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DOUBLE COSETS OF STABILIZERS OF TOTALLY ISOTROPIC SUBSPACES IN A SPECIAL UNITARY GROUP I

Abstract. Let D be a division algebra with a fixed involution and let V be the corresponding unitary space over D with T-condition (see [2]). For a pair of totally isotropic subspaces $u, v \leqslant V$ we consider the double cosets $P_u\gamma P_v$ of their stabilizers P_u, P_v in $\Gamma = \mathrm{SU}(V)$. We give a description of cosets $P_u\gamma P_v$ in the terms of the intersection distance $\mathrm{d_{in}}(u, \gamma(v))$ and the Witt index of $u + \gamma(v)$.

Introduction

In [8], the notion of a linear Kleiman group was introduced: a linear group $G \leqslant \mathrm{GL}(V)$ is called a Kleiman group, if for any pairs of linear subspaces $u, v \leqslant V$ there is an element $g \in G$ such that the subspaces $u, g(v)$ are in "general position" (that is, either $\dim u \cap g(v) = 0$ or $\dim v \cap g(v) = \dim u + \dim v - \dim V$). The "mutual position" of subspaces $u, g(u)$ is defined by one non-negative integer which can be taken equal to $\dim u \cap g(v)$ or $\dim(u + g(v))$, or equal to the "intersection distance" $\mathrm{d_{in}}(u, g(v)) = \max\{\dim u, \dim g(v)\} - \dim u \cap g(v)$ (the last notion was used in [8]) and the general position corresponds to the case where the intersection distance $\mathrm{d_{in}}(u, g(v))$ achieves the possible maximum. The mutual position of subspaces $u, g(v)$ is defined uniquely by the corresponding double coset $P_u g P_v$ of $\mathrm{GL}(V)$ where P_u, P_v are the stabilizers of u, v in $\mathrm{GL}(V)$. This is a general position if the double coset $\widetilde{P}_u g \widetilde{P}_v$ is Zariski open in GL where $\widetilde{P}_u, \widetilde{P}_v$ are stabilizers of u, v in the algebraic group GL.

It would be interesting to extend the investigation of the "Kleiman property" to the cases when $G \leqslant \Gamma$ is a subgroup of a classical group Γ. This seems to be a difficult task. Here we consider a step in this direction. Namely, let V be a linear space over a division algebra D with an involution and assume that a hermitian or skew-hermitian form corresponding

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to this involution is defined on V. Further, let $\Gamma = \mathrm{U}(V)$ or $\Gamma = \mathrm{SU}(V)$ and let O_u, O_v be Γ-orbits of linear subspaces of u, v. Then the group Γ acts naturally on $O_u \times O_v$ (namely, $\gamma(u', v') = (\gamma(u'), \gamma(v'))$ and there is a one-to-one correspondence of Γ-orbits of this action and double cosets $P_u \gamma P_v$ of the group Γ where P_u, P_v are stabilizers of u, v (respectively). Namely, pairs $(u, g_1(v)), (u, g_2(v))$ belong to the same orbit if and only if g_1, g_2 belong to the same double coset $P_u \gamma P_v$. Hence the description of a double coset $P_u \gamma P_v$ can be obtained by means of geometrical characterisation of the pair $u, \gamma(v)$. However, in general this is a difficult question and here we restrict ourselves to the case where u, v are totally isotropic subspaces. In this case we have a finite number of double cosets $P_u \gamma P_v$. If Γ is the whole unitary group then it is not difficult to see that every such coset $P_u \gamma P_v$ is defined uniquely by the dimension and the Witt index of the space $u + \gamma(v)$ (see Comment 7.1). However, there is a special interest to consider the group $\Gamma = \mathrm{SU}(V)$ because for every simple algebraic group $\widetilde{\Gamma}$ of type A_n, B_n, C_n, D_n which is defined over a field K there is a linear representation of $\widetilde{\Gamma}$ such that $\widetilde{\Gamma}(K) = \Gamma = \mathrm{SU}(V)$ where V is an appropriate hermitian or skew-hermitian form over an appropriate division algebra D which contains K in the center.

Here we give the description of double cosets $P_u \gamma P_v$ in the case $\Gamma = \mathrm{SU}(V)$ (under the restriction that the hermitian form on V is a T-form (see [2]). This description (Theorem 1) is the same as for the whole unitary group except if $D = K$ is a field with trivial involution, the corresponding hermitian form is a symmetric bilinear form which is completely split over K, $\dim V = 2n$ and $\mathrm{SU}(V) = \mathrm{SO}_{2n}(K)$. We also give the description of double cosets $P_u \gamma P_v$ in this special case (Theorem 2). Note that the T-condition in this case demands $\mathrm{char} K \neq 2$. However, it is possible to avoid this restriction (see Comment 7.3).

Notations. The algebraic closure of a field K is denoted by $\overline{K}$.

$X \approx Y$ denotes an isomorphism of the algebraic structures X and Y.

E_m is the identity matrix of size $m \times m$.

$\mathbf{0}_{r \times s}$ is the zero matrix of size $r \times s$.

§1. Preliminaries

Let K be a field and let F/K be a separable extension of degree $\leqslant 2$.

Further, let D be a division algebra over the center F of index c and with a fixed anti-automorphism $x \to x^\star$ which is trivial on K (here we admit the case $D = K$ when $\star$ is trivial and the case $D = F$). We fix a natural embedding

$$i : D \to M_c(\overline{F})$$

which is induced by maps $D \to \overline{F} \otimes_F D \approx M_c(\overline{F})$.

Let V be a linear space over D of dimension d with a non-degenerate hermitian or skew-hermitian form $h = (\ ,\ )$ with respect to the involution $\star$ (here we also consider symmetric or skew-symmetric forms over K as hermitian or skew-hermitian forms with $D = F = K$). We will refer to any linear space over D with the form h (not necessary non-degenerate) as an h-space.

The dimension of a maximal totally isotropic subspace of V (Witt index) will be denoted by n. *We assume below* $n \geqslant 1$.

The group of isometries of V will be denoted by $\mathrm{U}(D, h)$ (unitary group).

Subspaces of V.

Let $L, U \leqslant V$ be linear subspaces and let $L \cap U = \{0\}$. Suppose $(L, U) = 0$. Then we will denote the sum $L + U$ by the symbol $L \perp U$. For a linear subspace $U \leqslant V$ we write $U^{\perp} := \{l \in V \mid (l, U) = 0\}$.

For a linear subspace $U \leqslant V$ we have

$$U = \operatorname{rad} U \perp U^{\natural}$$

where $\operatorname{rad} U = \{u \in U \mid (u, U) = 0\}$ is the radical of U and $U^{\natural}$ is a non-degenerate h-space ([2, IX, §1]). A maximal totally isotropic space U of V will be denoted by U_0. The codimension of U_0 in U will be denoted by $I(U)$.

H-hyperbolic planes and h-quasi-hyperbolic spaces.

An h-hyperbolic plane $U \leqslant V$ here will be a plane which has a basis consisting of two isotropic vectors e, f such that $(e, f) = 1$. We denote such a plane by the symbol H_2. The subspace

$$U \approx H_{2s} = \underbrace{H_2 \perp H_2 \perp \cdots \perp H_2}_{s-\text{times}}$$

will be called an h-hyperbolic space and the space U of the form

$$U = \text{rad } U \perp H_{2s}$$

will be called an h-quasi-hyperbolic space.

h-forms with the T-condition.

Below we assume that the form h on V satisfies the T-condition ([2, IX, §4]):

For every $x \in V$ there is an element $\alpha \in D$ such that $\alpha + \epsilon\alpha^\star = h(x, x)$, where $\epsilon = 1$ if h is hermitian and $\epsilon = -1$ if h is skew-hermitian form. In particular, the T-condition holds if h is skew-hermitian or char$K \neq 2$.

This condition gives the following properties of h-spaces ([2, IX, §4]):

h1. *Witt decomposition.* For every subspace $U \leqslant V$ there is a decomposition

$$U = \mathrm{rad}\, U \perp (U_\flat + U^\flat) \perp A_U$$

where $U_\flat, U^\flat$ are totally isotropic spaces, $U_\flat + U^\flat \approx H_{2s}$ for some $s \geqslant 0$, and A_U is an anisotropic subspace of U. In particular, $V = (V_\flat + V^\flat) \perp A_V$ where $V_\flat, V^\flat$ are maximal totally isotropic subspaces of V (below we fix a pair $V_\flat, V^\flat$ and their bases $V_\flat = \langle e_1, \ldots, e_n \rangle, V^\flat = \langle f_1, \ldots, f_n \rangle$ such that $(e_i, f_j) = \delta_{ij}$).

h2. *Witt Theorem.* Every h-isometry $f : U_1 \to U_2$ of subspaces $U_1, U_2 \leqslant V$ can be extended to an h-isometry $F : V \to V$ (that is, $F_{|U_1} = f$).

Reduced norm. For a matrix $X \in \mathrm{M}_a(D)$ the determinant of the $ca \times ca$-matrix $i^*(X) \in \mathrm{M}_{ca}(\overline{F})$ is called *the reduced norm* of X and we denote it by $\mathrm{Nrd}\, X$; note that $\mathrm{Nrd}\, X \in F$ (see [9])). Then

$$\mathrm{SL}_a(D) \stackrel{\mathrm{def}}{=} \{X \in \mathrm{M}_a(D) \mid \mathrm{Nrd}\, X = 1\}.$$

(Note that the symbol $\mathrm{SL}_a(D)$ is used sometimes for the kernel of the Dieudonné determinant det : $\mathrm{GL}_a(D) \to D^*/[D^*, D^*]$ (see [1])) which in general does not coincide with the kernel of the map Nrd : $\mathrm{GL}_a(D) \to \overline{F}$).)

Classical algebraic groups.

Let $\widetilde{\Gamma}$ be a simple algebraic group which is defined over a field K. Below we identify the group $\widetilde{\Gamma}$ with $\widetilde{\Gamma}(\overline{K})$. Below we consider the classical groups $\widetilde{\Gamma}$ (that are of the type A_n, B_n, C_n, D_n) such that

$$\Gamma := \widetilde{\Gamma}(K) = \mathrm{SU}(D, h) \tag{1.1}$$

where $\mathrm{SU}(D, h)$ is the corresponding special unitary group $\mathrm{SU}(D, h) = U(D, h) \cap \mathrm{SL}_d(D)$. Every algebraic group $\widetilde{\Gamma}$ of type B_n, C_n, D_n which is defined over K is a group (up to the center) which satisfies the condition (1.1). Moreover, in these cases, $F = K$. The same is true for group of outer type of type A_n but here $\deg F/K = 2$ (see [9]).

However, we will use the assumptions **h1, h2** which definitely holds if char$K \neq 2$.

We write

$$\widehat{\Gamma} = \mathrm{U}(D, h).$$

We will subdivide all possibilities for the groups Γ:
General Case. Including all cases except the

Special Case. $K = D$ and h is a completely split symmetric form over K, $d = 2n$, and $\Gamma = \mathrm{SO}(V) = \mathrm{SO}_{2n}(K)$.

Orbits of totally isotropic subspaces.
The set of all totally isotropic subspaces of V of dimension k will be denoted by $\mathcal{I}_k$. Note that the set $\mathcal{I}_k$ is a single Γ-orbit except in the Special Case when $k = n = \frac{1}{2}\dim V$ (see Proposition 3.1 below). In this case there is a basis $e_i, f_i, i = 1, \ldots, n$ such that $(e_i, f_j) = \delta_{i,j}$. Let

$$V_n^+ = V_\flat = \langle e_1, \ldots, e_n \rangle, \quad V_n^- = \langle e_1, \ldots, e_{n-1}, f_n \rangle.$$

Then V_n^+, V_n^- are maximal totally isotropic spaces which belongs to different Γ-orbits of $\mathcal{I}_n$, denoted respectively by $\mathcal{I}_n^+, \mathcal{I}_n^-$.

§2. Some properties of the reduced norm

In this chapter we collect some technical results for the reduced norm of elements of unitary groups. We will use here the notion of pseudo-reflections in unitary spaces.

Pseudo-reflections in h-spaces. Let $e \in V$ be an anisotropic vector and let $L_e = De$ be the corresponding line and $L_e^\perp$ be the orthogonal complement. Further, let $a = (e, e)$ and let $\alpha \in D^*$ be an element such that

$$\alpha a \alpha^\star = a.$$

Then we can define a linear transformation σ of V by the formula

$$\sigma(x) = \begin{cases} x & \text{if} \quad x \in L_e^\perp, \\ \alpha e & \text{if} \quad x = e. \end{cases}$$

Then $\sigma \in \mathrm{U}(D, h)$. We call such a unitary transformation – a pseudo-reflection (in [7]) it is called a quasi-symmetry). The unitary group $\mathrm{U}(D, h)$ is generated by pseudo-reflections except for the cases $\mathrm{U}(D, h) = \mathrm{Sp}_{2m}(K)$, $\mathrm{U}_2(F_4)$ ([6, II, §3]; [7]).

Lemma 2.1. *Let $e \in V$ be an anisotropic vector in V and let $\sigma \in \mathrm{U}(D,h)$ be the pseudo-reflection such that $\sigma(e) = \alpha e$. Further, let U be a subspace which is isomorphic to a hyperbolic space H_2. Then there exists an anisotropic vector $e' \in U$ and a pseudo-reflection $\tau \in \mathrm{U}(D,h)$ that corresponds to the e' such that $\tau(e') = \alpha e'$.*

Proof. Let e_1, e_2 be two isotropic vectors in U such that $(e_1, e_2) = 1$. Since the h-space V satisfies the T-condition we have $(e,e) = \lambda + \epsilon\lambda^\star$ for some $\lambda \in D$ and $\epsilon = \pm 1$. Put $e' = \lambda e_1 + e_2$. Then

$$(e', e') = (\lambda e_1 + e_2, \lambda e_1 + e_2) = \lambda + \epsilon\lambda^\star = (e,e).$$

Then $\alpha(e', e')\alpha^\star = (e', e')$ and there is a pseudo-reflection $\tau \in \mathrm{U}(D,h)$ such that $\tau(e') = \alpha e'$. □

Reduced norm of elements of unitary spaces.

We have the following properties of the reduced norm of an element $g \in \mathrm{U}(D,h)$:

n1. $\mathrm{Nrd}\, g(\mathrm{Nrd}\, g)^\star = 1$.

(Indeed, if X_g is a matrix of g in some basis and X_g^* is the dual matrix, that is, $X_g X_g^* = E_d$ then $X_g^* = S X_g^{\star T} S^{-1}$ for some $S \in \mathrm{GL}_d(D)$ ([2, IX, §1, n. 10]). Hence $\mathrm{Nrd}\, g(\mathrm{Nrd}\, g)^\star = 1$.)

n2. if $F = K = D$ then $\mathrm{Nrd}\, g = \pm 1$.

(Note $(\mathrm{Nrd}\, g))^2 = 1$ if $F = K$.)

n3. if $F = K \neq D$ then $\mathrm{Nrd}\, g = 1$.

(Indeed, if $K = F \neq D$ then for every pseudo-reflection $\tau \in \mathrm{U}(D,h)$ such that $\tau(e) = \alpha e$ for an anisotropic vector e we have $\alpha = \lambda^\star\lambda^{-1}$ for some $\lambda \in D^*$ ([7, Theorem 3]) and therefore

$$\mathrm{Nrd}\,\tau = \mathrm{Nrd}\,\alpha = (\mathrm{Nrd}\,\lambda)^\star(\mathrm{Nrd}\,\lambda)^{-1} = 1.$$

Since the group $\mathrm{U}(D,h)$ is generated by pseudo-reflections (except the group $\mathrm{Sp}_{2n}(K)$ where every element has the reduced norm $= 1$ and the group $U_2(F_4)$ where $F \neq K$) we have $\mathrm{Nrd}\, g = 1$ for every $g \in \mathrm{U}(D,h)$).

Some transformations of hyperbolic planes.

Lemma 2.2. *Let $D \neq K$ and let $V = \langle e, f\rangle \approx H_2$. Then there exists an element $\gamma \in \mathrm{U}(D,h)$ with $\mathrm{Nrd}\,\gamma = 1$ which is presented by the matrix (in the basis e, f)*

$$M_\gamma = \begin{pmatrix} 0 & \mu \\ \nu & 0 \end{pmatrix}$$

for some $\mu, \nu \in F^*$.

Proof. If h is a skew symmetric form we may take $\mu = 1, \nu = -1$. Then $\operatorname{Nrd}\gamma = \det i^*(M_\gamma) = 1$. Suppose h is a symmetric form and the index c of the division algebra D is even. Then we may take $\mu = \nu = 1$ and we have $\operatorname{Nrd}\gamma = \det i^*(M_\gamma) = (\det M_\gamma)^c = (-1)^c = 1$. Suppose c is odd. We may assume $F \neq K$ (because of **n3**) and $\operatorname{char} K \neq 2$. Then there is an element $\beta \in F^*$ such that $\beta^\star = -\beta$. Now take $\mu = \beta, \nu = -\beta^{-1}$. Then $\operatorname{Nrd}\gamma = \det i^*(M_\gamma) = (\det M_\gamma)^c = 1^c = 1$. □

Lemma 2.3. *Let* $\alpha = \operatorname{Nrd} g$ *for some* $g \in \mathrm{U}(D,h)$. *Further, let* $U \leqslant V$ *and*

$$H_2 \approx U = \langle e_1, f_1\rangle = \langle e_2, f_2\rangle, \quad (e_i, f_i) = 1, \quad (e_i, e_i) = (f_i, f_i) = 0$$

for $i = 1, 2$. *Then there exits an element* $\tau \in \mathrm{U}(D,h)$ *such that*

a. $\operatorname{Nrd}\tau = \alpha^{-1}$;

b. $\tau(U) = U, \quad \tau(U^\perp) = U^\perp$ *and* $\tau(x) = x$ *for every* $x \in U^\perp$.

If, in addition, $D \neq K$ *then* τ *satisfies also the condition*

c. $\tau(De_1) = De_2, \quad \tau(Df_1) = Df_2$.

Proof. Suppose there exist elements $\tau_1, \tau_2 \in \mathrm{U}(D,h)$ such that τ_1 satisfies **a, b** and τ_2 satisfies **b, c** and **a** with $\alpha = 1$. Then the element $\tau = \tau_2\tau_1$ satisfies **a, b, c** (indeed, we may take $e_1 := \tau_1(e_1), f_1 := \tau_1(f_1)$).

Let us prove the existence of τ_1. We may assume $\alpha \neq 1$ (otherwise we may take $\tau_1 = 1$). Then $\mathrm{U}(D,h) \neq \mathrm{Sp}_{2n}(K)$. Also, if $\mathrm{U}(D,h) = U_2(F_4)$ the conditions **a, b** obviously hold. Thus we may assume that the group $\mathrm{U}(D,h)$ is generated by pseudo-reflections. Hence

$$\alpha = \operatorname{Nrd} g = \operatorname{Nrd} g_1 \operatorname{Nrd} g_2 \cdots \operatorname{Nrd} g_s$$

where $g_1, \dots, g_s \in \mathrm{U}(D,h)$ are pseudo-reflections. Let $l_1, \dots, l_s \in V$ be anisotropic vectors corresponding to the reflections $g_1, \dots, g_s$ and $g_i(l_i) = \beta_i l_i$ where $1 \neq \beta_i \in D$ and let $\alpha_i = \operatorname{Nrd} g_i$. Lemma 2.1 implies that there are anisotropic vectors $f_1, \dots, f_s \in U$ and the corresponding pseudo-reflections $h_1, \dots, h_s \in \mathrm{U}(D,h)$ such that $h_i(f_i) = \beta_i^{-1} f_i$. Obviously, the element $\tau_1 = h_1 h_2 \cdots h_s \in \mathrm{U}(D,h)$ satisfies **a, b**.

Let us prove the existence of τ_2.

Suppose $Df_1 = Df_2$ and $De_1 \neq De_2$. Then $e_2 = \lambda_1 e_1 + \lambda_2 f_1$ where $\lambda_1, \lambda_2 \in D^*$. Put $\lambda = \lambda_1^{-1}\lambda_2$. Consider the linear transformation τ_2 that is defined by the following formulas

$$\tau_2(e_1) = e_1 + \lambda f_1 = \lambda_1^{-1} e_2, \quad \tau_2(f_1) = f_1, \quad \tau_2(x) = x \text{ for every } x \in U^\perp.$$

These formulas show that $\tau_2 \in \mathrm{U}(D,h)$ and the element τ_2 satisfies the conditions **b, c** and **a** with $\alpha = 1$.

Suppose $De_1 = Df_2$. Consider $\gamma \in \mathrm{SU}(D,h)$ such that $\gamma(x) = x$ for every $x \in U^\perp$ and the restriction γ on U satisfies the condition of Lemma 2.2. Then

$$\gamma(De_1) = Df_1, \quad \gamma(Df_1) = De_1.$$

Now take instead of the basis e_1, f_1 the basis $e_1' = \gamma(e_1) = \mu f_1, f_1' = \gamma(f_1) = \nu e_1$ where $\beta, \delta \in D^*$. We may assume $De_1' \neq De_2$ (otherwise we may take $\tau_2 = \gamma$). Then $Df_1' = De_1 = Df_2$ and we are in the previous case. We may find as above an element τ_2' that satisfies the conditions **b, c** (for pairs e_1', f_1' and e_2, f_2) and **a** with $\alpha = 1$. Then $\tau_2 = \tau_2'\gamma$ satisfies the conditions **b, c** and **a** with $\alpha = 1$.

Suppose $De_1 \neq Df_2 \neq Df_1$. Then $f_2 = \xi_1 e_1 + \xi_2 f_1$ for some $\xi_1, \xi_2 \in D^*$. Put $\xi = \xi_2^{-1}\xi_1$. Then we have the unitary transformation τ':

$$\tau'(e_1) = e_1, \tau'(f_1) = \xi_2^{-1} f_2 = \xi e_1 + f_1 \quad \text{and} \quad \mathrm{Nrd}\,\tau' = 1.$$

Now we may take $f_1 := \xi_2^{-1} f_2$ instead of f_1. Then we are in the case $D(f_1) = D(f_2)$.

□

§3. Γ-ORBITS OF H-QUASI-HYPERBOLIC SPACES

In this chapter we consider some properties of Γ-orbits of h-quasi-hyperbolic subspaces of V which we will use below to describe the Γ-orbits of pairs of totally isotropic subspaces $(u, v) \in \mathcal{I}_l \times \mathcal{I}_k$.

Put

$$\mathcal{V}_{m,q} := \{U \text{ is an h-quasi-hyperbolic subspace of } V \mid I\ (U) = q, \\ \dim U = m + q\}.$$

Note if U is an h-quasi-hyperbolic space then the numbers $\dim U, I(U)$ define the space U up to an isomorphism from the group $\widehat{\Gamma}$ (see **h2**). Hence $\mathcal{V}_{m,q}$ is a single $\widehat{\Gamma}$-orbit. Moreover,

Proposition 3.1. *The set $\mathcal{V}_{m,q}$ is a single Γ-orbit, except, when in the Special Case we have $m = n = \frac{1}{2}\dim V, q = 0$. In the latter case there are two different Γ-orbits $\mathcal{I}_n^+$ and $\mathcal{I}_n^-$.*

Proof. Let $U, U' \in \mathcal{V}_{m,q}$ and let $g \in \widehat{\Gamma}$ be an element such that $g(U) = U'$. Let $\alpha = \operatorname{Nrd} g$. We may assume $\alpha \neq 1$ (otherwise there is nothing to prove).

Suppose U is not a totally isotropic space. Since U is a h-quasi-hyperbolic space we have $U = U_1 \perp U_2$ where U_1 is a hyperbolic plane. Also, $V = U_1 \perp U_1^{\perp}$ and $U_2 \leqslant U_1^{\perp}$. Lemma 2.3 **a, b** implies that there is an element $\tau \in \widehat{\Gamma}$ such that

$$\tau(U) = U \quad \text{and } \operatorname{Nrd} \tau = \alpha^{-1}.$$

Then $g' = g\tau \in \Gamma$ and $g'(U) = U'$.

Now we may assume $U \leqslant V$ is a totally isotropic subspace. If U is not a maximal totally isotropic space then there is a hyperbolic plane $H_2 \leqslant V$ such that $H_2 \cap U = \{0\}$ and $H_2 \leqslant U^{\perp}$. Then the arguments which are used above show then $g'(U) = U'$ for some $g' \in \Gamma$.

Assume U is a maximally isotropic subspace of V. We have $U = L \perp Dl$ for some $l \in U$ and $L \leqslant U$ and $\dim L = n - 1$. There exists an element $\gamma \in \Gamma$ such that $\gamma(L) \leqslant U'$ (we see above that non-maximal totally isotropic subspaces of the same dimension are in the same Γ-orbit). If $\gamma(l) \in U'$ than $\gamma(U) = U'$ (because, $\dim U' = \dim U = n$). Hence we may assume $f = \gamma(l) \notin U'$. Thus there is a vector $e \in U' \setminus \gamma(L)$ such that $\langle e, f \rangle \approx H_2$ and $(e, f) = 1$. Now we have

$$V = \langle e,f\rangle \perp \langle e, f\rangle^{\perp},\ \gamma(L) \leqslant \langle e, f\rangle^{\perp},\ \langle f, \gamma(L)\rangle = \gamma(U),\ \langle e, \gamma(L)\rangle = U'. \quad (3.1)$$

Let $D \neq K$. Then Lemma 2.3 **a, b, c** implies that there is an element $\tau \in \widehat{\Gamma}$ such that

$$\operatorname{Nrd} \tau = 1,\ \tau(De) = Df,\ \tau(Df) = De,\ \tau(x) = x \ \text{ for every } x \in \langle e, f\rangle^{\perp}. \quad (3.2)$$

From (3.1) and (3.2) we get $g' = \tau\gamma \in \Gamma$ and $g'(U) = U'$.

Suppose $K = D$, and $\langle e, f \rangle$ is a symplectic space. Then we obviously have an element $\tau \in \widehat{\Gamma}$ which satisfies conditions (3.2) and therefore we will get the element $g' \in \Gamma$ such that $g'(U) = U'$.

Suppose $K = D$, and V is an orthogonal space (then $\operatorname{char} K \neq 2$ because of T-condition). Hence $\operatorname{Nrd}(\widehat{\Gamma}) = \{\pm 1\}$ (see **n2**) and therefore $\alpha = -1$. If we are not in the Special Case then there exists an anisotropic vector $l \in U^{\perp}$ and the reflection $\rho, \rho(l) = -l$. Then $\rho(U) = U, \operatorname{Nrd} \rho = -1$ and $g' = g\rho \in \Gamma$, $g'(U) = U'$. Now assume that we are in the Special Case. Then there are exactly two Γ-orbits of maximal totally isotropic subspaces of V, namely, $\mathcal{I}_n^+, \mathcal{I}_n^-$ ([4, VIII, §13]). □

Proposition 3.2. *Assume we are in the Special Case and let* $v, u \in \mathcal{I}_n$*. Further, let*

$$\mathrm{sign}(v,u) = (-1)^{\mathrm{d_{in}}(v,u)}.$$

Then

$$v \in \mathcal{I}_n^{\pm} \Leftrightarrow u \in \mathcal{I}_n^{\pm \mathrm{sign}(v,u)}.$$

Proof. We may assume $v = V_n, u = g(V_n)$ or $u = g(V_n^-)$ for some $g \in \Gamma$. Let P_n, P_n^- be the stabilizers of V_n, V_n^- respectively. Then P_n, P_n^- are the standard parabolic subgroups with respect to the maximal torus of Γ which is diagonizable in the basis e_i, f_i (see [5]). The group Γ is decomposed into disjoint double cosets of the form $P_n\dot{w}P_n$ (resp. $P_n\dot{w}P_n^-$) where w runs through some subset of the Weyl group W.

Let $g \in P_n\dot{w}P_n$ (or $g \in P_n\dot{w}P_n^-$) for some $w \in W$. Since $\mathrm{sign}(v,u) = \mathrm{sign}(\gamma(v),\gamma(u))$ for every $\gamma \in \Gamma$ we may assume $u = \dot{w}(V_n)$ (or $u = \dot{w}(V_n^-)$). The number of e_i such that $\dot{w}(e_i) = f_j$ is even (this can be checked by consideration of actions of elementary reflections $\dot{w}_\alpha$). Hence the number $\mathrm{d_{in}}(V_n, \dot{w}(V_n))$ is even and $\mathrm{d_{in}}(V_n, \dot{w}(V_n^-))$ is odd. □

§4. Γ-orbits of sums of totally isotropic subspaces

Sums of totally isotropic spaces.

Lemma 4.1. *Let* U_1, U_2 *be totally isotropic subspaces of* V*. Then* $U = U_1 + U_2$ *is h-quasi-hyperbolic.*

Proof. For the case $I(U) = 0$ the statement is trivial. Now let $I(U) > 0$. Then there exist vectors $l_1 \in U_1, l_2 \in U_2$ such that $(l_1, l_2) \neq 0$. Then $\langle l_1, l_2\rangle = H_2$ is an h-hyperbolic plane. Further,

$$U_1 = U_{1,2} \perp \langle l_1\rangle, \quad U_2 = U_{2,1} \perp \langle l_2\rangle$$

for some $U_{1,2} \leqslant U_1, U_{2,1} \leqslant U_2$. Hence

$$U = \langle l_1, l_2\rangle \perp (U_{1,2} + U_{2,1}).$$

Now our statement can be proved by the induction on $I(U)$. □

Lemma 4.2. *Let* U_1, U_2 *be totally isotropic subspaces of* V *and* $U = U_1 + U_2$*. Further, let* $U_1 + U_2'$ *and* $U_1 + U_2''$ *be two maximal totally isotropic spaces of* U *where* $U_2', U_2'' \leqslant U_2$*. Then*

$$U_1 + U_2' = U_1 + U_2''.$$

Proof. Let $l \in U_2''$. We have $(l,x) = 0$ for every $x \in U_1$ (because $U_1 + U_2''$ is an isotropic space) and for every $x \in U_2'$ (because $U_2' \leqslant U_2$ and $l \in U_2$). Hence $(l,x) = 0$ for every $x \in U_1 + U_2'$. Since $U_1 + U_2'$ is a maximal totally isotropic subspace of U we have $l \in U_1 + U_2'$ and therefore $U_1 + U_2'' \subset U_1 + U_2'$. The latter inclusion obviously implies the equality $U_1 + U_2' = U_1 + U_2''$. □

Lemma 4.3. *Let U_1, U_2 be totally isotropic subspaces of V. Then*

$$U = U_1 + U_2 = U_1^0 \perp (U_1 \cap U_2) \perp U_2^0 \perp \underbrace{(U_1^1 + U_2^1)}_{h\text{-hyperbolic space}}$$

where

$$U_1^0 \leqslant U_1,\ U_2^0 \leqslant U_2,\ \textit{and}$$

$$U_1^0 \perp (U_1 \cap U_2) \perp U_2^0 \perp U_1^1 \ \textit{is a maximal totally isotropic subspace of}\ U,$$

$$U_1^1 \leqslant U_1, \quad U_2^1 \leqslant U_2, \quad \dim U_1^1 = \dim U_2^1 = q = I(U),\ \textit{and}$$

$$(U_1^1 + U_2^1)\ \textit{is a maximal h-hyperbolic subspace of}\ U.$$

Proof. Since U is an h-quasi-hyperbolic space (Lemma 4.1) we have

$$U = \operatorname{rad} U \perp H_{2q}$$

where H_{2q} is a maximal h-hyperbolic subspace of dimension $2q$. Since U_1, U_2 are totally isotropic spaces and $U = U_1 + U_2$ we may assume

$$H_{2q} = (U_1^1 + U_2^1) \quad \text{for some} \quad U_1^1 \leqslant U_1, \quad U_2^1 \leqslant U_2.$$

Obviously, $(U_1 \cap U_2) \leqslant \operatorname{rad} U$. Let $l = l_1 + l_2 \in \operatorname{rad} U$ where $l_1 \in U_1$ and $l_2 \in U_2$. Then $l_1, l_2 \in \operatorname{rad} U$. (Indeed, if $l_1 \notin \operatorname{rad} U$ then there is a vector $l' = l_1' + l_2' \in U$, where $l_1' \in U_1$ and $l_2' \in U_2$ and $(l_1, l') \neq 0$. Hence $(l_1, l_2') \neq 0$ and therefore $(l, l_2') \neq 0$. It is a contradiction with the choice of l.) Thus we may chose the complement of $(U_1 \cap U_2)$ in $\operatorname{rad} U$ in the form $U_1^0 \perp U_2^0$ for some $U_1^0 \leqslant U_1,\ U_2^0 \leqslant U_2$. □

Spaces v_g.
Let $(u,v) \in \mathcal{I}_l \times \mathcal{I}_k$. For $g \in \Gamma$ define

$$v_g = u + g(v).$$

Since v, u are totally split spaces the space v_g is h-quasi-hyperbolic (Lemma 4.1).

Proposition 4.4. *Let $g_1, g_2 \in \Gamma$. Further, let $v_{g_1} = \sigma(v_{g_2})$ for some $\sigma \in \Gamma$. Then there exists an element $g \in \widehat{\Gamma}$ such that*

$$g(u) = u, \quad g(g_2(v)) = g_1(v).$$

If, in addition, we are in the General Case we may find such an element $g \in \Gamma$.

Proof. Lemma 4.3 implies that for $i = 1, 2$ we have

$$v_{g_i} = u_i^0 \perp (u \cap g_i(v)) \perp g_i(v)^0 \perp (u_i^1 + g_i(v)^1) \tag{4.1}$$

where

$$u_i^0 \leqslant u, \quad g_i(v)^0 \leqslant g_i(v), \quad u_i^0 \perp (u \cap g_i(v)) \perp g_i(v)^0 = \operatorname{rad} v_{g_i}$$

and

$$u_i^1 \leqslant u, \quad g_i(v)^1 \leqslant g_i(v), \quad \dim u_i^1 = \dim g_i(v)^1 = I(v_{g_i})$$

and $(u_i^1 + g_i(v)^1)$ is a maximal hyperbolic subspace in v_{g_i}. Since $v_{g_1} = \sigma(v_{g_2})$ the dimensions of all components of (4.1) for $i = 1$ equal corresponding components for $i = 2$. Then there is an isomorphism $v_{g_2} \to v_{g_1}$ of h-spaces such that each component of (4.1) for $i = 2$ maps isomorphically to the corresponding component of (4.1) for $i = 1$. Hence there is an element $g \in \widehat{\Gamma}$ such that

$$g(u_2^0) = u_1^0, \quad g(u \cap g_2(v)) = u \cap g_1(v), \quad g(g_2(v)^0) = g_1(v)^0,$$
$$g(u_2^1) = u_i^1, \quad g(g_2(v)^1) = g_1(v)^1$$

(see **h2**) and therefore

$$g(u) = u, \quad g(g_2(v)) = g_1(v), \quad g(v_{g_2}) = v_{g_1}. \tag{4.2}$$

Let $\operatorname{Nrd} g = \alpha \in \operatorname{Nrd}(\widehat{\Gamma})$. We may assume $\alpha \neq 1$ (otherwise it is nothing to prove).

Let $D = K$. Then $\alpha = -1$. Sine we are in General Case there is an anisotropic vector e which is orthogonal to v_{g_1} and the reflection $\tau \in \widehat{\Gamma}, \tau(e) = -1$. Hence $g := \tau g \in \Gamma$ and the element g satisfies (4.2).

Let $D \neq K$. Suppose $q(v_{g_1}) \neq 0$. Then there exist vectors $e \in u, f \in g_1(v)$ such that $\langle e, f \rangle \approx H_2$ and $v_{g_1} = \langle e, f \rangle \perp (U_1 + U_2)$ for some $U_1, U_2 \leqslant V$ and $\langle e, U_1 \rangle = u, \quad \langle f, U_2 \rangle = g_1(v)$. Lemma 2.3 implies that there is an element $\tau \in \widehat{\Gamma}$ such that $\operatorname{Nrd} \tau = \alpha^{-1}, \quad \tau(De) = De, \quad \tau(Df) = Df, \quad \tau(x) = x$ for every $x \in U_1 + U_2$. Then $\tau(u) = u, \tau(g_1(v)) = g_1(v)$ and therefore the element $g := \tau g \in \Gamma$ satisfies (4.2).

Let $D \neq K$. Suppose $q(v_{g_1}) = 0$. Then we may take $e \in u$ and $f \in V$ such that $\langle e, f\rangle \approx H_2$ and $v_{g_1} = De \perp U$ where $U \leqslant (Df)^{\perp}$. Now we may use the same arguments as above. □

Now we consider the equality $\sigma(v_{g_2}) = v_{g_1}$ in the Special Case. According to Lemma 4.2 in the space $v_{g_i} = u + g_i(v)$ there exists the unique maximal totally isotropic subspace of v_{g_i} which contains u. We denote this subspace as $v^u_{g_i}$. Since $\sigma(v_{g_2}) = v_{g_1}$ we have $\dim v^u_{g_2} = \dim v^u_{g_1}$.

In the Proposition below we preserve the notation of Proposition 4.4.

Proposition 4.5. *Suppose we are in the Special Case and $\sigma(v_{g_2}) = v_{g_1}$ for some $\sigma \in \Gamma$. Suppose that one of the following conditions holds:*

i. $\dim u = n$;

ii. $\dim v^u_{g_i} < n$;

iii. $\dim v^u_{g_i} = n$ *and* $\operatorname{sign}(v^u_{g_1}, v^u_{g_2}) = 1$.

Then there exists an element $g \in \Gamma$ such that $g(u) = u$ and $g(g_2(v)) = g_1(v)$.

Proof. According to Proposition 4.4 there exits an element $g \in \widehat{\Gamma}$ such that $g(u) = u$ and $g(g_2(v)) = g_1(v)$. We may assume $\operatorname{Nrd} g = -1$ (otherwise there is nothing to prove).

i. We may assume $u = V^+_n = V_\flat = \langle e_1, \ldots, e_n\rangle$. Since $\operatorname{Nrd} g = -1$ we have $g = g'\sigma$ where $g' \in \Gamma$ and $\sigma \in \widehat{\Gamma}$ is an involution such that $\sigma(e_i) = e_i, \sigma(f_i) = f_i$ for every $i < n$ and $\sigma(e_n) = f_n, \sigma(f_n) = e_n$. We have $\sigma(V^+_n) = V^-_n$. Then $g(u) = g(V^+_n) = g'(V^-_n) \in \mathcal{I}^-_n$ and therefore the element g cannot belong to the stabilizer of $u = V^+_n$ in $\widehat{\Gamma}$. This is a contradiction with the choice of g. Hence $g \in \Gamma$ in this case.

ii. In this case we may find an h-hyperbolic plane $H_2 \leqslant V$ which is orthogonal to v_{g_1}. Then there is a reflection $\rho \in \widehat{\Gamma}$ which corresponds to an anisotropic vector $e \in H_2$. The reflection ρ acts trivially on v_{g_1}. Hence $\gamma = \rho g$ is an appropriate element of Γ.

iii. We may assume $v^u_{g_2} = V^+_n$. Since $g(u) = u$ and $g(v_{g_2}) = v_{g_1}$ we have $g(v^u_{g_2}) = v^u_{g_1}$ (see Lemma 4.2). Since $\operatorname{sign}(v^u_{g_1}, v^u_{g_2}) = 1$ the spaces $v^u_{g_1}, v^u_{g_2}$ are in the same Γ-orbit (Proposition 3.2). But elements of $\widehat{\Gamma}$ which have reduced norm -1 change Γ-orbits of maximal totally isotropic spaces (as we have seen in the proof of **i**.). Hence the assumption $\operatorname{Nrd} g = -1$ leads us to a contradiction. Thus, $\gamma = g \in \Gamma$ is an appropriate element. □

Proposition 4.6. *If* $\dim v^u_{g_i} = n$ *and* $\operatorname{sign}(v^u_{g_1}, v^u_{g_2}) = -1$ *then the pairs* $(u, g_1(v))$ *and* $(u, g_2(v))$ *are in different* Γ*-orbits.*

Proof. Suppose $g(u) = u, g(g_2(v)) = g_1(v)$ for some $g \in \Gamma$. Then $g(v^u_{g_2}) = v^u_{g_1}$ and therefore $\operatorname{sign}(v^u_{g_1}, v^u_{g_2}) = 1$ (see the proof of Proposition 4.5, **iii**.). □

§5. Theorem 1 (The General Case)

Here we consider the General Case. Below V is an h-space of index n which satisfies the conditions **h1, h2** (we preserve the notations and assumptions of the previous chapter for $K, F, D, \star, V, h, \Gamma, \mathcal{I}_k, \mathcal{I}_l$).

For a given pair of integers $0 < k \leqslant l \leqslant n$ we define the set

$$X_{pq} = \{(p, q) \ \mid 0 \leqslant p \leqslant \min\{k, n-l,\},\ 0 \leqslant q \leqslant k-p\}.$$

Now let $v \in \mathcal{I}_k, u \leqslant \mathcal{I}_l$ where $k \leqslant l$ and let P_v, P_u be the stabilizers of v, u in Γ.

Theorem 1. *The double cosets* $P_u\gamma P_v$ *can be enumerated as follows:*

i) $\Gamma = \bigcup_{(p,q)\in X_{pq}} P_u\gamma_{pq}P_v$;

ii) $g \in P_u\gamma_{pq}P_v \Leftrightarrow \mathrm{d}_{\mathrm{in}}(g(v), u) = l-k+p+q$ *and* $I(u+g(v)) = q$.

Proof.

Lemma 5.1.

$$g_1, g_2 \in P_u\gamma P_v \Leftrightarrow v_{g_2} = \sigma(v_{g_1}) \quad \textit{for some} \quad \sigma \in \Gamma.$$

Proof. Let $g_1, g_2 \in P_u\gamma P_v$. Then $v_{g_1} = v_{p_1\gamma} = p_1(v_\gamma),\ v_{g_2} = v_{p_2\gamma} = p_2(v_\gamma)$ for some $p_1, p_2 \in P_u$. Hence $v_{g_1} = p_1p_2^{-1}(v_{g_2})$.

Now let $v_{g_2} = \sigma(v_{g_1})$ for some $\sigma \in \Gamma$. Then $g(u) = u, \quad g(g_2(v)) = g_1(v)$ for some $g \in \Gamma$ (Proposition 4.4). Thus, the pairs $(u, g_1(v))$ and $(u, g_2(v))$ are in the same Γ-orbit and therefore g_1, g_2 are in the same double coset $P_u\gamma P_v$ (recall, that there is a one-to-one correspondence between double cosets $P_u\gamma P_v$ and Γ-orbits of pairs (u', v') where $u' = g_1(u), v' = g_2(v)$ for some $g_1, g_2 \in \Gamma$; (see [8]). □

Remark 5.2. The proof of Lemma 5.1 implies that the implication

$$g_1, g_2 \in P_u\gamma P_v \Rightarrow v_{g_2} = \sigma(v_{g_1}) \ \text{ for some } \ \sigma \in \Gamma$$

of Lemma 5.1 also holds in the Special Case. The implication

$$g_1, g_2 \in P_u\gamma P_v \Leftarrow v_{g_2} = \sigma(v_{g_1}) \ \text{ for some } \ \sigma \in \Gamma$$

holds in the Special Case if and only if v_{g_1}, v_{g_2} satisfies one of the conditions of Proposition 4.5 (see Propositions 4.5, 4.6).

Let $g \in P_u\gamma P_v$ and let $v_g = u + g(v)$. Then Proposition 3.1 and Lemma 5.1 imply that the double coset $g \in P_u\gamma P_v$ is defined uniquely by $m + q = \dim v_g$ and $q = I(v_g)$. The dimension of a maximal isotropic subspace of v_g is equal to m and this dimension $\geqslant l = \dim u$. Put

$$p = m - l.$$

Then

$$l + p + q = \dim v_g \leqslant l + k$$

and, therefore, $p + q \leqslant k$. Hence the pair (m, q) is uniquely defined by the pair (p, q) of non-negative integers such that $p+q \leqslant k$. Thus we may mark $\gamma = \gamma_{pq}$. Moreover,

$$\mathrm{d_{in}}(u, g(v)) = \dim v_g - \min\{\dim u, \dim v\} = l + p + q - k = (l - k) + p + q.$$

Let us show that for every $p \leqslant \min\{k, n - l\}$ and $p + q \leqslant k$ there is a double coset $P_u\gamma_{pq}P_v$ which satisfies condition ii. We may decompose the space u in the following way

$$u = u' + u'_q + u'_{k-p-q},$$
$$\dim u'_q = q, \quad \dim u' = l - k + p, \quad \dim u'_{k-p-q} = k - p - q.$$

Since $q \leqslant k \leqslant l \leqslant n$ we may find a totally isotropic space $v'_q \leqslant V$ such that $\dim v'_q = q$ and

$$u'_q + v'_q = H_{2q}$$

and v'_q is orthogonal to u', u'_{k-p-q}. Further, one can find a totally isotropic space $v'_p \leqslant V$ such that $\dim v'_p = p$ and v'_p is orthogonal to $u + v'_q$. Now we have

$$\begin{aligned} u + v' &= u + v'_q + v'_p \\ &= u' + \underbrace{u'_{k-p-q} + v'_p + v'_q}_{:=v'} + u'_q = \underbrace{u' + u'_{k-p-q} + v'_p}_{\mathrm{rad}(u+v')} + \underbrace{u'_q + v'_q}_{H_{2q}}. \end{aligned} \tag{5.1}$$

Since v' is a totally isotropic space of the dimension k there exists an element $g \in \Gamma$ such that $g(v) = v'$. Then (5.1) implies that the space $v_g = u + g(v)$ satisfies the condition $\dim v_g = l + p + q$, $I(v_g) = q$ and therefore $g \in P_u\gamma_{pq}P_v$ for some element $\gamma_{pq} \in \Gamma$ which satisfies the condition ii. $\square$

§6. Theorem 2. The Special Case

We preserve the notation of the previous chapter. Note that since our proof is based on the general assumption of having a T-form, we make here the following restriction $\mathrm{char} K \neq 2$ (however, there is a different approach for the cases of split groups which allows us to avoid this restriction; see Comment 7.3).

Theorem 2. *Let* $\Gamma = \mathrm{SO}(V) = \mathrm{SO}_{2n}(K)$ *be a split orthogonal group of the dimension* $2n$. *The double cosets* $P_u\gamma P_v$ *can be numerated in the following way:*

i1) *If* $0 < n - l \leqslant k$ *then*

$$\Gamma = \Bigg(\bigcup_{\substack{0\leqslant p\leqslant n-l,\\ 0\leqslant q\leqslant k-p}} P_u\gamma_{pq}P_v\Bigg) \cup \Bigg(\bigcup_{q\leqslant k+l-n} P_u\gamma^-_{n-l\ q}P_v\Bigg).$$

i2) *If* $k < n - l$ *then*

$$\Gamma = \bigcup_{\substack{0\leqslant p\leqslant k,\\ 0\leqslant q\leqslant k-p}} P_u\gamma_{pq}P_v.$$

i3) *If* $l = n, k < n$ *then*

$$\Gamma = \bigcup_{q\leqslant k} P_u\gamma_q P_v.$$

i4) *If* $k = l = n$ *then*

$$\begin{cases} \Gamma = \bigcup_{0\leqslant q=2m\leqslant n} P_u\gamma_q P_v & \text{if} \quad \mathrm{sign}(v,u) = 1,\\ \Gamma = \bigcup_{1\leqslant q=2m+1\leqslant n} P_u\gamma_q P_v & \text{if} \quad \mathrm{sign}(v,u) = -1. \end{cases}$$

ii1) $g \in P_u\gamma_{pq}P_v$, $p \neq n-l \Leftrightarrow \mathrm{d_{in}}(g(v),u) = l-k+p+q$ *and* $I(u+g(v)) = q$, $g \in P_u\gamma_{n-l\ q}P_v \Leftrightarrow \mathrm{d_{in}}(g(v),u) = n-k+q$ *and* $I(u+g(v)) = q$, *and* $v^u_g \in \mathcal{I}^+_n$, $g \in P_u\gamma^-_{n-l\ q}P_v \Leftrightarrow \mathrm{d_{in}}(g(v),u) = n-k+q$ *and* $I(u+g(v)) = q$, *and* $v^u_g \in \mathcal{I}^-_n$.

ii2) $g \in P_u\gamma_{pq}P_v \Leftrightarrow \mathrm{d_{in}}(g(v),u) = l-k+p+q$ *and* $I(u+g(v)) = q$.

ii3) $g \in P_u\gamma_q P_v \Leftrightarrow \mathrm{d_{in}}(g(v),u) = n-k+q$ *and* $I(u+g(v)) = q$.

ii4) $g \in P_u\gamma_q P_v \Leftrightarrow \mathrm{d_{in}}(g(v),u) = q = I(u+g(v))$.

Proof.

i1) Let $0 < n-l \leqslant k$. Let $v_{g_1} = u + g_1(v), v_{g_2} = u + g_2(v)$ where $g_1, g_2 \in P_u\gamma P_v$. Then $v_{g_1} = \sigma(v_{g_2})$ for some $\sigma \in \Gamma$ (see Remark 5.2). Then we have $g(u) = u, g(g_2(v)) = g_1(v)$ for some $g \in \widehat{\Gamma}$ (Proposition 4.4). Hence

$$\mathrm{d_{in}}(u, g_1(v)) = \mathrm{d_{in}}(u, g_2(v)), \quad I(v_{g_1}) = I(v_{g_2}) = q \tag{6.1}$$

for some non-negative integer q. Also,

$$\dim v^u_{g_1} = \dim v^u_{g_2} = l + p \tag{6.2}$$

for some non-negative integer q. The equalities (6.1), (6.2) imply

$$p \leqslant n - l, \quad q \leqslant k - p, \quad \mathrm{d_{in}}(u, g_1(v)) = \mathrm{d_{in}}(u, g_2(v)) = l - k + p + q. \tag{6.3}$$

Suppose

$$\dim v^u_{g_i} = l + p < n. \tag{6.4}$$

Then the parameters p, q determine the double coset $P_u \gamma P_v$ uniquely, that is, if

$$\mathrm{d_{in}}(u, g(v)) = l - k + p + q, \quad I(v_g) = q$$

for some $g \in \Gamma$ then $g \in P_u \gamma P_v$ (Proposition 3.1, Proposition 4.5, Remark 5.2). Hence we may put $\gamma = \gamma_{pq}$.

For every p, q which satisfies the conditions $p \leqslant n - l, \quad q \leqslant k - p$ one can find a totally isotropic subspace v' such that $\dim v' = k, \mathrm{d_{in}}(u, v') = l - k + p + q, I(u + v') = q$ (see the proof of Theorem 1.) Since $n - l > 0$ we have $k \leqslant l < n$ and therefore there exists an element $g \in \Gamma$ such that $g(v) = v'$ (Proposition 3.1). Thus, for every pair p, q which satisfies the conditions $p \leqslant n - l, \quad q \leqslant k - p$ there is the corresponding double coset $P_u \gamma_{pq} P_v$.

Suppose

$$\dim v^u_{g_i} = l + p = n. \tag{6.5}$$

Hence $p = n - l$. We may assume

$$u = \langle e_1, \ldots, e_l \rangle, v^u_{g_1} = V_\flat = \langle e_1, \ldots, e_n \rangle, \quad v_{g_1} = \langle e_1, \ldots, e_n, f_1, \ldots, f_q \rangle, \quad l, q < n$$

and put $\gamma = \gamma_{n-l\ q}$. Let $g \in \Gamma$ be an element such that $\mathrm{d_{in}}(u, g(v)) = l - k + p + q, I(v_g) = q$. Then

$$g \in P_u \gamma_{n-l\ q} P_v \Leftrightarrow \mathrm{sign}(v^u_{g_1}, v^u_g) = 1$$

(see Remark 5.2). If $\mathrm{sign}(v^u_{g_1}, v^u_g) = -1$ and $g \in P_u \omega P_v$ for some $\omega \in \Gamma$ we put $\omega = \gamma^-_{n-l\ q}$. Note if $\mathrm{d_{in}}(u, g'(v)) = l - k + p + q$, $I(v_{g'}) = q$ for some $g' \in \Gamma$ and $\mathrm{sign}(v^u_{g_1}, v^u_{g'}) = -1$ then $\mathrm{sign}(v^u_g, v^u_{g'}) = 1$ and therefore $g' \in P_u \gamma_{n-l\ q} P_v$.

The same arguments as above show that double cosets $P_u\gamma_{n-l\ q}P_v$, $P_u\gamma^-_{n-l\ q}P_v$ exist for every $q \leqslant k-p = k-(n-l) = k+l-n$.

i2) Let $k < n-l$. Then for elements $g_1, g_2 \in P_u\gamma P_v$ which satisfy (6.3) we also have the inequality (6.4) (because $p \leqslant k < n-l$) and therefore we may use the arguments above for the case (6.4).

i3) Let $l = n, k < n$. Let $g \in P_u\gamma P_v$ and $v_g = u + g(v)$. Then $u = v_g^u$ and therefore $p = \dim v_g^u - \dim u = 0$. Propositions 3.1, 4.5 and Remark 5.2 imply that the double coset $P_u\gamma P_v$ is determined uniquely by the one parameter $q = \dim I(v_g)$. Thus we may put $\gamma = \gamma_q$. Note that for every $q \leqslant k$ one can find $g \in \Gamma$ such that $I(v_g) = q$ (the same argument as above). Hence we may numerate double cosets $P_u\gamma_q P_v$ by the set of non-negative integers $q \leqslant k$. Also, in this case

$$I(v_g) = q, \quad \mathrm{d_{in}}(u, g(v)) = n-k+q. \tag{6.6}$$

i4) Let $l = k = n$. We may assume

$$u = \langle e_1, \ldots, e_n \rangle = V_\flat.$$

Then in the same way as above one can see that the double cosets $P_u\gamma P_v$ can be numerated by one non-integer number $q \leqslant n$. However, the space v may belong to the different Γ-orbits. For $g \in \Gamma$ we have $\mathrm{sign}(u, g(v)) = (-1)^{\mathrm{d_{in}}(v,u)}$ (see Proposition 3.2) and therefore

$$\mathrm{d_{in}}(u, g(v)) = q = \begin{cases} 2m \leqslant n & \text{if } \ \mathrm{sign}(v,u) = 1, \\ 2m+1 \leqslant n & \text{if } \ \mathrm{sign}(v,u)) = -1. \end{cases} \tag{6.7}$$

Also, we can get every appropriate $0 \leqslant q = 2m \leqslant n$ (resp. $1 \leqslant q = 2m+1$) (indeed, one can take $u = V_\flat$ and $v = V^+ = V_\flat$ or $v = V^- = \langle e_1, \ldots, e_{n-1}, f_n \rangle$ and $g \in \Gamma$ such that

$$g(v) = \langle f_1, \ldots, f_{2m}, e_{2m+1}, \ldots, e_n \rangle,$$
$$\text{or} \quad g(v) = \langle f_1, \ldots, f_{2m}, e_{2m+1}, \ldots e_{n-1}, f_n \rangle.$$

ii. 1–4) follows from the definitions of the numeration and (6.3), (6.6), and (6.7). □

§7. Some comments

7.1. *Case of Unitary group.* If we change the group Γ from $\mathrm{SU}(V)$ to $U(V)$, the difference between the General Case and the Special Case will disappear (in the Special Case we would have $\Gamma = O_{2n}(K)$) and Theorem 1 holds for this Γ (the proof here is much easier: we can use the same arguments, based on Witt's theorem).

7.2 *Adherence of double cosets.* Here we did not touch the question of adherence of double cosets $\widetilde{P}_u\gamma_{pq}\widetilde{P}_v$ where $\widetilde{P}_u, \widetilde{P}_v$ are the corresponding parabolic subgroups of $\widetilde{\Gamma}$ such that $\widetilde{P}_v(K) = P_v, \widetilde{P}_u(K) = P_u$. We consider this question in the next paper.

7.3. *Split Case.* Let $\widetilde{\Gamma} = \mathrm{SO}, \mathrm{Sp}$ be a split algebraic group over a field K. Then we may describe the decomposition of $\Gamma = \cup P_u\gamma P_v$ using the language of root systems. The choice of representatives of double cosets can be obtained by the theory of double cosets of parabolic subgroups of Chevalley groups (see [5], Chapter 2.7). This method does not depend on the characteristic of the field K. Hence we can avoid the restriction for the Special Case. However, here we do not write down this proof and give only the interpretation of representatives of double cosets as elements of the corresponding Weyl group.

Example. Special Case. Let

$$V = V_\flat + V^\flat = H_{2n} = \langle e_1, \ldots, e_n, f_1, \ldots, f_n\rangle$$

(see Ch1). Then $\Gamma = \mathrm{SO}(V) = \widetilde{\Gamma}(K)$ where $\widetilde{\Gamma}$ is the simple algebraic group of type D_n over a field K which is split over K (here we assume that $\widetilde{\Gamma}$ is considered as a linear group in V and the basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ corresponds to weight vectors of the weights $\epsilon_1, \ldots, \epsilon_n, -\epsilon_n, \ldots, -\epsilon_1$ in the notations of [3].) Let W be the Weyl group of Γ. For an element $w \in W$ we may correspond an element $\dot{w} \in \Gamma$ (see [5]), in particular, for the roots

$$\alpha = \epsilon_i - \epsilon_j, \beta = \epsilon_i + \epsilon_j$$

we may identify the correspondend reflections w_α, w_β with the operators $\dot{w}_\alpha, \dot{w}_\beta \in \mathrm{SO}(V)$ which are defined in the following way

$$\dot{w}_\alpha(e_r) = \dot{w}_\beta(e_r) = e_r \ \dot{w}_\alpha(f_r) = \dot{w}_\beta(f_r) = f_r \quad \text{if } r \neq i, j,$$

$$\dot{w}_\alpha(e_i) = e_j, \quad \dot{w}_\alpha(e_j) = e_i, \quad \dot{w}_\beta(e_i) = f_j, \quad \dot{w}_\alpha(f_j) = e_i,$$

$$\dot{w}_\alpha(f_i) = f_j, \quad \dot{w}_\alpha(f_j) = f_i, \quad \dot{w}_\beta(f_i) = e_j, \quad \dot{w}_\alpha(e_j) = f_i.$$

We may assume $u=\langle e_1,\dots,e_l\rangle$. For the simplicity of notations we also assume $v=\langle e_1,\dots,e_k\rangle$ if $k<n$. If $k=l=n$ we may take $v=\langle e_1,\dots e_n\rangle=V_n^+$ or $v=\langle e_1,\dots e_{n-1},f_n\rangle=V_n^-$. Then we may chose the representatives of double cosets $P_u\gamma P_v$ among elements of the form $\dot{w}$.

1) Let $0<n-l\leqslant k$.

Put

$$\alpha_1=\epsilon_1-\epsilon_{l+1},\alpha_2=\epsilon_2-\epsilon_{l+2},\dots,\alpha_p=\epsilon_l-\epsilon_{l+p},$$

$$\mu_k=\epsilon_k-\epsilon_n,\dots,\mu_{q-k+1}=\epsilon_{k-q+1}-\epsilon_n,$$

$$\nu_k=\epsilon_k+\epsilon_n,\dots,\mu_{q-k+1}=\epsilon_{k-q+1}+\epsilon_n,$$

$$w_{pq}=w_{\alpha_1}w_{\alpha_2}\cdots w_{\alpha_p}(w_{\mu_k}w_{\nu_k})(w_{\mu_{k-1}}w_{\nu_{k-1}})\cdots(w_{\mu_{k-q+1}}w_{\nu_{k-q+1}})\in W.$$

Then

$$\dot{w}_{pq}(v)=\begin{cases}\langle e_{p+1},\dots,e_{k-q},e_{l+1},\dots,e_{l+p},f_{(k-q+1)},\dots f_k\rangle & \text{if } p+1\leqslant k-q,\\ \langle e_{l+1},\dots,e_{l+p},f_{(k-q+1)},\dots f_k\rangle & \text{if } p+1>k-q,\end{cases}$$

and therefore

$$\mathrm{d_{in}}(u,\dot{w}_{pq}(v))=l-k+p+q,\quad I(u+\dot{w}_{pq}(v))=q. \tag{7.1}$$

If $p=n-l$ put $\zeta^-=\epsilon_1-\epsilon_n,\zeta^+=\epsilon_1+\epsilon_n$ and

$$w^-=w_{\zeta^-}w_{\zeta^+},\ w^-_{n-l\ q}=w^-w_{n-l\quad q}$$

Then

$$\dot{w}^-_{pq}(v)$$
$$=\begin{cases}\langle e_{n-l+1},\dots,e_{k-q},e_{l+1},\dots,e_{n-1},f_n,f_{(k-q+1)},\dots,f_k\rangle & \text{if } n-l+1\leqslant k-q,\\ \langle e_{l+1},\dots,e_{n-1},f_n,f_{(k-q+1)},\dots,f_k\rangle & \text{if } n-l+1>k-q.\end{cases}$$

and therefore

$$\mathrm{d_{in}}(u,\dot{w}_{pq}(v))=n-k+q,\quad I(u+\dot{w}_{pq}(v))=q. \tag{7.2}$$

Now Theorem 2 (i1)) can be written in the form

$$\Gamma=\mathrm{SO}(V)=\Bigg(\bigcup_{\substack{0\leqslant p\leqslant n-l,\\ 0\leqslant q\leqslant k-p}}P_u\dot{w}_{pq}P_v\Bigg)\cup\Bigg(\bigcup_{q\leqslant k+l-n}P_u\dot{w}^-_{n-l,\ q}P_v\Bigg)$$

where $\dot{w}_{pq},\dot{w}^-_{n-l,\ q}$ satisfies (7.1), (7.2) (Theorem 2 (ii1)).

2) Let $k < n-l$. Then $p \leqslant k < n-l$.Then we only have representatives of the form $\dot{w}_{pq}$ (Theorem 2.(i2)):

$$\Gamma = \bigcup_{0\leqslant p\leqslant k, 0\leqslant q\leqslant k-p} P_u \dot{w}_{pq} P_v,$$

where $\dot{w}_{pq}$ satisfy (7.1), (7.2) (Theorem 2 (ii2)).

3) Let $l = n, k < n$ then $p = 0$ and we may put $\dot{w}_q = \dot{w}_{pq}$. Hence (Theorem 1(i3))

$$\Gamma = \bigcup_{q\leqslant k} P_u \dot{w}_q P_v$$

and $\mathrm{d_{in}}(u, \dot{w}_q(v)) = l - k + q,\ I(u + \dot{w}_q(v)) = q$ (Theorem 1(i3)).

4) Let $k = l = n$. Let $2m \leqslant n \leqslant 2m + 1$. Put

$$\mu'_1 = \epsilon_1 - \epsilon_2, \mu'_2 = \epsilon_3 - \epsilon_4, \dots, \mu'_m = \epsilon_{2m-1} - \epsilon_{2m},$$
$$\nu'_1 = \epsilon_1 + \epsilon_2, \nu'_2 = \epsilon_3 + \epsilon_4, \dots, \nu'_m = \epsilon_{2m-1} + \epsilon_{2m}.$$

For $q = 2m$ or $q = 2m + 1$ we put

$$w'_q = w_{\mu_1} w_{\nu_1} \cdots w_{\mu_m} w_{\nu_m}.$$

If $v = V^+$ we have $0 < q = 2m \leqslant n$ and

$$\dot{w}'_q(v) = \langle f_1, f_2, \dots, f_{2m-1}, f_{2m}, e_{2m+1}, \dots, e_n\rangle.$$

If $v = V_n^- = \langle e_1, \dots, e_{n-1}, f\rangle$ we have $1 < q = 2m + 1 \leqslant n$ and

$$\dot{w}'_q(v) = \langle f_1, f_2, \dots, f_{2m-1}, f_{2m}, e_{2m+1}, \dots, e_n\rangle.$$

Thus, (Theorem 2 (i4))

$$\begin{cases} \Gamma = \bigcup_{0\leqslant q=2m\leqslant n} P_u \dot{w}'_q P_v & \text{if } \mathrm{sign}(v, u) = 1, \\ \Gamma = \bigcup_{1\leqslant q=2m+1\leqslant n} P_u \dot{w}'_q P_v & \text{if } \mathrm{sign}(v, u) = -1. \end{cases}$$

and

$$\mathrm{d_{in}}(u, \dot{w}'_q(v)) = n - q, \quad I(u + \dot{w}'_q(v)) = q$$

(Theorem 2(ii4)).

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