, \ldots, f. \qquad \square$$

If $\mathcal{Q}$ is a partition of $[1, e] \cup [1', e']$ which determines a fusion, then its symmetrization $\mathcal{Q}^+ := \{(Q \cup Q') \cap [1, e] \mid Q \in \mathcal{Q}\}$ determines a fusion of $\mathcal{R}^+$. This gives us a homomorphism which maps the lattice of fusion schemes of $\mathcal{R}$ into the one of $\mathcal{R}^+$. It turns out that the lattice of fusions of $\mathcal{R}$ may be reconstructed from the lattice of fusions of $\mathcal{R}^+$.

Proposition 5.3. *Let $\mathcal{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_f\}$ be a partition of $[1, e]$ which determines a fusion scheme of $\mathcal{R}^+$. Then any admissible partition $\mathcal{Q}$ of $[1, e] \cup [1', e']$ with $\mathcal{Q}^+ = \mathcal{P}$ determines a fusion of $\mathcal{R}$.*

Proof. Let $\mathcal{Q}$ be an admissible partition of $[1, e] \cup [1', e']$ with $g \geq 0$ pairs of anti-symmetric classes, that is $\mathcal{Q} = \{\mathcal{Q}_1, \ldots, \mathcal{Q}_g, \mathcal{Q}'_1, \ldots, \mathcal{Q}'_g, \mathcal{Q}_{g+1}, \ldots, \mathcal{Q}_f\}$.

It follows from $\mathcal{Q}^+ = \mathcal{P}$ that (after renumbering) $\mathcal{Q}^+_i = \mathcal{P}_i, i = 1, \ldots, f$. For any algebraic automorphism $\tau \in T$ a partition $\mathcal{Q}^\tau$ is also admissible. Moreover, $\mathcal{Q}$ determines a fusion of $\mathcal{R}$ iff $\mathcal{Q}^\tau$ does. Since there always exists $\tau \in T$ with $\bigcup_{i=1}^{g} \mathcal{Q}^\tau_i \subseteq [1, e]$, we may assume, without loosing generality, that $\bigcup_{i=1}^{g} \mathcal{Q}_i \subseteq [1, e]$. In this case $\mathcal{Q}_i = \mathcal{P}_i, i = 1, \ldots, g$, and $\mathcal{Q}_j = \mathcal{P}_j \cup \mathcal{P}'_j, j = g+1, \ldots, f$. By Proposition 5.2 the partition $\mathcal{P} \cup \mathcal{P}'$ determines a fusion of

$\mathcal{R}$. Therefore, a coarsening of $\mathcal{P} \cup \mathcal{P}'$ by any subgroup $S \leq T$ determines a fusion of $\mathcal{R}$. Now it remains to notice that $\mathcal{Q}$ is a coarsening of $\mathcal{P} \cup \mathcal{P}'$ with respect to a subgroup generated by the elements $\sigma_i := \prod_{j \in \mathcal{P}_i} \tau_j, i = g+1, \ldots, f$. □

Corollary 5.4. *If a partition $\mathcal{P}$ of $[1, e]$ determines a fusion of $\mathcal{R}^+$, then $|\mathcal{P}|$ is odd. In particular, both $\mathcal{R}$ and $\mathcal{R}^+$ are primitive schemes.*

Proof. By Proposition 5.2 $\mathcal{P} \cup \mathcal{P}'$ determines a fusion of $\mathcal{R}$ which is T-amorphous. By Corollary 2.2 $|\mathcal{P}|$ is an odd number. Since $\mathcal{R}^+$ is a symmetrization of $\mathcal{R}$, they are primitive or imprimitive simultaneously. Since imprimitive schemes always have a fusion with two classes, we conclude that $\mathcal{R}$ is primitive. □

Proposition 5.5. $\frac{\sqrt{v+1}}{2} < v_i$ *for each* $i \neq 0$.

Proof. Since $\mathcal{R}$ is a primitive scheme, the eigenvalues P_{ij} satisfy the inequality $|P_{ij}| < v_j$. Taking $i = j$ we obtain $|P_{ii}|^2 < v_i^2$. By Proposition 4.1 we obtain $(P_{ii}^+)^2 + v < 4v_i^2$. The rest follows from $P_{ii}^+ \equiv 1 (\mathrm{mod}\ 2)$. □

Question. It follows from Proposition 5.5 that $d < 2\sqrt{v}$. Does there exist an infinite series of T-amorphous schemes with $d > \alpha(\sqrt{v})$, where a constant α is independent of v.

§6. Cyclotomic T-amorphous schemes.

The main goal of this section is to give necessary and sufficient conditions for a cyclotomic scheme to be T-amorphous. Let $\mathbb{F}_q, q = p^f$, be a finite field and d be a divisor of $q-1$. For a generator $g \in \mathbb{F}_q^*$ we define the d-th cyclotomic classes as follows $C_i := \langle g^d \rangle g^i$, where $i = 0, 1, \ldots, d-1$. The basic relations of the cyclotomic scheme are $R_0 := I_{\mathbb{F}_q}$ and $R_{i+1} := \{(x,y) \in \mathbb{F}_q^2 \,|\, x - y \in C_i\}, i = 0, 1, \ldots, d-1$. One of the important properties of cyclotomic schemes is that a cyclic permutation $i \mapsto i+1 (\mathrm{mod}\ d)$ is an algebraic automorphism of the scheme. Let us call a scheme $\mathfrak{X} = (\Omega, \mathcal{R} = \{R_0, R_1, \ldots, R_d\})$ *pseudo-cyclotomic* if it admits a cyclic group H of algebraic automorphisms which acts regularly on the set $\{1, \ldots, d\}$. Since H is transitive on $[1, d]$, the scheme $\mathfrak{X}$ is either symmetric or anti-symmetric. In the latter case d is even and the following statement holds

Proposition 6.1. *Let $\mathfrak{X}$ be antisymmetric. Then $i^z = i'$ where $z \in H$ is a unique element of order* 2.

Proof. Let $z \in H$ be a unique element which maps 1 to $1'$, that is $1^z = 1'$. Since $(i')^h = (i^h)'$ holds for each $h \in H$ and $i \in [1,d]$, we obtain that $i^z = i'$ holds for each $i \in [1,d]$. This implies that $z^2 = 1$. □

According to [7] the group H acts regularly on the set of primitive idempotents $E_1, \ldots, E_d$. For each irreducible character $\chi \in \mathsf{Irr}(H)$ we define the eigenspace $\mathcal{A}_\chi$ of H as follows $\mathcal{A}_\chi := \{A \in \mathcal{A} \mid A^g = \chi(g)A\}$. Notice that $\mathcal{A}_\chi$ is one-dimensional if χ is non-principal and two-dimensional if $\chi = 1_H$ (the principal character of H). The subspace A_{1_H} is a BM-algebra of a trivial scheme with the standard bases $\{I, J-I\}$. An algebra $\mathcal{A}$ is a direct sum of the subspaces $\mathcal{A}_\chi, \chi \in \mathsf{Irr}(H)$. Since H is a group of algebraic automorphisms, the decomposition $\mathcal{A} = \bigoplus_{\chi \in \mathsf{Irr}(H)} \mathcal{A}_\chi$ is a grading of $\mathcal{A}$ with respect to both multiplications $\cdot$ and $\circ$, that is

$$\begin{array}{rcl} \mathcal{A}_\chi \cdot \mathcal{A}_\psi & \subseteq & \mathcal{A}_{\chi\psi}, \\ \mathcal{A}_\chi \circ \mathcal{A}_\psi & \subseteq & \mathcal{A}_{\chi\psi}. \end{array} \tag{3}$$

For each $i \in [1,d]$ and $\chi \in \mathsf{Irr}(H)$ we set $A_{i,\chi} := \sum\limits_{h \in H} \chi(h^{-1}) A_{i^h}$ and $E_{i,\chi} := \sum\limits_{h \in H} \chi(h^{-1}) E_{i^h}$. A direct check shows that for each $h \in H$ one has

$$\begin{array}{l} (A_{i,\chi})^h = A_{i^h,\chi} = \chi(h) A_{i,\chi}, \\ (E_{i,\chi})^h = E_{i^h,\chi} = \chi(h) E_{i,\chi}. \end{array} \tag{4}$$

If $\chi \neq 1_H$, then both $E_{i,\chi}$ and $A_{j,\chi}$ span $\mathcal{A}_\chi$. Therefore, they are proportional, that is $A_{j,\chi} = \lambda E_{i,\chi}$ for some complex number λ. Changing j and i leads to a multiplication of λ by a complex d-th root of unity. In a cyclotomic scheme this coefficient (under certain choice of i and j) is known as Gauss sum. For this reason we denote this coefficient as $G(\chi, i, j)$ and call it Gauss sum as well. Thus,

$$A_{i,\chi} = G(\chi, i, j) E_{j,\chi}. \tag{5}$$

Theorem 6.2. *Let $\mathfrak{X}$ be an antisymmetic pseudo-cyclotomic scheme with $d = 2e$ classes, e odd, and with $H = \langle h_0 \rangle, h_0 = (1, \ldots, e, 1', \ldots, e')$. Then $\mathfrak{X}$ is T-amorphous iff there exists $j \in [1,e] \cup [1',e']$ and $\varepsilon \in \{\pm 1\}$ such that for any character $\chi \in \mathsf{Irr}(H)$ of even order $G(\chi, 1, j) = \varepsilon \iota \sqrt{v}$.*

Proof. Necessity. In this case $A_1 - A_{1'} = \iota\sqrt{v}(E_j - E_{j'})$ for some $j \in [1,e] \cup [1',e']$. Therefore, for each $h \in H$ one has

$$A_{1^h} - A_{(1^h)'} = \iota\sqrt{v}(E_{j^h} - E_{(j^h)'}) \iff A_{1^h} - A_{1^{zh}} = \iota\sqrt{v}(E_{j^h} - E_{j^{hz}}),$$
$$\text{where} \quad z := h_0^e.$$

Next, let $\chi \in \mathsf{Irr}(H)$ be a character of even order. Then $\chi(z) = -1$ and

$$\begin{aligned} A_{1,\chi} &= \sum_{h\in H} \chi(h) A_{1^h} = \sum_{h\in O} \chi(h)(A_{1^h} - A_{1^{hz}}) \\ &= \iota\sqrt{v} \sum_{h\in O} \chi(h)(E_{j^h} - E_{j^{hz}}) = \iota\sqrt{v} E_{j,\chi}, \end{aligned}$$

where O is a unique subgroup of H of order e.

Sufficiency. Let $\chi \in \mathsf{Irr}(H)$ be a character of even order. Then $A_{1,\chi} = \varepsilon\iota\sqrt{v}E_{j,\chi}$ or, equivalently,

$$\begin{aligned} &\sum_{h\in O} \chi(h)(A_{1^h} - A_{1^{hz}}) = \varepsilon\iota\sqrt{v} \sum_{h\in O} \chi(h)(E_{j^h} - E_{j^{hz}}) \\ \iff &\sum_{h\in O} \chi(h)(A_{1^h} - A_{(1^h)'}) = \varepsilon\iota\sqrt{v} \sum_{h\in O} \chi(h)(E_{j^h} - E_{(j^h)'}). \end{aligned}$$

Since the restriction map $\chi \mapsto \chi_O$ is a bijection between irreducible H-characters of even order and $\mathsf{Irr}(O)$, we conclude that the equality

$$\sum_{h\in O} \psi(h)(A_{i^h} - A_{(i^h)'}) = \varepsilon\iota\sqrt{v} \sum_{h\in O} \psi(h)(E_{j^h} - E_{(j^h)'})$$

holds for each $\psi \in \mathsf{Irr}(O)$. Since the matrix $(\psi(h))_{\psi\in\mathsf{Irr}(O),h\in O}$ is invertible, we obtain that $A_{i^h} - A_{(i^h)'} = \varepsilon\iota\sqrt{v}(E_{j^h} - E_{(j^h)'})$ holds for each $h \in O$. Now the claim follows from the fact that $\{i^h\}_{h\in O}$ is an $'$-transversal. □

Corollary 6.3. *Let $\mathfrak{X}$ be a cyclotomic scheme with $d = 2e, e$ odd, classes over a finite field $\mathbb{F}_q, q = p^f$. Assume that $q \equiv 3 (\mathrm{mod}\ 4)$. Then $\mathfrak{X}$ is T-amorphous if and only if there exists $\varepsilon \in \{\pm 1\}$ such that $G(\chi) = \varepsilon\iota\sqrt{q}$ holds for all multiplicative characters χ of $\mathbb{F}_q^*$ of order $2e_1, e_1 \mid e$.*

Proof. The Gauss sum of the cyclotomic scheme $G(\chi)$ coincides with $G(\chi, 1, 1)$. Sufficiency follows directly from Theorem 6.2. Thus, we have to prove necessity. First, we show that $\gcd(d, p-1) = 2$. Consider the fusion $\mathfrak{X}'$ of $\mathfrak{X}$ with respect to the subgroup of H of index $\gcd(d, p-1)$. This is an anti-symmetric fusion with $d_0 := \gcd(d, p-1)$ classes. Thus, $\mathfrak{X}'$ is a cyclotomic T-amorphous scheme with $d_0 = 2e_0$ classes. By Theorem 6.2

there exists $j \in [1, d_0]$ and $\varepsilon \in \{\pm 1\}$ such that $G(\chi, 1, j) = \varepsilon\iota\sqrt{q}$ for each multiplicative character of order d_0. Since $G(\chi) = G(\chi, 1, 1)$ and $G(\chi, 1, j)$ differs by a factor which is a complex d_0-th root of unity, we conclude that $G(\chi)/\sqrt{q}$ is a complex root of unity. By [12] for each $a \in [0, d_0 - 1]$ coprime to d_0 one has $\sum_{i=0}^{m-1} (p^i a)_{d_0} = md_0/2 = me_0$, where[2] $m = \mathrm{ord}_{d_0}(p)$ and x_{d_0} is the remainder of x modulo d_0. Since d_0 divides $p-1$, we obtain that $(p^i a)_{d_0} = a$ for each $a \in [0, d_0 - 1]$. Therefore, $a = e_0$ for each $a \in [0, d_0 - 1]$ coprime to d_0. It follows that $\mathbb{Z}_{d_0}^* = \{e_0\}$ and thus $d_0 = 2$.

Let π denote an algebraic automorphism of $\mathfrak{X}$ induced by the Frobenius automorphism of the field. Since $\gcd(d, p-1) = 2$, the Frobenius automorphism fixes only two cyclotomic classes C_0 and C_e. Therefore, in its action on the first (second) standard basis of $\mathfrak{X}$ the automorphism π fixes only two elements A_1 and $A_{1'} = A_e$ (E_1 and $E_{1'} = E_e$ respectively).

By Theorem 6.2 there exist $\varepsilon \in \{\pm 1\}$ and $j \in [1, d]$ such that $G(\chi, 1, j) = \varepsilon\iota\sqrt{q}$ for each character χ of $\mathbb{F}_q^*$ of order $2e_1 \mid d$. We have to show that $j = 1$. It follows from the proof of Theorem 6.2 that $A_1 - A_{1'} = \varepsilon\iota\sqrt{v}(E_j - E_{j'})$ Applying π to both sides we conclude that $j^\pi = j$. Therefore, $j \in \{1, e\}$. If $j = e$, then $A_{1,\chi} = G(\chi, 1, e)E_{e,\chi} = \varepsilon\iota\sqrt{q}E_{e,\chi} = \varepsilon\iota\sqrt{q}E_{1^z,\chi} = \varepsilon\iota\sqrt{q}\chi(z)E_{1,\chi} = -\varepsilon\iota\sqrt{q}E_{1,\chi}$ implying $G(\chi, 1, 1) = -\varepsilon\iota\sqrt{v}$. □

As a direct consequence of Davenport–Hasse Theorem we obtain the following

Corollary 6.4. *If a cyclotomic scheme over a field $\mathbb{F}_q$ with d classes is T-amorphous, then for any odd m a d-class cyclotomic scheme over $\mathbb{F}_{q^m}$ is also T-amorphous.*

Another corollary follows immediately from the proof of Corollary 6.3.

Corollary 6.5. *If a cyclotomic scheme over $\mathbb{F}_q$ with d classes is T-amorphous, then* $\gcd(p-1, d) = 2$.

This fact implies that the number d of classes in a T-amorphous cyclotomic scheme is at least 14. Indeed, if $d < 14$, then $d = 6, 10$. Since $\mathrm{ord}_d(p)$ is odd and $\mathbb{Z}_6^*, \mathbb{Z}_{10}^*$ are 2-groups, we conclude that in any of these two cases $p \equiv 1 (\mathrm{mod}\ d)$ which contradicts Corollary 6.5. Cyclotomic T-amorphous schemes with 14 classes do exist – the smallest example found by Feng and Xiang has 14 classes over 11^3 points.

[2]Here the sum is in $\mathbb{Z}$.

Problem. Find a critetion for a cyclotomic scheme to be T-amorphous in terms of p, f and d (without Gauss sums) like it is done for amorphic cyclotomic schemes.

§7. T-amorphous schemes with 6 classes

By Corollary 2.2 a minimal number of classes of a T-amorphous scheme is 6. An example of such a scheme occurs when one fuses classes of a cyclotomic scheme with 14 classes on 11^3 points (see [11]) by the Frobenius automorphism. In this section we make an attempt to find the character table of such a scheme. Let $\mathfrak{X} = (\Omega, \{R_0, R_1, R_2, R_3, R_{1'}, R_{2'}, R_{3'}\})$ be a T-amorphous scheme. As before,

$$A_0, A_1, A_2, A_3, A_{1'}, A_{2'}, A_{3'} \text{ and } E_0, E_1, E_2, E_3, E_{1'}, E_{2'}, E_{3'}$$

stand for the first and the second standard bases of the BM-algebra $\mathcal{A}$ of $\mathfrak{X}$, respectively. The elements of the standard bases are numbered in such a way that $A_i - A_{i'} = \iota\sqrt{v}(E_i - E_{i'})$ holds for $i = 1, 2, 3$. The symmetrized scheme $\mathfrak{X}^+$ has three classes $R_i^+ = R_i \cup R_{i'}, i = 1, 2, 3$, and its standard bases are A_0, A_1^+, A_2^+, A_3^+, E_0, E_1^+, E_2^+, E_3^+. Its valencies and multiplicities are $m_i^+ = v_i^+ = 2v_i, i = 1, 2, 3$. Following Hanaki [3] we call a symmetric scheme a *B-scheme* if there exists an ordering (called a *B-ordering*) of its standard bases such that $v_i = m_i$. There are two natural classes of B-schemes – those which are (formally) self-dual and pseudocyclic schemes. But these classes do not exhaust all possible examples of B-schemes. Arjana Žitnik pointed out that there exists a feasible parameter set of a diameter three distance regular graph whose scheme is a B-scheme. So, it seems reasonable to consider this class of schemes in general.

7.1. B-schemes. We start with the following

Proposition 7.1. *Let $\mathfrak{X}$ be a B-scheme with three classes with B-ordering A_0, A_1, A_2, A_3 and E_0, E_1, E_2, E_3 of its standard bases. Let C be its cosine matrix. Then either C is symmetric (and the mapping $E_i \mapsto A_i$ is a duality) or there exist real numbers a, t such that*

$$a^2 v + t^2 v_1 v_2 v_3 = (a-2)^2 \tag{6}$$

and

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -\frac{a(v_2+v_3)+2}{2v_1} & \frac{a+v_3t}{2} & \frac{a-v_2t}{2} \\ 1 & \frac{a-v_3t}{2} & -\frac{a(v_1+v_3)+2}{2v_2} & \frac{a+v_1t}{2} \\ 1 & \frac{a+v_2t}{2} & \frac{a-v_1t}{2} & -\frac{a(v_1+v_2)+2}{2v_3} \end{pmatrix}. \tag{7}$$

Proof. By orthogonality relations for the first row and column we have

$$\begin{aligned} 1 + v_1C_{11} + v_2C_{12} + v_3C_{13} = 0; \\ 1 + v_1C_{11} + v_2C_{21} + v_3C_{31} = 0 \end{aligned} \tag{8}$$

and

$$\begin{aligned} 1 + v_1C_{11}^2 + v_2C_{12}^2 + v_3C_{13}^2 = \frac{v}{v_1}; \\ 1 + v_1C_{11}^2 + v_2C_{21}^2 + v_3C_{31}^2 = \frac{v}{v_1}. \end{aligned} \tag{9}$$

Therefore,

$$\begin{aligned} v_2(C_{12} - C_{21}) = v_3(C_{31} - C_{13}); \\ v_2(C_{12}^2 - C_{21}^2) = v_3(C_{31}^2 - C_{13}^2) \end{aligned} \tag{10}$$

implying that either $C_{12} = C_{21}, C_{13} = C_{31}$ or $C_{12} + C_{21} = C_{13} + C_{31}$. Writing similar relations for the second and the third row/column we conclude that either C is symmetric or $C_{ij} + C_{ji}$ is constant. Let us denote this constant by a. Then

$$C_{12} + C_{21} = C_{13} + C_{31} = C_{23} + C_{32} = a.$$

It follows from

$$\begin{aligned} v_2(C_{12} - C_{21}) &= v_3(C_{31} - C_{13}), \\ v_1(C_{12} - C_{21}) &= v_3(C_{23} - C_{32}), \\ v_2(C_{23} - C_{32}) &= v_1(C_{31} - C_{13}) \end{aligned}$$

that

$$\frac{C_{12} - C_{21}}{v_3} = \frac{C_{23} - C_{32}}{v_1} = \frac{C_{31} - C_{13}}{v_2}.$$

Denoting this common value by t we obtain that

$$\begin{aligned} C_{12} - C_{21} &= v_3t, & C_{12} + C_{21} &= a, \\ C_{23} - C_{32} &= v_1t, & C_{23} + C_{32} &= a, \\ C_{31} - C_{13} &= v_2t, & C_{31} + C_{13} &= a, \end{aligned}$$

implying

$$C_{12} = \frac{a+v_3t}{2}, \qquad C_{21} = \frac{a-v_3t}{2},$$
$$C_{23} = \frac{a+v_1t}{2}, \qquad C_{32} = \frac{a-v_1t}{2},$$
$$C_{31} = \frac{a+v_2t}{2}, \qquad C_{13} = \frac{a-v_2t}{2}.$$

Adding the equations in (8) we obtain $2+2v_1C_{11}+a(v_2+v_3) = 0$. Therefore, $C_{11} = -\frac{a(v_2+v_3)+2}{2v_1}$. Analogously, $C_{22} = -\frac{a(v_1+v_3)+2}{2v_1}$, $C_{33} = -\frac{a(v_1+v_2)+2}{2v_3}$. Thus, the cosine matrix has the following form

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -\frac{a(v_2+v_3)+2}{2v_1} & \frac{a+v_3t}{2} & \frac{a-v_2t}{2} \\ 1 & \frac{a-v_3t}{2} & -\frac{a(v_1+v_3)+2}{2v_2} & \frac{a+v_1t}{2} \\ 1 & \frac{a+v_2t}{2} & \frac{a-v_1t}{2} & -\frac{a(v_1+v_2)+2}{2v_3} \end{pmatrix}.$$

Substituting the above expressions for C_{1i} into (9) we obtain equation (6). □

Corollary 7.2. *A B-scheme with three classes is self-dual if and only if C is symmetric or at least two of the valencies v_1, v_2, v_3 are equal.*

Proof. Assume that C is non-symmetric. If the scheme is self-dual, then the duality permutation ψ on $1,2,3$ is non-identical. W.l.o.g. $\psi(1) \neq 1$. Then $v_{\psi(1)} = m_{\psi(1)} = v_1$.

Now, assume that some of v_i's are equal, say $v_1 = v_2$. Then $\psi = (12)$ is a duality permutation. □

Proposition 7.3. *Let F be the decomposition field of $\mathfrak{X}$. If $F \neq \mathbb{Q}$, then $\mathsf{Gal}(F/\mathbb{Q}) \cong \mathbb{Z}_3$ and $\mathfrak{X}$ is pseudocyclic.*

Proof. The Galois group $\mathsf{Gal}(F/\mathbb{Q})$ acts faithfully on the set of primitive idempotents E_1, E_2, E_3 of the scheme. Assume that $\mathsf{Gal}(F/\mathbb{Q})$ contains an automorphism σ which acts on E_1, E_2, E_3 as a transposition, say $E_1^\sigma = E_2, E_2^\sigma = E_1, E_3^\sigma = E_3$. This implies that

$$\left(\frac{a-v_2t}{2}\right)^\sigma = \frac{a+v_1t}{2}, \quad \left(\frac{a+v_1t}{2}\right)^\sigma = \frac{a-v_2t}{2}.$$

Together with $v_1 = v_2$ we obtain that $a \in \mathbb{Q}$. From $E_3^\sigma = E_3$ it follows that $C_{31}^\sigma = C_{31}$, or, equivalently, $\left(\frac{a+v_2t}{2}\right)^\sigma = \frac{a+v_2t}{2}$. Hence $t \in \mathbb{Q}$ and $F = \mathbb{Q}$. A contradiction.

Thus, the group $\mathsf{Gal}(F/\mathbb{Q})$ is either trivial or acts regularly on E_1, E_2, E_3. In the latter case the scheme is pseudocyclic. $\square$

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Поступило 12 января 2017 г.