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CORRELATION FUNCTIONS OF REAL ZEROS OF RANDOM POLYNOMIALS

Abstract. We give an explicit formula for the correlation functions of real zeros of a random polynomial with arbitrary independent continuously distributed coefficients.

§1. Introduction

Let $\xi_0, \xi_1, \dots, \xi_n$ be independent random variables with probability density functions $f_0, \dots, f_n$. Consider a random polynomial

$$G(x) = \xi_n x^n + \xi_{n-1} x^{n-1} + \dots + \xi_1 x + \xi_0, \quad x \in \mathbb{R}^1.$$

With probability one, all zeros of G are simple. Denote by μ the empirical measure counting the real zeros of G:

$$\mu = \sum_{x : G(x)=0} \delta_x,$$

where δ_x is the unit point mass at x. The distribution of μ can be described by its *correlation functions* (also known as *joint intensities*). Recall (see, e.g., [6]) that the correlation functions of μ are functions (if well-defined) $\rho_k : \mathbb{R}^k \to \mathbb{R}^+$ for $k = 1, \dots, n$, such that for any family of mutually disjoint Borel subsets $B_1, \dots, B_k \subset \mathbb{R}^1$,

$$\mathbf{E}\Big[\prod_{i=1}^{k} \mu(B_i)\Big] = \int\limits_{B_1} \dots \int\limits_{B_k} \rho_k(x_1, \dots, x_k)\, dx_1 \dots dx_k.$$

Key words and phrases: Random polynomial, correlation between zeros, joint intensities, Coarea formula.

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A standard tool for evaluating ρ_k is the following extension of the Kac–Rice formula (see [1, 2]):

$$\rho_k(x_1, \dots, x_k) = \int\limits_{\mathbb{R}^k} |t_1 \dots t_k| D_k(\mathbf{0}, \mathbf{t}, x_1, \dots, x_k)\, dt_1 \dots dt_k, \tag{1}$$

where $\mathbf{t} = (t_1, \dots, t_k)$ and $D_k(\,\cdot\,, \cdot\,, x_1, \dots, x_k)$ is the joint probability density function of the random vectors

$$(G(x_1), \dots, G(x_k)) \quad \text{and} \quad (G'(x_1), \dots, G'(x_k)).$$

The goal of this paper is to provide more explicit expressions for $\rho_k(\mathbf{x})$. The main tool that we will use is the *Coarea formula* (see Lemma 5.2).

Our methods can be applied to the case of dependent coefficients having arbitrary joint probability density function. For simplicity, we consider only the case of independent coefficients.

§2. Main result

Let us start with some notation. Denote

$$\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}^k.$$

We use the following notation for the elementary symmetric polynomials:

$$\sigma_i(\mathbf{x}) := \begin{cases} \sum\limits_{1 \le j_1 < \dots < j_i \le k} x_{j_1} x_{j_2} \dots x_{j_i} & \text{if} \quad 0 \le i \le k, \\ 0, & \text{otherwise.} \end{cases}$$

Denote by $V(\mathbf{x})$ the Vandermonde matrix

$$V(\mathbf{x}) = \begin{pmatrix} 1 & x_1 & \dots & x_1^{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_k & \dots & x_k^{k-1} \end{pmatrix}.$$

To formulate the first result, consider the random function $\eta = (\eta_0, \dots, \eta_{k-1})^T : \mathbb{R}^k \to \mathbb{R}^k$ defined as

$$\eta(\mathbf{x}) = -V^{-1}(\mathbf{x}) \begin{pmatrix} \sum\limits_{j=k}^{n} \xi_j x_1^j \\ \vdots \\ \sum\limits_{j=k}^{n} \xi_j x_k^j \end{pmatrix}. \tag{2}$$

Theorem 2.1. *We have*

$$\rho_k(\mathbf{x}) = \prod_{1\le i<j\le k} |x_i - x_j|^{-1} \times \mathbf{E}\left[\prod_{i=1}^{k}\left|\sum_{j=0}^{k-1} j\eta_j(\mathbf{x})x_i^{j-1} + \sum_{j=k}^{n} j\xi_j x_i^{j-1}\right|\prod_{i=0}^{k-1} f_i(\eta_i(\mathbf{x}))\right]. \tag{3}$$

Theorem 2.1 has been stated in [11] without detailed proof.

It is possible to obtain an explicit expression for $\eta(\mathbf{x})$ in terms of the *Schur functions*. Recall that for a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of length $\le k$ the Schur function $S_\lambda(\mathbf{x})$ is given by

$$S_\lambda(\mathbf{x}) = \frac{\det(x_i^{\lambda_{k-j+1}+j-1})_{1\le i,j\le k}}{\prod\limits_{1\le i<j\le k}(x_j - x_i)}.$$

Proposition 2.2. *For $i = 1, \dots, k$, we have*

$$\eta_i(\mathbf{x}) = (-1)^{k-i}\sum_{j=k}^{n} \xi_j S_{\lambda_{ij}}(\mathbf{x}),$$

where the partition λ_{ij} is defined as

$$\lambda_{ij} = (j-k+1, \underbrace{1,\dots,1}_{k-i-1}, \underbrace{0,\dots,0}_{i}).$$

For the basic properties of the Schur functions see, e.g., [9, Section 1.3].

Now we are ready to state our main result.

Theorem 2.3. *We have*

$$\rho_k(\mathbf{x}) = \prod_{1\le i<j\le k} |x_i - x_j| \times \int\limits_{\mathbb{R}^{n-k+1}} \prod_{i=0}^{n} f_i\Big(\sum_{j=0}^{n-k}(-1)^{k-i+j}\sigma_{k-i+j}(\mathbf{x})t_j\Big)\prod_{i=1}^{k}\left|\sum_{j=0}^{n-k} t_j x_i^j\right| dt_0\dots dt_{n-k}.$$

Corollary 2.4. *For $k = n$ we have*

$$\rho_n(\mathbf{x}) = \prod_{1\le i<j\le n} |x_i - x_j| \int_{-\infty}^{\infty} |t|^n \prod_{i=0}^{n} f_i\Big((-1)^{n-i}\sigma_{n-i}(\mathbf{x})t\Big)\, dt. \tag{4}$$

§3. Uniformly distributed coefficients

In algebraic number theory, random polynomials with independent and uniformly distributed on $[-1,1]$ coefficients are of special interest (see [4, 5, 7]). Let us apply Theorem 2.3 to this case.

Suppose that

$$f_i = \frac{1}{2}\mathbb{1}[-1,1], \quad i = 0,\dots,n.$$

Then it follows from Theorem 2.3 that

$$\rho_k(\mathbf{x}) = 2^{-n-1} \prod_{1\le i<j\le k} |x_i - x_j| \int\limits_{D_{\mathbf{x}}} \prod_{i=1}^{k} \left| \sum_{j=0}^{n-k} t_j x_i^j \right| dt_0 \dots dt_{n-k},$$

where the domain of integration $D_{\mathbf{x}}$ is defined as

$$D_{\mathbf{x}} = \left\{ (t_0,\dots,t_{n-k}) \in \mathbb{R}^{n-k+1} \ : \ \max_{0\le i\le n} \left| \sum_{j=0}^{n-k} (-1)^{i-j} \sigma_{i-j}(\mathbf{x}) t_j \right| \le 1 \right\}.$$

In particular,

$$\rho_n(\mathbf{x}) = \frac{2^{-n}}{n+1} \cdot \frac{\prod\limits_{1\le i<j\le k} |x_i - x_j|}{(\max_{0\le i\le n} |\sigma_i(\mathbf{x})|)^{n+1}}.$$

§4. The n-point correlation function

It follows from the properties of the correlation functions (see, e.g., [6]) that

$$\mathbf{E}\,\left[\mu(\mathbb{R}^1)(\mu(\mathbb{R}^1)-1)\dots(\mu(\mathbb{R}^1)-n+1)\right] = \int\limits_{\mathbb{R}^n} \rho_n(\mathbf{x})\,d\mathbf{x}.$$

Since $\mu(\mathbb{R}^1) \le n$, we can calculate the probability that all zeros of G are real:

$$\begin{aligned}\mathbf{P}(\mu(\mathbb{R}^1) = n) &= \frac{1}{n!} \int\limits_{\mathbb{R}^n} \rho_n(\mathbf{x})\,d\mathbf{x} \\ &= \frac{1}{n!} \int\limits_{\mathbb{R}^n} \prod_{1\le i<j\le n} |x_i - x_j| \int\limits_{-\infty}^{\infty} |t|^n \prod_{i=0}^{n} f_i\Big((-1)^{n-i}\sigma_{n-i}(\mathbf{x})t\Big)\,dt\,d\mathbf{x}.\end{aligned}$$

This formula has been obtained earlier in [10].

Let us calculate ρ_n for some specific distributions.

4.1. Gaussian distribution. Suppose that

$$f_i(t) = \frac{1}{\sqrt{2\pi} v_i} \exp\left(-\frac{t^2}{2v_i^2}\right), \quad i = 0, \dots, n.$$

Using the formula for the n-th absolute moment of the Gaussian distribution, it follows from (4) that

$$\rho_n(\mathbf{x}) = \frac{\sqrt{2}\Gamma\left(\frac{n+1}{2}\right)}{(2\pi)^{n/2} v_0 \dots v_n} \left(\sum_{i=0}^{n} \frac{\sigma_{n-i}^2(\mathbf{x})}{v_i^2}\right)^{-(n+1)/2} \prod_{1 \le i < j \le n} |x_i - x_j|.$$

In particular, for $v_i = v_j$ we have

$$\rho_n(\mathbf{x}) = \frac{\sqrt{2}\Gamma\left(\frac{n+1}{2}\right)}{(2\pi)^{n/2}} \left(\sum_{i=0}^{n} \sigma_{n-i}^2(\mathbf{x})\right)^{-(n+1)/2} \prod_{1 \le i < j \le n} |x_i - x_j|.$$

4.2. Exponential distribution. Suppose that

$$f_i(t) = \exp(-t)\mathbb{1}\{t \ge 0\}, \quad i = 0, \dots, n.$$

Then with probability one G does not have positive real zeros. Hence $\rho_n(\mathbf{x}) > 0$ only if $\mathbf{x}$ lies in the negative orthant $\mathbb{R}_-^k$. In this case we have $(-1)^i \sigma_i(\mathbf{x}) \ge 0$ and by some elementary transformations, (4) implies

$$\rho_n(\mathbf{x}) = n! \left(\sum_{i=0}^{n} (-1)^i \sigma_i(\mathbf{x})\right)^{-(n+1)} \prod_{1 \le i < j \le n} |x_i - x_j| \, \mathbb{1}\{\mathbf{x} \in \mathbb{R}_-^k\}.$$

Using the well-known identity

$$\sum_{i=0}^{n} (-1)^i \sigma_i(\mathbf{x}) = \prod_{i=1}^{n} (1 - x_i),$$

we obtain

$$\rho_n(\mathbf{x}) = n! \, \frac{\prod_{1 \le i < j \le n} |x_i - x_j|}{\prod_{i=1}^{n} (1 - x_i)^{n+1}} \, \mathbb{1}\{\mathbf{x} \in \mathbb{R}_-^k\}.$$

§5. Proof of Theorem 2.1

Obviously, $G(x_1) = \cdots = G(x_k) = 0$ if and only if

$$\begin{pmatrix} 1 & x_1 & \dots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_k & \dots & x_k^n \end{pmatrix} \begin{pmatrix} \xi_0 \\ \vdots \\ \xi_n \end{pmatrix} = \mathbf{0}, \tag{5}$$

which is equivalent to

$$\begin{pmatrix} x_1^k & x_1^{k+1} & \dots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ x_k^k & x_k^{k+1} & \dots & x_k^n \end{pmatrix} \begin{pmatrix} \xi_k \\ \vdots \\ \xi_n \end{pmatrix} = -V(\mathbf{x}) \begin{pmatrix} \xi_0 \\ \vdots \\ \xi_{k-1} \end{pmatrix}.$$

Recalling (2), we obtain that (5) is equivalent to

$$\eta(\mathbf{x}) = \begin{pmatrix} \xi_0 \\ \vdots \\ \xi_{k-1} \end{pmatrix}.$$

Denote by $J_\eta(\mathbf{x})$ the Jacobian matrix of η at the point $\mathbf{x}$.

Lemma 5.1.

$$\det J_\eta(\mathbf{x}) = -\frac{\prod_{i=1}^{k} \left(\sum_{j=0}^{k-1} j\eta_j(\mathbf{x}) x_i^{j-1} + \sum_{j=k}^{n} j\xi_j x_i^{j-1} \right)}{\prod_{1\le i<j\le k} (x_j - x_i)}.$$

Proof. Differentiating

$$V(\mathbf{x})\eta(\mathbf{x}) = -\begin{pmatrix} x_1^k & x_1^{k+1} & \dots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ x_k^k & x_k^{k+1} & \dots & x_k^n \end{pmatrix} \begin{pmatrix} \xi_k \\ \vdots \\ \xi_n \end{pmatrix},$$

we obtain

$$\begin{aligned} V(\mathbf{x})J_\eta(\mathbf{x}) &+ \operatorname{diag}\left(\sum_{j=0}^{k-1} j\eta_j(\mathbf{x}) x_1^{j-1}, \dots, \sum_{j=0}^{k-1} j\eta_j(\mathbf{x}) x_k^{j-1} \right) \\ &= -\operatorname{diag}\left(\sum_{j=k}^{n} j\xi_j x_1^{j-1}, \dots, \sum_{j=k}^{n} j\xi_j x_k^{j-1} \right). \end{aligned}$$

We finish the proof by taking the second term from the left hand side to the right hand side and using

$$\det V(\mathbf{x}) = \prod_{1\le i<j\le k} (x_j - x_i). \qquad \square$$

Lemma 5.2 (Coarea formula). *Let $B \subset \mathbb{R}^k$ be a region. Let $u : B \to \mathbb{R}^k$ be a Lipschitz function and $h : \mathbb{R}^k \to \mathbb{R}^1$ be an L^1-function. Then*

$$\int_{\mathbb{R}^k} \#\{\mathbf{x} \in B \,:\, u(\mathbf{x}) = \mathbf{y}\}\, h(\mathbf{y})\, d\mathbf{y} = \int_B |\det J_u(\mathbf{x})|\, h(u(\mathbf{x}))\, d\mathbf{x},$$

where $J_u(\mathbf{x})$ is the Jacobian matrix of $u(\mathbf{x})$.

Proof. See [3, pp. 243–244]. $\square$

Let $B_1, \dots, B_k$ be a family of mutually disjoint Borel subsets in $\mathbb{R}^1$. Denote $B = B_1 \times \dots \times B_k \subset \mathbb{R}^k$. We have

$$\begin{aligned}\mathbf{E}\left[\prod_{i=1}^{k} \mu(B_i)\right] &= \mathbf{E}\,\#\{\mathbf{x} \in B \,:\, \eta(\mathbf{x}) = (\xi_0, \dots, \xi_{k-1})\} \\ &= \mathbf{E} \int_{\mathbb{R}^k} \#\{\mathbf{x} \in B \,:\, \eta(\mathbf{x}) = \mathbf{y}\} f_0(y_0) \dots f_{k-1}(y_{k-1})\, d\mathbf{y}.\end{aligned}$$

Applying Lemma 5.2 and Fubini's theorem to the right hand side, we obtain

$$\mathbf{E}\left[\prod_{i=1}^{k} \mu(B_i)\right] = \int_B \mathbf{E}\,|\det J_\eta(\mathbf{x})|\, f_0(\eta_0(\mathbf{x})) \dots f_{k-1}(\eta_{k-1}(\mathbf{x}))\, d\mathbf{x}.$$

Now the proof of Theorem 2.1 follows from Lemma 5.1.

§6. Proof of Theorem 2.3

Theorem 2.1 states that

$$\begin{aligned}\rho_k(\mathbf{x}) = &\prod_{1\le i<j\le k} |x_j - x_i|^{-1} \\ &\times \int_{\mathbb{R}^{n-k+1}} \prod_{i=1}^{k} \left|\sum_{j=0}^{n} j a_j x_i^{j-1}\right| \prod_{i=0}^{n} f_i(a_i)\, da_k\, da_{k+1} \dots da_n,\end{aligned} \tag{6}$$

where $a_0, \dots, a_{k-1}$ are functions of $a_k, \dots, a_n$ and $x_1, \dots, x_k$:

$$\begin{pmatrix} a_0 \\ \vdots \\ a_{k-1} \end{pmatrix} = -V^{-1}(\mathbf{x}) \begin{pmatrix} \sum\limits_{j=k}^{n} a_j x_1^j \\ \vdots \\ \sum\limits_{j=k}^{n} a_j x_k^j \end{pmatrix}. \tag{7}$$

To prove the theorem, we will use the ideas from [8, pp. 58–59] and [7, Lemmas 5 and 6]. Equation (7) means that $x_1, x_2, \dots, x_k$ are zeros of the polynomial $\sum\limits_{j=0}^{n} a_j x^j$. Hence there exists a unique polynomial $\sum\limits_{j=0}^{n-k} b_j x^j$ such that

$$\begin{aligned} \sum_{j=0}^{n} a_j x^j &= \prod_{i=1}^{k} (x - x_i) \Big(\sum_{j=0}^{n-k} b_j x^j \Big) \\ &= \Big(\sum_{j=0}^{k} (-1)^{k-j} \sigma_{k-j}(\mathbf{x}) x^j \Big) \Big(\sum_{j=0}^{n-k} b_j x^j \Big). \end{aligned} \tag{8}$$

The variables $a_0, \dots, a_n$ are uniquely defined by $\mathbf{x}$ and $b_0, \dots, b_{n-k}$ from (8):

$$a_i = \sum_{j=0}^{n-k} (-1)^{k-i+j} \sigma_{k-i+j}(\mathbf{x}) b_j. \tag{9}$$

Thus we can change variables in (6) substituting $a_k, \dots, a_n$ by their expressions in terms of $\mathbf{x}$ and $b_0, \dots, b_{n-k}$ from (9). The Jacobian of this substitution is a lower triangle matrix with unities in the diagonal. Hence its determinant is equal to one.

Differentiating (8) at the point x_i we get

$$\begin{aligned} &\sum_{j=0}^{n} j a_j x_i^{j-1} \\ &\qquad = \prod_{j \neq i} (x_i - x_j) \left(b_{n-k} x_i^{n-k} + \dots + b_1 x_i + b_0 \right), \ i = 1, \dots k. \end{aligned} \tag{10}$$

Substituting (9) and (10) in (6) finishes the proof.

§7. Proof of Proposition 2.2

For $0 \le i \le k-1$ and $j \ge k$, denote by $V^*_{ij}(\mathbf{x})$ the matrix obtained from $V(\mathbf{x})$ by substitution of the i-th column by $(x_1^j, \dots, x_k^j)^T$:

$$V^*_{ij}(\mathbf{x}) = \begin{pmatrix} 1 & x_1 & \dots & x_1^{r-1} & x_1^j & x_1^{r+1} & \dots & x_1^{k-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & x_k & \dots & x_k^{r-1} & x_k^j & x_k^{r+1} & \dots & x_k^{k-1} \end{pmatrix}.$$

Then by Cramer's rule, we have

$$\eta_r(\mathbf{x}) = -\frac{1}{\det V(\mathbf{x})} \sum_{j=k}^{n} \xi_j \det V^*_{ij}(\mathbf{x}).$$

It is easily seen that

$$\frac{V^*_{ij}(\mathbf{x})}{\det V(\mathbf{x})} = (-1)^{k-i-1} S_{\lambda_{ij}}(\mathbf{x}),$$

and the proof follows.

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