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LOCAL-GLOBAL PRINCIPLE FOR GENERAL QUADRATIC AND GENERAL HERMITIAN GROUPS AND THE NILPOTENCE OF KH_1

Abstract. In this article we establish an analog of the Quillen–Suslin's local-global principle for the elementary subgroup of the general quadratic group and the general Hermitian group. We show that unstable K_1-groups of general Hermitian groups over module finite rings are nilpotent-by-abelian. This generalizes earlier results of A. Bak, R. Hazrat, and N. Vavilov.

Dedicated to the memory of late Professor Amit Roy

§1. Introduction

The vigorous study of general linear groups and more generally algebraic K-theory was stimulated in mid-sixties by the desire to solve Serre's problem on projective modules (*cf.* Faisceaux Algébriques Coherents, 1955). This prominent problem in commutative algebra asks whether *finitely generated projective modules over a polynomial ring over a field are free.* The beautiful book *Serre's problem on projective modules* by T. Y. Lam gives a comprehensive account of the mathematics surrounding Serre's problem and its solution. Later we see analogs of Serre's Problem for modules with forms and for other classical groups in the work of H. Bass, A. Suslin, L. N. Vaserstein, V. I. Kopeiko, R. Parimala and others in [11, 26, 28, 29, 37, 38]. In this current paper, we are interested in the context of modules with forms in certain problems related to Serre's Problem, *viz.* normality of the elementary subgroup of the full automorphism group, Suslin's local-global principle for classical-like groups, stabilization for K_1-functors of classical-like groups, and the structure of unstable K_1-groups of classical-like groups.

Difficulties one has in handling the quadratic version of Serre's Problem in characteristic 2 were first noted by Bass in [11]. In fact, in many cases it was difficult to handle classical groups over fields of characteristic 2,

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rather than classical groups over fields of char $\neq 2$. (For details see [19]). In 1969, A. Bak resolved this problem by introducing *form rings* and *form parameter*. He introduced the general quadratic group or Bak's unitary group, which covers many different types of classical-like groups. We also see some results in this direction in the work of Klein, Mikhalev, Vaserstein *et al.* in [24, 25, 43]. The concept of *form parameter* also appears in the work of K. McCrimmon, and plays an important role in his classification theory of Jordan algebras (*cf.* [27]), for details see ([22, footnote pg. 190]) and [23]. In his seminal work "K-theory of forms", Bak has established analog of many problems related to Serre's problem in a very explicit and rigorous manner. But, Bak's definition of the general quadratic group does not include many other types of classical-like groups, *viz.* odd dimensional orthogonal groups, exceptional groups of types E_6, E_7, E_8 *etc.* In 2000, G. Tang, in his Ph.D thesis, established analog of many results for the general Hermitian groups. Very recently, in 2005, Victor Petrov using Bak's concept of *doubly parametrized form parameter* has resolved this problem by introducing *odd unitary groups*, which also includes Bak's unitary and general Hermitian groups; *cf.* [30]. Also, he has established many analogous results for his group.

In 1976, D. Quillen came up with a localization method which was one of the main ingredients for the proof of Serre's problem (now widely known as Quillen–Suslin Theorem). Shortly after the original proof Suslin introduced the following matrix theoretic version of Quillen's local-global principle.

Suslin's Local-Global Principle: *Let R be a commutative ring with identity, X a variable and $\alpha(X) \in \mathrm{GL}(n, R[X])$ with $\alpha(0) = \mathrm{I}_n$, $n \geqslant 3$. If $\alpha_{\mathfrak{m}}(X) \in \mathrm{E}(n, R_{\mathfrak{m}}[X])$ for every maximal ideal $\mathfrak{m} \in \mathrm{Max}(R)$, then $\alpha(X) \in \mathrm{E}(n, R[X])$.*

Soon after he gave the K_1-analog of Serre's problem, which says,

for a polynomial ring in r variables over a field K elementary subgroup of $\mathrm{GL}(n, R)$, $n \geqslant 3$, coincides with the special linear group. i.e.,

$$\mathrm{E}(n, K[X_1, \dots, X_r]) = \mathrm{SL}(n, K[X_1, \dots, X_r]).$$

In connection with this theorem he proved the normality of the elementary subgroup $\mathrm{E}(n, A)$ in the general linear group $\mathrm{GL}(n, A)$, over a module finite ring A, for $n \geqslant 3$; (*cf.* [41]). Later analogous results for the symplectic and orthogonal groups were proven by Suslin and Kopeiko in [35] and [37] and by Fu An Li in [16], and for arbitrary Chevalley groups by Abe (*cf.* [1]) in the local case, and by Taddei (*cf.* [40]) in general. Later we see a simpler and more general treatment in works of Ambily, Bak, Hazrat, Petrov, Rao, Stavrova, Stepanov, Suzuki, Vavilov, and others.

We see generalization of the above local-global principle for the symplectic group in [26], and for the orthogonal group in [37]. The normality of the general quadratic groups is known from the work of A. Bak and N. Vavilov, *cf.* [8]. In [39], G. Tang has proved the normality property for the general Hermitian groups. In [12], we have shown that the question of normality of the elementary subgroup of the general linear group, symplectic and orthogonal groups, is equivalent to the above local-global principle, where the base ring is associative with identity and finite over its center. In that article above three classical groups were treated uniformly. Motivated by the work of A. Bak, R. G. Swan, L. N. Vaserstein and others, in [6], the author with A. Bak and R. A. Rao has established an analog of Suslin's local-global principle for the transvection subgroup of the automorphism group of projective, symplectic and orthogonal modules of global rank at least 1 and local rank at least 3, under the assumption that the projective module has constant local rank and that the symplectic and orthogonal modules are locally an orthogonal sum of a constant number of hyperbolic planes. In this article we have proved the equivalence of the local-global principle with the normality property. Since normality holds in the above cases, this establishes that the local global principle also holds. In fact, following Suslin–Vaserstein's method we establish an analogous local-global principle for the general quadratic and general Hermitian groups.

We treat these two groups uniformly and give explicit proofs of those results. We have overcome many technical difficulties which come in the Hermitian case due to the elements $a_1, \ldots, a_r$ (with respect to these elements we define the Hermitian groups). We assume $a_1 = 0$. The rigorous study of the general Hermitian groups can be found in [39]. In [8], we get an excellent survey on this area in a joint work of A. Bak and N. Vavilov. We refer to [20] for an alternative approach to localization, [21] for a general overview, and to [14] for relative cases. Also, for commutative rings with identity Quillen–Suslin's local-global principle is in the work of V. Petrov and A. Stavrova (*cf.* [31]), which covers, in particular, classical groups of Witt index $\geqslant 2$ or $\geqslant 3$, depending on the type.

In [12], it has been shown that the normality criterion of the elementary subgroup of the general linear group is equivalent to the above local-global principle. In this paper we establish the analogous local-global principle for the general quadratic and Hermitian group, and prove an equivalence. More precisely, we prove (§6, Theorem 6.7, and §7, Theorem 7.10)

Theorem 1 (Local-Global Principle). *Let k be a commutative ring with identity and R an associative k-algebra such that R is finite as a left k-module. If $\alpha(X) \in \mathrm{G}(2n, R[X], \Lambda[X])$, $\alpha(0) = \mathrm{I}_n$ and*

$$\alpha_{\mathfrak{m}}(X) \in \mathrm{E}(2n, R_{\mathfrak{m}}[X], \Lambda_{\mathfrak{m}}[X])$$

for every maximal ideal $\mathfrak{m} \in \mathrm{Max}(k)$, then

$$\alpha(X) \in \mathrm{E}(2n, R[X], \Lambda[X]).$$

Theorem 2. *Let k be a commutative ring with identity and R an associative k-algebra such that R is finite as a left k-module. Then for size at least 6 in the quadratic case and at least $2(r+3)$ in the Hermitian case:*

(Normality of the elementary subgroup)

$\equiv$

(Local-Global Principle)

To give a complete picture about the K_1-functors we shall shortly discuss the progress in the stabilization problem for K_1-functors. The study of this problem first appeared in the work of Bass–Milnor–Serre, and then we see it in the work by A. Bak, M. Stein, L. N. Vaserstein, and others for the symplectic, orthogonal and general quadratic groups. For details *cf.* [2,36,42–44]. In 1998, R. A. Rao, and W. van der Kallen studied this problem for the linear groups over an affine algebra in [32]. The result settled for the general quadratic and the general Hermitian groups by A. Bak, G. Tang and V. Petrov in [5] and [4]. The result by Bak–Petrov–Tang has been improved by Sergei Sinchuk, (*cf.* [34]). It has been observed that over a regular affine algebra Vaserstein's bounds for the stabilization can be improved for the transvection subgroup of the full automorphism group of projective, and symplectic modules. But they cannot be improved for the orthogonal case in general. For details *cf.* [33]. We refer to the recent breakthrough result by J. Fasel, R. A. Rao, and R. G. Swan ([15, Corollary 7.7]). A very recent result of Weibo Yu gives a similar bound for the odd unitary groups, (*cf.* [46]). In this paper we don't prove any new result in this direction.

Though the study of stability for K_1-functors started in mid-sixties, initially the structure of K_1-group below the level of stable range was not much studied. In 1991, A. Bak showed that the group $\mathrm{GL}(n, R)/\mathrm{E}(n, R)$

is nilpotent-by-abelian for $n \geqslant 3$; (*cf.* [3]). In [17], R. Hazrat proved the similar result for the general quadratic groups over module finite rings. The paper of Hazrat and Vavilov [18] redoes this for ordinary classical Chevalley groups (that is types A, C, and D) and then extends it further to the exceptional Chevalley groups (that is types E, F, and G). They have shown the following: Let Φ be a reduced irreducible root system of rank $\geqslant 2$ and R be a commutative ring such that its Bass–Serre dimension $\delta(R)$ is finite. Then for any Chevalley group $\mathrm{G}(\Phi, R)$ of type Φ over R the quotient $\mathrm{G}(\Phi, R)/\mathrm{E}(\Phi, R)$ is nilpotent-by-abelian. In particular, $\mathrm{K}_1(\Phi, R)$ is nilpotent of class at most $\delta(R)+1$. They use the localization-completion method of A. Bak in [3]. In [6], the author with Bak and Rao gave a uniform proof for the transvection subgroup of the full automorphism group of projective, symplectic and orthogonal modules of global rank at least 1 and local rank at least 3. Our method of proof shows that for classical groups the localization part suffices. Recently, in (*cf.* [7]) Bak, Vavilov and Hazrat proved the relative case for the unitary and Chevalley groups. But, to my best knowledge, so far there is no definite result for the general Hermitian groups. I observe that using the above local-global principle, arguing as in [6], it follows that the unstable K_1 of general Hermitian group is nilpotent-by-abelian. We follow the line of Theorem 4.1 in [6]. More precisely, we prove (Theorem 8.1)

Theorem 3. *For the general Hermitian group of large size over a commutative ring R with identity, the quotient group*

$$\mathrm{SH}(2n, R, a_1, \dots, a_r)/\mathrm{EH}(2n, R, a_1, \dots, a_r)$$

is nilpotent for $n \geqslant r + 3$.

We conclude with a brief description of the organization of the rest of the paper. Section 1 of the paper serves as an introduction. In Section 2 we recall the notion of form rings, in Section 3 we discuss general quadratic groups over form rings and their elementary subgroups, in Section 4 we introduce general Hermitian groups and their elementary subgroups, Section 5 provides preliminary results regarding the groups above, in Section 6 we establish the local-global principle for the elementary subgroup of the general quadratic and general Hermitian group, and in Section 7 we prove equivalence of normality of the elementary subgroup and the local-global principle for the elementary subgroup. Finally, Section 8 culminates with the proof of the nilpotent by abelian structure of non-stable K_1 of the general Hermitian group.

§2. Form Rings

Definition. *Let us first recall the concept of Λ-quadratic forms introduced by A. Bak in his Ph.D. thesis (cf.* [2]*) in order to overcome the difficulties that arise for the characteristic 2 cases.*

Let R be an (not necessarily commutative) associative ring with identity, and with involution $- : R \to R$, $a \mapsto \overline{a}$. Let $\lambda \in C(R)$ = center of R be an element with the property $\lambda\overline{\lambda} = 1$. We define additive subgroups of R

$$\Lambda_{\max} = \{a \in R \,|\, a = -\lambda\overline{a}\} \quad \& \quad \Lambda_{\min} = \{a - \lambda\overline{a} \,|\, a \in R\}.$$

One checks that $\Lambda_{\max}$ and $\Lambda_{\min}$ are closed under the conjugation operation $a \mapsto \overline{x}ax$ for any $x \in R$. A λ-form parameter on R is an additive subgroup Λ of R such that $\Lambda_{\min} \subseteq \Lambda \subseteq \Lambda_{\max}$, and $\overline{x}\Lambda x \subseteq \Lambda$ for all $x \in R$. A pair (R, Λ) is called a form ring.

Examples:

(1) $\Lambda_{\min} = 0 \Leftrightarrow \lambda = 1$, and involution is trivial. In particular, $\Lambda = 0 \Leftrightarrow \lambda = 1$, involution is trivial, and R is commutative.
(2) If R is a commutative integral domain, and involution is trivial, then $\lambda^2 = 1 \Leftrightarrow \lambda = \pm 1$. If $\lambda = 1$ and $\mathrm{char} R \neq 2$, then $\Lambda_{\max} = 0$, and so 0 is the only form parameter. If $\lambda = -1$ and $\mathrm{char} R \neq 2$, then Λ contains $2R$, and closed under multiplication by squares. If R is a field, then we get $\Lambda = R$. If R is a $\mathbb{Z}$, then we get $\Lambda = 2\mathbb{Z}$ and $\mathbb{Z}$. If $\mathrm{char} R = 2$, then R^2 is a subring of R, and $\Lambda = R^2$-submodules of R.
(3) The ring of $n \times n$ matrices $(\mathrm{M}(n, R), \Lambda_n)$ is a form ring.

Remark. An earlier version of λ-form parameter due to K. McCrimmon plays an important role in his classification theory of Jordan algebras. He defined it for the wider class of alternative rings (not just associative rings), but for associative rings it is a special case of Bak's concept. (For details, *cf.* N. Jacobson; Lectures on Quadratic Jordan Algebras, TIFR, Bombay 1969). The excellent work of Hazrat–Vavilov in [19] is a very good source to understand the historical motivation behind the concept of form rings. And, an excellent source to understand the theory of form rings is the book [22] by A. J. Hahn and O. T. O'Meara.

§3. General Quadratic Group

Let V be a right R-module and $\mathrm{GL}(V)$ the group of all R-linear automorphisms of V. A map $f : V \times V \to R$ is called *sesqulinear form* if $f(u+v, z+w) = f(u,z) + f(u,w) + f(v,z) + f(v,w)$ and $f(ua, vb) = \overline{a}f(u,v)b$ for all $u,v \in V$, $a,b \in R$. We define Λ-*quadratic form* q on V, and associated λ-*Hermitian form* and as follows:

$$q : V \to R/\Lambda, \quad \text{given by} \quad q(v) = f(v,v) + \Lambda, \quad \text{and}$$

$$h : V \times V \to R, \quad \text{given by} \quad h(u,v) = f(u,v) + \lambda\overline{f(v,u)}.$$

A Quadratic Module over (R, Λ) is a triple (V, h, q).

Definition. *"Bak's Unitary Groups" or "The Unitary Group of a Quadratic Module" or "General Quadratic Group"* $\mathrm{GQ}(V, q, h)$ *is defined as follows:*

$$\mathrm{GQ}(V,q,h) \;=\; \{\alpha \in \mathrm{GL}(V) \mid h(\alpha u, \alpha v) = h(u,v),\ q(\alpha v) = q(v)\}.$$

Examples: Traditional Classical Groups

(1) By taking $\Lambda = \Lambda_{\max} = R$, $\lambda = -1$, and trivial involution we get the symplectic group $\mathrm{GQ}(2n, R, \Lambda) = \mathrm{Sp}(2n, R)$.
(2) By taking $\Lambda = \Lambda_{\min} = 0$, $\lambda = 1$, trivial involution we get the quadratic or the orthogonal group $\mathrm{GQ}(2n, R, \Lambda) = \mathrm{O}(2n, R)$.
(3) For the general linear group, let R^o be the ring opposite to R, and $R^e = R \oplus R^o$. Define involution as follows: $(x, y^o) \mapsto (y, x^o)$. Let $\lambda = (1, 1^o)$ and $\Lambda = \{(x, -x^o) \mid x \in R\}$. Then identify

$$\mathrm{GQ}(2n, R^e, \Lambda) = \big\{(g, g^{-1}) \,|\, g \in \mathrm{GL}(n,R)\big\}$$

with $\mathrm{GL}(n, R)$.

Free Case: Let V be a free right R-module of rank $2n$ with ordered basis $e_1, e_2, \ldots, e_n, e_{-n}, \ldots, e_{-2}, e_{-1}$. Consider the sesqulinear form $f : V \times V \longrightarrow R$, defined by $f(u,v) = \overline{u}_1 v_{-1} + \cdots + \overline{u}_n v_{-n}$. Let h be Hermitian form, and q be the Λ-quadratic form defined by f. So, we have

$$h(u,v) = \overline{u}_1 v_{-1} + \cdots + \overline{u}_n v_{-n} + \lambda\overline{u}_{-n} v_n + \cdots + \lambda\overline{u}_{-1} v_1,$$

$$q(u) = \Lambda + \overline{u}_1 u_{-1} + \cdots + \overline{u}_n u_{-n}.$$

Using this basis we can identify $\mathrm{GQ}(V, h, q)$ with a subgroup of $\mathrm{GL}(2n, R)$ of rank $2n$. We denote this subgroup by $\mathrm{GQ}(2n, R, \Lambda)$.

By fixing a basis $e_1, e_2, \dots, e_n, e_{-1}, e_{-2}, \dots, e_{-n}$, we define the form

$$\psi_n = \begin{pmatrix} 0 & \lambda \mathrm{I}_n \\ \mathrm{I}_n & 0 \end{pmatrix}.$$

Hence, $\mathrm{GQ}(2n, R, \Lambda) = \{\sigma \in \mathrm{GL}(2n, R, \Lambda) \mid \overline{\sigma}\psi_n \sigma = \psi_n\}$.

For $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2n, R, \Lambda)$, one can show that $\sigma \in \mathrm{GQ}(2n, R, \Lambda)$ ($\alpha, \beta, \gamma, \delta$ are $n \times n$ block matrices) if and only if $\overline{\gamma}\alpha, \overline{\delta}\beta \in \Lambda$. For more details see ([2, 3.1 and 3.4]).

A typical element in $\mathrm{GQ}(2n, R, \Lambda)$ is denoted by a $2n \times 2n$ matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, where $\alpha, \beta, \gamma, \delta$ are $n \times n$ block matrices.

There is a standard embedding, $\mathrm{GQ}(2n, R, \Lambda) \to \mathrm{GQ}(2n+2, R, \Lambda)$, given by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & 1 & 0 & 0 \\ \gamma & 0 & \delta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

called the *stabilization* map. This allows us to identify $\mathrm{GQ}(2n, R, \Lambda)$ with a subgroup in $\mathrm{GQ}(2n+2, R, \Lambda)$.

Elementary Quadratic Matrices: Let ρ be the permutation, defined by $\rho(i) = n + i$ for $i = 1, \dots, n$. Let e_{ij} be the matrix with 1 in the ij-th position and 0's elsewhere. For $a \in R$, and $1 \leqslant i, j \leqslant n$, we define

$$q\varepsilon_{ij}(a) = \mathrm{I}_{2n} + ae_{ij} - \overline{a}e_{\rho(j)\rho(i)} \quad \text{for } i \neq j,$$

$$qr_{ij}(a) = \begin{cases} \mathrm{I}_{2n} + ae_{i\rho(j)} - \lambda\overline{a}e_{j\rho(i)} & \text{for } i \neq j, \\ \mathrm{I}_{2n} + ae_{\rho(i)j} & \text{for } i = j, \end{cases}$$

$$ql_{ij}(a) = \begin{cases} \mathrm{I}_{2n} + ae_{\rho(i)j} - \overline{\lambda}\overline{a}e_{\rho(j)i} & \text{for } i \neq j, \\ \mathrm{I}_{2n} + ae_{\rho(i)j} & \text{for } i = j. \end{cases}$$

(Note that for the second and third type of elementary matrices, if $i = j$, then we get $a = -\lambda\overline{a}$, and hence it forces that $a \in \Lambda_{\max}(R)$. One checks that these above matrices belong to $\mathrm{GQ}(2n, R, \Lambda)$; *cf.* [2].)

n-th Elementary Quadratic Group $\mathrm{EQ}(2n, R, \Lambda)$**:** The subgroup generated by $q\varepsilon_{ij}(a)$, $qr_{ij}(a)$ and $ql_{ij}(a)$, for $a \in R$ and $1 \leqslant i, j \leqslant n$.

It is clear that the stabilization map takes generators of $\mathrm{EQ}(2n, R, \Lambda)$ to the generators of $\mathrm{EQ}(2(n+1), R, \Lambda)$.

Commutator Relations: There are standard formulas for the commutators between quadratic elementary matrices. For details we refer [2, Lemma 3.16], and [17, §2]. In later sections we shall repeatedly use those relations.

§4. Hermitian Group

We assume that Λ is a λ-form parameter on R. For a matrix $M = (m_{ij})$ over R we define $\overline{M} = (\overline{m}_{ij})^t$. For $a_1, \ldots, a_r \in \Lambda$ and $n > r$ let

$$A_1 = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & a_{r-1} & 0 \\ 0 & \cdots & 0 & 0 & a_r \end{pmatrix} = [a_1, \ldots, a_r]$$

denote the diagonal matrix whose ii-th diagonal coefficient is a_i. Let $A = A_1 \perp \mathrm{I}_{n-r}$. We define the forms

$$\psi_n^h = \begin{pmatrix} A & \lambda \mathrm{I}_n \\ \mathrm{I}_n & 0 \end{pmatrix}, \qquad \psi_n^q = \begin{pmatrix} 0 & \lambda \mathrm{I}_n \\ \mathrm{I}_n & 0 \end{pmatrix}.$$

Definition: General Hermitian Group of the elements $a_1, \ldots, a_r$ is defined as follows: $\mathrm{GH}(2n, R, a_1, \ldots, a_r, \Lambda)$: The group generated by the all non-singular $2n \times 2n$ matrices

$$\{\sigma \in \mathrm{GL}(2n, R) \,|\, \overline{\sigma} \psi_n^h \sigma = \psi_n^h\}.$$

As before, there is an obvious embedding

$$\mathrm{GH}(2n, R, a_1, \ldots, a_r, \Lambda) \hookrightarrow \mathrm{GH}(2n+2, R, a_1, \ldots, a_r, \Lambda).$$

To define **elementary Hermitian matrices**, we need to consider the set $C = \{(x_1, \ldots, x_r)^t \in (R^r)^t \,|\, \sum_{i=1}^{r} \overline{x}_i a_i x_i \in \Lambda_{\min}(R)\}$ for $a_1, \ldots, a_r$ as above. In order to overcome the technical difficulties caused by the elements $a_1, \ldots, a_r$, we shall finely partition a typical matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ of

$\mathrm{GH}(2n, R, a_1, \ldots, a_r, \Lambda)$ into the form

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \beta_{11} & \beta_{12} \\ \alpha_{21} & \alpha_{22} & \beta_{21} & \beta_{22} \\ \gamma_{11} & \gamma_{12} & \delta_{11} & \delta_{12} \\ \gamma_{21} & \gamma_{22} & \delta_{21} & \delta_{22} \end{pmatrix}$$

where $\alpha_{11}, \beta_{11}, \gamma_{11}, \delta_{11}$ are $r \times r$ matrices, $\alpha_{12}, \beta_{12}, \gamma_{12}, \delta_{12}$ are $r \times (n-r)$ matrices, $\alpha_{21}, \beta_{21}, \gamma_{21}, \delta_{21}$ are $(n-r) \times r$ matrices, and $\alpha_{22}, \beta_{22}, \gamma_{22}, \delta_{22}$ are $(n-r) \times (n-r)$ matrices. By ([39, Lemma 3.4]),

$$\text{the columns of } \alpha_{11} - \mathrm{I}_r, \alpha_{12}, \beta_{11}, \beta_{12}, \overline{\beta}_{11}, \overline{\beta}_{21}, \overline{\delta}_{11} - \mathrm{I}_r, \overline{\delta}_{21} \in C. \tag{1}$$

It is a straightforward check that the subgroup of $\mathrm{GH}(2n, R, a_1, \ldots, a_r, \Lambda)$ consisting of

$$\left\{ \begin{pmatrix} \mathrm{I}_r & 0 & 0 & 0 \\ 0 & \alpha_{22} & 0 & \beta_{22} \\ 0 & 0 & \mathrm{I}_r & 0 \\ 0 & \gamma_{22} & 0 & \delta_{22} \end{pmatrix} \in \mathrm{GH}(2n, R, a_1, \ldots, a_r) \right\}$$

$$\cong \mathrm{GH}(2(n-r), R, a_1, \ldots, a_r, \Lambda).$$

Elementary Hermitian Matrices: The first three kinds of generators are taken for the most part from $\mathrm{GQ}(2(n-r), R, \Lambda)$, which is embedded, as above, as a subgroup of $\mathrm{GH}(2n, R)$ and the last two kinds are motivated by the result (1) concerning the column of a matrix in $\mathrm{GH}(2n, R)$. For $a \in R$, we define

$$h\varepsilon_{ij}(a) = \mathrm{I}_{2n} + ae_{ij} - \overline{a}e_{\rho(j)\rho(i)} \quad \text{for } r+1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n, i \neq j,$$

$$hr_{ij}(a) = \begin{cases} \mathrm{I}_{2n} + ae_{i\rho(j)} - \lambda\overline{a}e_{j\rho(i)} & \text{for } r+1 \leqslant i, j \leqslant n, i \neq j \\ \mathrm{I}_{2n} + ae_{i\rho(j)} & \text{for } r+1 \leqslant i, j \leqslant n, i = j, \end{cases}$$

$$hl_{ij}(a) = \begin{cases} \mathrm{I}_{2n} + ae_{\rho(i)j} - \overline{\lambda}\overline{a}e_{\rho(j)i} & \text{for } 1 \leqslant i, j \leqslant n, i \neq j \\ \mathrm{I}_{2n} + ae_{\rho(i)j} & \text{for } 1 \leqslant i, j \leqslant n, i = j. \end{cases}$$

(Note that for the second and third type of elementary matrices, if $i = j$, then we get $a = -\lambda\overline{a}$, and hence it forces that $a \in \Lambda_{\max}(R)$). One checks that the above matrices belong to $\mathrm{GH}(2n, R, a_1, \ldots, a_r, \Lambda)$; *cf.* [39].

For $\zeta = (x_1, \ldots, x_r)^t \in C$, let $\zeta_f \in R$ be such that $\zeta_f + \lambda\overline{\zeta}_f = \sum_{i=1}^{r} \overline{x}_i a_i x_i$. (The element ζ_f is not unique in general). We define

$$hm_i(\zeta) = \begin{pmatrix} \mathrm{I}_r & \alpha_{12} & 0 & 0 \\ 0 & \mathrm{I}_{n-r} & 0 & 0 \\ 0 & -\overline{A}_1\alpha_{12} & \mathrm{I}_r & 0 \\ 0 & \gamma_{22} & -\overline{\alpha}_{12} & \mathrm{I}_{n-r} \end{pmatrix}$$

for $\zeta \in C$ and $r+1 \leqslant i \leqslant n$ to be the $2n \times 2n$ matrix, where α_{12} is the $r \times (n-r)$ matrix with ζ as its $(i-r)$th column and all other column's zero, and γ_{22} is the $(n-r) \times (n-r)$ matrix with $\overline{\zeta}_f$ in $(i-r, i-r)$th position and 0's elsewhere. Let e_k denote the column vector of length $(n-r)$ with 1 in the kth position and 0's elsewhere, and $e_{t\,s}$ denote a $(n-r) \times (n-r)$ matrix with 1 in the tsth position and 0's elsewhere.

As above, we define

$$hr_i(\zeta) = \begin{pmatrix} \mathrm{I}_r & 0 & 0 & \beta_{12} \\ 0 & \mathrm{I}_{n-r} & -\lambda\overline{\beta}_{12} & \beta_{22} \\ 0 & 0 & \mathrm{I}_r & -\overline{A}_1\beta_{12} \\ 0 & 0 & 0 & \mathrm{I}_{n-r} \end{pmatrix}$$

for $\zeta \in C$ and $r+1 \leqslant i \leqslant n$ to be a $2n \times 2n$ matrix, where β_{12} is the $r \times (n-r)$ matrix with ζ as its $(i-r)$th column and all other column's zero, and β_{22} is the $(n-r) \times (n-r)$ matrix with $\lambda\overline{\zeta}_f$ in $(i-r, i-r)$th position and 0's elsewhere.

Note that if $\eta = e_{pq}(a)$ is an elementary generator in $\mathrm{GL}(s, R)$, then the matrix $(\mathrm{I}_{n-s} \perp \eta \perp \mathrm{I}_{n-s} \perp \eta^{-1}) = h\varepsilon_{ij}(a)$. It has been shown in [39] (§5) that each of the above matrices is in $\mathrm{GH}(2n, R, a_1, \ldots, a_r, \Lambda)$.

Definition: nth Elementary Hermitian Group of the elements $a_1, \ldots, a_r$; $\mathrm{EH}(2n, R, a_1, \ldots, a_r, \Lambda)$: The group generated by $h\varepsilon_{ij}(a)$, $hr_{ij}(a)$, $hl_{ij}(a)$, $hm_i(\zeta)$ and $hr_i(\zeta)$, for $a \in R$, $\zeta \in C$ and $1 \leqslant i, j \leqslant n$.

The stabilization map takes generators of $\mathrm{EH}(2n, R, a_1, \ldots, a_r, \Lambda)$ to the generators of $\mathrm{EH}(2(n+1), R, a_1, \ldots, a_r, \Lambda)$.

Commutator Relations: There are standard formulas for the commutators between quadratic elementary matrices. For details we refer [39].

§5. Preliminaries and Notations

Blanket Assumption: We always assume that $2n \geqslant 6$ and $n > r$ while dealing with the Hermitian case. We do not want to put any restriction on the elements of C. Therefore we assume that $a_i \in \Lambda_{\min}(R)$ for $i = 1, \dots, r$, as in that case $C = R^r$. We always assume $a_1 = 0$.

Notation 5.1. In the sequel $\mathrm{M}(2n, R)$ will denote the set of all $2n \times 2n$ matrices. By $\mathrm{G}(2n, R, \Lambda)$ we shall denote either the quadratic group $\mathrm{GQ}(2n, R, \Lambda)$ or the Hermitian group $\mathrm{GH}(2n, R, a_1, \dots, a_r, \Lambda)$ of size $2n \times 2n$. By $\mathrm{S}(2n, R, \Lambda)$ we shall denote respective subgroups $\mathrm{SQ}(2n, R, \Lambda)$ or $\mathrm{SH}(2n, R, a_1, \dots, a_r, \Lambda)$ with matrices of determinant 1, in the case when R will be commutative. Then, by $\mathrm{E}(2n, R, \Lambda)$ we shall denote the corresponding elementary subgroups $\mathrm{EQ}(2n, R, \Lambda)$ and $\mathrm{EH}(2n, R, a_1, \dots, a_r, \Lambda)$. To treat uniformly we denote the elementary generators of $\mathrm{EQ}(2n, R, \Lambda)$, and the first three types of elementary generators of $\mathrm{EH}(2n, R, \Lambda)$ by $\vartheta_{ij}(\star)$, for some $\star \in R$. To express the last two types of generators of $\mathrm{EH}(2n, R, \Lambda)$ we shall use the notation $\vartheta_i(\star)$, where $\star$ is a column vector of length r defined over the ring R, *i.e.*, we will have two types of elementary generators, namely ϑ_{ij}(ring element) and ϑ_i(column vector). Let $\Lambda[X]$ denote the λ-form parameter on $R[X]$ induced from (R, Λ), *i.e.*, λ-form parameter on $R[X]$ generated by Λ, *i.e.*, the smallest form parameter on $R[X]$ containing Λ. Let Λ_s denote the λ-form parameter on R_s induced from (R, Λ).

For any column vector $v \in (R^{2n})^t$ we define the row vectors $\widetilde{v}_q = \overline{v}^t \psi_n^q$ and $\widetilde{v}_h = \overline{v}^t \psi_n^h$.

Definition 5.2. We define the map $\mathrm{M} : (R^{2n})^t \times (R^{2n})^t \to \mathrm{M}(2n, R)$ and the inner product $\langle \, , \rangle$ as follows:

$$\begin{aligned}
\mathrm{M}(v, w) &= v.\widetilde{w}_q - \overline{\lambda}\overline{w}.\widetilde{v}_q, \quad \text{when } \mathrm{G}(2n, R) = \mathrm{GQ}(2n, R, \Lambda) \\
&= v.\widetilde{w}_h - \overline{\lambda}\overline{w}.\widetilde{v}_h, \quad \text{when } \mathrm{G}(2n, R) = \mathrm{GH}(2n, R, a_1, \dots, a_r, \Lambda), \\
\langle v, w \rangle &= \widetilde{v}_q.w, \quad \text{when } \mathrm{G}(2n, R) = \mathrm{GQ}(2n, R, \Lambda) \\
&= \widetilde{v}_h.w, \quad \text{when } \mathrm{G}(2n, R) = \mathrm{GH}(2n, R, a_1, \dots, a_r, \Lambda).
\end{aligned}$$

Note that the elementary generators of the both groups $\mathrm{EQ}(2n, R)$ and $\mathrm{EH}(2n, R)$ are of the form $\mathrm{I}_{2n} + \mathrm{M}(\star_1, \star_2)$ for suitable chosen standard basis vectors.

We recall the following well known facts:

Lemma 5.3. (*cf.* [2,39]) *The group* $\mathrm{E}(2n, R, \Lambda)$ *is perfect for* $n \geqslant 3$ *in the quadratic case, and for* $n \geqslant r+3$ *in the Hermitian case, i.e.,*

$$[\mathrm{E}(2n, R, \Lambda), \mathrm{E}(2n, R, \Lambda)] = \mathrm{E}(2n, R, \Lambda).$$

Lemma 5.4. (Splitting property): *For all elementary generators of the general quadratic group* $\mathrm{GQ}(2n, R, \Lambda)$ *and for the first three types elementary generators of the Hermitian group* $\mathrm{GH}(2n, R, a_1, \ldots, a_r, \Lambda)$ *we have:*

$$\vartheta_{ij}(x+y) = \vartheta_{ij}(x)\vartheta_{ij}(y)$$

for all $x, y \in R$.

For the last two types of elementary generators of Hermitian group we have the following relation:

$$hm_i(\zeta)hm_i(\xi) = hm_i(\zeta+\xi)hl_{ii}(\bar{\zeta}_f + \bar{\xi}_f + \bar{\zeta}\bar{A}_1\xi - \overline{(\zeta+\xi)}_f),$$

$$hr_i(\zeta)hr_i(\xi) = hr_i(\zeta+\xi)hr_{ii}((\zeta+\xi)_f - \xi_f - \zeta_f - \bar{\xi}A_1\zeta).$$

Proof. See [2, pp. 43–44, Lemma 3.16] for the $\mathrm{GQ}(2n, R, \Lambda)$ and [39, Lemma 8.2] for the group $\mathrm{GH}(2n, R, a_1, \ldots, a_r, \Lambda)$. □

Lemma 5.5. *Let* G *be a group, and* a_i, $b_i \in G$, *for* $i = 1, \ldots, n$. *Then for* $r_i = \prod_{j=1}^{i} a_j$, *we have* $\prod_{i=1}^{n} a_i b_i = \prod_{i=1}^{n} r_i b_i r_i^{-1} \prod_{i=1}^{n} a_i$.

Notation 5.6. By $\mathrm{G}(2n, R[X], \Lambda[X], (X))$ we shall mean the group of all quadratic and Hermitian matrices over $R[X]$ which are I_n modulo (X).

Lemma 5.7. *The group* $\mathrm{G}(2n, R[X], \Lambda[X], (X)) \cap \mathrm{E}(2n, R[X], \Lambda[X])$ *is generated by the elements of the types* $\varepsilon\vartheta_{ij}(\star_1)\varepsilon^{-1}$ *and* $\varepsilon\vartheta_i(\star_2)\varepsilon^{-1}$, *where* $\varepsilon \in \mathrm{E}(2n, R, \Lambda)$, $\star_1 \in R[X]$, $\star_2 \in ((R[X])^{2n})^t$ *with both* $\vartheta_{ij}(\star_1)$ *and* $\vartheta_i(\star_2)$ *congruent to* I_{2n} *modulo* (X).

We give a proof of this Lemma for the Hermitian group. The proof for the quadratic case is similar, but easier.

Proof of Lemma 5.7. Let $a_1(X), \ldots, a_r(X)$ be r elements in the polynomial ring $R[X]$ with respect to which we are considering the Hermitian group $\mathrm{GH}(2n, R[X], a_1(X), \ldots, a_r(X), \Lambda[X])$.

Let $\alpha(X) \in \mathrm{EH}(2n, R[X], a_1(X), \ldots, a_r(X), \Lambda[X])$ be such that $\alpha(X)$ is congruent to I_{2n} modulo (X). Then we can write $\alpha(X)$ as a product of elements of the form $\vartheta_{ij}(\star_1)$, where $\star_1$ is a polynomial in $R[X]$, and of the form $\vartheta_i(\star_2)$, where $\star_2$ is a column vector of length r defined over $R[X]$.

We write each $\star_1$ as a sum of a constant term and a polynomial which is identity modulo (X). Hence by using the splitting property described in Lemma 5.4 each elementary generator $\vartheta_{ij}(\star_1)$ of first three type can be written as a product of two such elementary generators with the left one defined on R and the right one defined on $R[X]$ which is congruent to I_{2n} modulo (X).

For the last two types of elementary generators we write each vector $\star_2$ as a sum of a column vector defined over the ring R and a column vector defined over $R[X]$ which is congruent to the zero vector of length r modulo (X). In this case, as shown in Lemma 5.4, we get one extra term involving elementary generator of the form hl_{ii} or hr_{ii}. But that extra term is one of the generator of first three types. And then we can split that term again as above. Therefore, $\alpha(X)$ can be expressed as a product of following types of elementary generators:

$$\vartheta_{ij}(\star_1(0))\vartheta_{ij}(X\star_1) \text{ with } \star_1(0) \in R \text{ and } \vartheta_{ij}(X\star_1) = \mathrm{I}_{2n} \text{ modulo } (X),$$

$$\vartheta_{i}(\star_2(0))\vartheta_{i}(X\star_2) \text{ with } \star_2(0) \in R \text{ and } \vartheta_{i}(X\star_2) = \mathrm{I}_{2n} \text{ modulo (X)}.$$

Now result follows by using the identity described in Lemma 5.5. □

§6. Suslin's Local-Global Principle

In his remarkable thesis (*cf.* [2]) A. Bak showed that for a form ring (R, Λ) the elementary subgroup $\mathrm{EQ}(2n, R, \Lambda)$ is perfect for $n \geqslant 3$ and hence is a normal subgroup of $\mathrm{GQ}(2n, R, \Lambda)$. As we have noted earlier, this question is related to Suslin's local-global principle for the elementary subgroup. In [39], G. Tang has shown that for $n \geqslant r + 3$ the elementary Hermitian group $\mathrm{EH}(2n, R, a_1, \ldots, a_r, \Lambda)$ is a normal subgroup of $\mathrm{GH}(2n, R, a_1, \ldots, a_r, \Lambda)$. In this section we deduce an analogous local-global principle for the elementary subgroup of the general quadratic and Hermitian groups, when R is module finite, *i.e.*, finite over its center. We use this result in §8 to prove the nilpotent property of the unstable Hermitian group KH_1. Furthermore, we show that if R is finite over its center, then the normality of the elementary subgroup is equivalent to the local-global principle. This generalizes our result in [12].

The following is the key Lemma, and it tells us the reason why we need to assume that the size of the matrix is at least 6. In [12], proof is given for the general linear group. Arguing in similar manner by using identities of commutator laws result follows in the unitary and Hermitian cases. A list of commutator laws for elementary generators is stated in ([2, pp. 43–44, Lemma 3.16]) for the unitary groups and in ([39, pp. 237–239, Lemma 8.2]) for the Hermitian groups. For a direct proof we refer [30, Lemma 5].

Lemma 6.1. *Suppose ϑ is an elementary generator of the general quadratic (Hermitian) group* $\mathrm{G}(2n, R[X], \Lambda[X])$, $n \geqslant 3$. *Let ϑ be congruent to identity modulo (X^{2m}), for $m > 0$. Then, if we conjugate ϑ with an elementary generator of the general quadratic (Hermitian) group* $\mathrm{G}(2n, R, \Lambda)$, *we get the resulting matrix as a product of elementary generators of general quadratic (Hermitian) group* $\mathrm{G}(2n, R[X], \Lambda[X])$, *each of which is congruent to identity modulo (X^m).*

Corollary 6.2. *In Lemma 6.1 we can take ϑ as a product of elementary generators of the general quadratic (general Hermitian) group*

$$\mathrm{G}(2n, R[X], \Lambda[X]).$$

Lemma 6.3. *Let (R, Λ) be a form ring and $v \in \mathrm{E}(2n, R, \Lambda)e_{2n}$. Let $w \in R^{2n}$ be a column vector such that $\langle v, w\rangle = 0$. Then $\mathrm{I}_{2n} + \mathrm{M}(v, w) \in \mathrm{E}(2n, R, \Lambda)$.*

Proof. Let $v = \varepsilon e_{2n}$, where $\varepsilon \in \mathrm{E}(2n, R, \Lambda)$. Then it follows that $\mathrm{I}_{2n} + \mathrm{M}(v, w) = \varepsilon(\mathrm{I}_{2n} + \mathrm{M}(e_{2n}, w_1))\varepsilon^{-1}$, where $w_1 = \varepsilon^{-1}w$. Since $\langle e_{2n}, w_1\rangle = \langle v, w\rangle = 0$, we get $w_1^t = (w_{11}, \ldots, w_{1\,n-1}, 0, w_{1\,n+1}, \ldots, w_{1\,2n})$. Therefore, as $\lambda\overline{\lambda} = \overline{\lambda}\lambda = 1$,

$$\mathrm{I}_{2n} + \mathrm{M}(v,w)$$

$$= \begin{cases} \prod\limits_{\substack{1\leqslant j\leqslant n\\ 1\leqslant i\leqslant n-1}} \varepsilon\, ql_{in}(-\overline{\lambda}\overline{w}_{n+1\,i})\, q\varepsilon_{jn}(-\overline{\lambda}\overline{w}_{1\,j})ql_{n\,n}^{-1}(*)\varepsilon^{-1} \\ \prod\limits_{\substack{1\leqslant k\leqslant r\\ r+1\leqslant j\leqslant n\\ 1\leqslant i\leqslant n-1}} \varepsilon hl_{in}(-\overline{\lambda}\overline{w}_{n+1\,i})h\varepsilon_{jn}(-\overline{\lambda}\overline{w}_{1\,j})hm_n(-\overline{w}_{1\,k)})hl_{n\,n}^{-1}(*)\varepsilon^{-1} \end{cases}$$

(in the quadratic and Hermitian cases respectively),
where $\overline{w}_{1\ n+k} = (w_{1\,n+k}, 0, \ldots, 0)$. Hence the result follows. □

Note that the above implication is true for any associative ring with identity. From now onwards we assume that R is finite over its center $C(R)$. Let us recall

Lemma 6.4. *Let A be a Noetherian ring and $s \in A$. Then there exists a natural number k such that the homomorphism $\mathrm{G}(A, s^kA, s^k\Lambda) \to \mathrm{G}(A_s, \Lambda_s)$ (induced by localization homomorphism $A \to A_s$) is injective.*

For the proof of the above lemma we refer ([18, Lemma 5.1]). Also, we recall that any module finite ring R is the direct limit of its finitely generated subrings. Thus, one may assume that $C(R)$ is Noetherian.

Let (R,Λ) be a (module finite) form ring with identity.

Lemma 6.5. (Dilation Lemma) *Let $\alpha(X)\in \mathrm{G}(2n,R[X],\Lambda[X])$, with $\alpha(0)=\mathrm{I}_{2n}$. If $\alpha_s(X)\in \mathrm{E}(2n,R_s[X],\Lambda_s[X])$, for some non-nilpotent $s\in R$, then $\alpha(bX)\in \mathrm{E}(2n,R[X],\Lambda[X])$, for $b\in s^l\mathrm{C}(R)$, and $l\gg 0$.*

Remark 6.6. (In the above Lemma we actually mean there exists some $\beta(X)\in \mathrm{E}(2n,R[X],\Lambda[X])$ such that $\beta(0)=\mathrm{I}_{2n}$ and $\beta_s(X)=\alpha(bX)$.)

Proof. Given that $\alpha_s(X)\in \mathrm{E}(2n,R_s[X],\Lambda_s[X])$. Since $\alpha(0)=\mathrm{I_{2n}}$, using Lemma 5.7 we can write $\alpha_s(X)$ as a product of the matrices of the form $\varepsilon\vartheta_{ij}(\star_1)\varepsilon^{-1}$ and $\varepsilon\vartheta_i(\star_2)\varepsilon^{-1}$, where $\varepsilon\in \mathrm{E}(2n,R_s,\Lambda_s)$, $\star_1\in R_s[X]$, $\star_2\in ((R_s[X])^{2n})^t$ with both $\vartheta_{ij}(\star_1)$ and $\vartheta_i(\star_2)$ congruent to I_{2n} modulo (X). Applying the homomorphism $X\mapsto XT^d$, where $d\gg 0$, from the polynomial ring $R[X]$ to the polynomial ring $R[X,T]$, we look on $\alpha(XT^d)$. Note that $R_s[X,T]\cong (R_s[X])[T]$. As $C(R)$ is Noetherian, it follows from Lemma 6.4 and Corollary 6.2 that over the ring $(R_s[X])[T]$ we can write $\alpha_s(XT^d)$ as a product of elementary generators of general quadratic (Hermitian) group such that each of those elementary generators is congruent to identity modulo (T). Let l be the maximum of the powers occurring in the denominators of those elementary generators. Again, as $C(R)$ is Noetherian, by applying the homomorphism $T\mapsto s^mT$, for $m\geqslant l$, it follows from Lemma 6.4 that over the ring $R[X,T]$ we can write $\alpha_s(XT^d)$ as a product of elementary generators of general quadratic (Hermitian) group such that each of those elementary generator is congruent to identity modulo (T), for some $b\in(s^l)C(R)$, *i.e.*, we get there exists some $\beta(X,T)\in \mathrm{E}(2n,R[X,T],\Lambda[X,T])$ such that $\beta(0,0)=\mathrm{I}_{2n}$ and $\beta_s(X,T)=\alpha(bXT^d)$. Finally, the result follows by putting $T=1$. $\square$

Theorem 6.7. (Local-Global Principle) *If $\alpha(X)\in \mathrm{G}(2n,R[X],\Lambda[X])$, $\alpha(0)=\mathrm{I}_n$ and*

$$\alpha_{\mathfrak{m}}(X)\in \mathrm{E}(2n,R_{\mathfrak{m}}[X],\Lambda_{\mathfrak{m}}[X]),$$

for every maximal ideal $\mathfrak{m}\in \mathrm{Max}\,(\mathrm{C(R)})$, then

$$\alpha(X)\in \mathrm{E}(2n,R[X],\Lambda[X]).$$

(*Note that $R_{\mathfrak{m}}$ denotes $S^{-1}R$, where $S=C(R)\setminus\mathfrak{m}$*).

Proof. Since $\alpha_{\mathfrak{m}}(X) \in \mathrm{E}(2n, R_{\mathfrak{m}}[X], \Lambda_{\mathfrak{m}}[X])$, for all $\mathfrak{m} \in \mathrm{Max}(C(R))$, for each $\mathfrak{m}$ there exists $s \in C(R) \setminus \mathfrak{m}$ such that $\alpha_s(X) \in \mathrm{E}(2n, R_s[X], \Lambda_s[X])$. Using Noetherian property we can consider a finite cover of $C(R)$, say $s_1 + \cdots + s_r = 1$. Let $\theta(X,T) = \alpha_s(X+T)\alpha_s(T)^{-1}$. Then

$$\theta(X,T) \in \mathrm{E}(2n, (R_s[T])[X], \Lambda_s[T][X])$$

and $\theta(0,T) = \mathrm{I}_n$. By Dilation Lemma, applied with base ring $R[T]$, there exists $\beta(X) \in \mathrm{E}(2n, R[X,T], \Lambda[X,T])$ such that

$$\beta_s(X) = \theta(bX, T). \tag{2}$$

Since for $l \gg 0$, the ideal $\langle s_1^l, \ldots, s_r^l \rangle = R$, we chose $b_1, b_2, \ldots, b_r \in C(R)$, with $b_i \in (s^l)C(R)$, $l \gg 0$ such that (2) holds and $b_1 + \cdots + b_r = 1$. Then there exists $\beta^i(X) \in \mathrm{E}(2n, R[X,T], \Lambda[X,T])$ such that $\beta^i_{s_i}(X) = \theta(b_i X, T)$. Therefore,

$$\prod_{i=1}^{r} \beta^i(X) \in \mathrm{E}(2n, R[X,T], \Lambda[X,T]).$$

But,

$$\alpha_{s_1 \cdots s_r}(X) = \left(\prod_{i=1}^{r-1} \theta_{s_1 \ldots \widehat{s_i} \ldots s_r}(b_i X, T)|_{T = b_{i+1}X + \cdots + b_r X} \right) \theta_{s_1 \ldots \ldots s_{r-1}}(b_r X, 0).$$

Since $\alpha(0) = \mathrm{I}_n$, and as a consequence of the Lemma 6.4 it follows that the map $\mathrm{E}(R, s^k R, s^k \Lambda) \to \mathrm{E}(R_s, \Lambda_s)$ in injective, we conclude $\alpha(X) \in \mathrm{E}(2n, R[X], \Lambda[X])$. $\square$

§7. Equivalence of Normality and Local-Global Principle

Next we are going to show that if k is a commutative ring with identity and R is an associative k-algebra such that R is finite as a left k-module, then the normality criterion of elementary subgroup is equivalent to Suslin's local-global principle for above two classical groups. (Remark: One can also consider R as a right k-algebra.)

One of the crucial ingredients in the proof of the above theorem is the following result which states that the group E acts transitively on unimodular vectors. The precise statement of the fact is the following:

Definition 7.1. A vector $(v_1, \ldots, v_{2n}) \in R^{2n}$ is said to be unimodular if there exists another vector $(u_1, \ldots, u_{2n}) \in R^{2n}$ such that $\sum_{i=1}^{2n} v_i u_i = 1$.

The set of all unimodular vector in R^{2n} is denoted by $\mathrm{Um}(2n, R)$.

Theorem 7.2. *Let R be a semilocal ring (not necessarily commutative) with involution and $v = (v_1, \dots, v_{2n})^t$ be a unimodular and isotropic vector in R^{2n}. Then $v \in \mathrm{E}(2n, R)e_{2n}$ for $n \geqslant 2$, i.e., $\mathrm{E}(2n, R)$ acts transitively on the set of isotropic vectors in* $\mathrm{Um}(2n, R)$.

Let us first recall some known facts before we give a proof of the theorem.

Definition 7.3. An associative ring R is said to be **semilocal** if $R/\mathrm{rad}(R)$ is Artinian semisimple.

We recall the following three lemmas.

Lemma 7.4. (H. Bass) *Let A be an associative B-algebra such that A is finite as a left B-module and B be a commutative local ring with identity. Then A is semilocal.*

Proof. Since B is local, $B/\mathrm{rad}(B)$ is a division ring by definition. That implies $A/\mathrm{rad}(A)$ is a finite module over the division ring $B/\mathrm{rad}(B)$ and hence is a finitely generated vector space. Thus $A/\mathrm{rad}(A)$ Artinian as $B/\mathrm{rad}(B)$ module and hence $A/\mathrm{rad}(A)$ Artinian as $A/\mathrm{rad}(A)$ module, so it is an Artinian ring.

It is known that an artin ring is semisimple if its radical is trivial. Thus $A/\mathrm{rad}(A)$ is semisimple, as $\mathrm{rad}(A/\mathrm{rad}(A)) = 0$. Hence $A/\mathrm{rad}(A)$ Artinian semisimple. Therefore, A is semilocal by definition. $\square$

Lemma 7.5. (H. Bass) ([10, Lemma 4.3.26]) *Let R be a semilocal ring (may not be commutative), and let I be a left ideal of R. Let a in R be such that $Ra + I = R$. Then the coset $a + I = \{a + x \,|\, x \in I\}$ contains a unit of R.*

Proof. We give a proof due to R. G. Swan. We can factor out the radical and assume that R is semisimple Artinian. Let $I = (Ra \cap I) \oplus I'$. Replacing I by I' we can assume that $R = Ra \oplus I$. Let $f : R \to Ra$ by $r \mapsto ra$, for $r \in R$. Therefore, we get a split exact sequence

$$0 \longrightarrow J \longrightarrow R \xrightarrow{f} Ra \longrightarrow 0,$$

for some ideal J in R which gives us a map $g : R \to J$ such that $R \xrightarrow{(f,g)} Ra \oplus J$ is an isomorphism. Since $Ra \oplus J \cong R \cong Ra \oplus I$ cancellation (using Jordan–Hölder or Krull–Schmidt) shows that $J \cong I$. If $h : R \cong J \cong I$, then $R \xrightarrow{(f,g)} Ra \oplus I \cong R$ is an isomorphism sending 1 to (a, i) to $a + i$, where $i = h(1)$. Hence it follows that $a + i$ is a unit. $\square$

Lemma 7.6. *Let R be a semisimple Artinian ring and I be a left ideal of R. Let $J = Ra + I$. Write $J = Re$, where e is an idempotent (possible since J is projective. For detail cf.* [13, Theorem 4.2.7]). *Then there is an element $i \in I$ such that $a + i = ue$, where u is a unit in R.*

Proof. Since $R = J + R(1-e) = Ra + I + R(1-e)$, using Lemma 7.5 we can find a unit $u = a + i + x(1-e)$ in R, for some $x \in R$. Since $a + i \in Re$, it follows that $ue = a + i$. $\square$

Corollary 7.7. *Let R be a semisimple Artinian ring and $(a_1, \ldots, a_n)^t$ be a column vector over R, where $n \geqslant 2$. Let $\Sigma Ra_i = Re$, where e is an idempotent. Then there exists $\varepsilon \in \mathrm{E}_n(R)$ such that $\varepsilon(a_1, \ldots, a_n)^t = (0, \ldots, 0, e)^t$.*

Proof. By Lemma 7.6 we can write $ue = \Sigma_{i=1}^{n-1} b_i a_i + a_n$, where u is a unit. Therefore, applying an elementary transformation we can assume that $a_n = ue$. Multiplying from the left by $(\mathrm{I}_{n-2} \perp u \perp u^{-1})$ we can make $a_n = e$. Since all a_i are left multiple of e, further elementary transformations reduce our vector to the required form. $\square$

The following observation will be needed to do the case $2n = 4$.

Lemma 7.8. *Let R be a semisimple Artinian ring and e be an idempotent. Let $f = 1-e$, and b be an element of R. If $fRb \subseteq Re$, then we have $b \in Re$.*

Proof. Since R is a product of simple rings, it will suffice to do the case in which R is simple. If $e = 1$, we are done. Otherwise RfR is a non-zero two sided ideal, and hence $RfR = R$. Since $Rb = RfRb \subseteq Re$, we have $b \in Re$. $\square$

Lemma 7.9. *Let R be a semisimple Artinian ring and let $- : R \to R$ be a λ-involution on R. Let $(x\ y)^t$ be a unimodular row of length $2n$, where $2n \geqslant 4$, and $x, y \in R^n$. Then there exists an element $\varepsilon \in \mathrm{E}(2n, R)$ such that $\varepsilon(x\ y)^t = (x'\ y')^t$, where x'_1 is a unit in R.*

Proof. Let $x = (x_1, \ldots, x_n)^t$ and $b = (y_1, \ldots, y_n)^t$. We claim that there exists $\varepsilon \in \mathrm{E}(2n, R)$ such that $\varepsilon(x\ y)^t = (x'\ y')$, where x' is a unit in R. Among all $(x'\ y')^t$ of this form, choose one for which the ideal $I = \Sigma Rx'_i$ is maximal. Replacing the original $(x\ y)^t$ by $(x'\ y')^t$ we can assume that $I = \Sigma Rx_i$ is maximal among such ideals. Write $I = Re$, where e is an idempotent in R. By Corollary 7.7 we can find an element $\eta \in \mathrm{E}_n(R)$ such that $\eta x = (0, 0, \ldots, e)^t$. So we can modify x by elementary generators of the form $q\varepsilon_{ij}(\star)$ or $h\varepsilon_{ij}(\star)$ and hence we assume that $x = (0, 0, \ldots, e)^t$. We claim that $y_i \in Re$ for all $i \geqslant 1$.

First we consider the case $2n \geqslant 6$. Assume $y_1 \notin I$, but $y_i \in I$ for all $i \geqslant 2$. If we apply $q\varepsilon_{1n}(1)$ in the quadratic case then this replaces y_n to $y_n - y_1$ but not changes e and y_1. On the other hand for the Hermitian case we do not have the generator $q\varepsilon_{1n}(1)$. But if we apply $hm_n(1, \ldots, 1)$, then it changes y_2 but does not changes e and b_1. Therefore, in both the cases we can therefore assume that some y_i with $i > 1$ is not in I. (Here recall that we have put no restriction on C, *i.e.*, for us $C = R^r$). Apply $qr_{ii}(1)$ with $2 \leqslant i \leqslant n$ in the quadratic case. This changes $x_i = 0$ (for $i > 1$) to y_i while $x_n = e$ is preserved. The ideal generated by the entries of x now contains $Re + Ry_i$, which is larger than I, a contradiction, as I is maximal. In the Hermitian case if we apply suitable $hr_i(1, \ldots, 1)$ then also we see that the ideal generated by the entries of x now contains $Re + Ry_i$, hence a contradiction.

If $2n = 4$, we can argue as follows. Let $f = 1 - e$. Let us assume that $y_1 \neq I$ as above. Then by Lemma 7.8 it will follow that we can find some $s \in R$ such that $fsy_1 \neq Re$. First consider the quadratic case. Applying $qr_{21}(fs)$ replaces $x_2 = e$ by $c = e + fsy_1$. As $ec = e$, $I = Re \subset Rc$. Also, $fc = fsy_1 \in Rc$ but $fc \notin I$. Hence $I \subsetneq Rc$, a contradiction. We can get the similar contradiction for y_2 by applying $qr_{22}(fs)$. In the Hermitian case, apply $hr_1(1)$ to get the contradiction for y_1. Now note that in this $r = 1$ as we have assume $r < n$. Hence we can apply $qr_{22}(fs)$ to get the contradiction.

Since all y_i lie in Re, the left ideal generated by the all entries of $(x\ y)^t$ is Re, but as this column vector is unimodular, we get $Re = R$, and therefore $e = 1$. $\square$

Proof of Theorem 7.2. Let J be the Jacobson radical of R. Since the left and the right Jacobson radical are same, J is stable under the involution which therefore passes to R/J. Let ε be as in Lemma 7.9 for the image $(x'\ y')^t$ of $(x\ y)^t$. By lifting ε from R/J to R and applying it to $(x\ y)^t$ we reduce to the case where x_n is a unit in R. Let $\alpha = x_n \perp x_n^{-1}$. Then applying $(\mathrm{I}_{n-2} \perp \alpha \perp \mathrm{I}_{n-2} \perp \alpha^{-1})$ we can assume that $x_n = 1$.

Next applying $\Pi_{i=1}^{n-1} ql_{ni}(-y_i)$ and $\Pi_{i=1}^{n-1} hl_{ni}(-y_i)$ in the respective cases we get $y_1 = \cdots = y_{n-1} = 0$. As isotropic vector remains isotropic under elementary quadratic (Hermitian) transformation, we have $y_n + \lambda\overline{y}_n = 0$, hence $ql_{11}(\overline{\lambda y}_n)$ and $hl_{11}(\overline{\lambda y}_n)$ are defined and applying it reduces y_n to 0 in both the cases. Now we want to make $x_i = 0$ for $i = 1, \ldots, n$. In the quadratic case it can be done by applying $\Pi_{i=1}^{n-1} h\varepsilon_{in}(-x_i)$. Note that this transformation does not affect any y_i's, as $y_i = 0$. In the Hermitian case we can make $x_{r+1} = \cdots = x_n = 0$ as before applying $\Pi_{i=r+1}^{n-1} q\varepsilon_{in}(-x_i)$. To make $x_1 = \cdots = x_r = 0$ we have to recall that the set $C = R^r$, *i.e.*, there is no restriction on the set C. Hence $hr_n(-x_1, \ldots, -x_r)$ is defined and applying it we get $x_1 = \cdots = x_r = 0$. Also note that other x_i's and y_i's remain unchanged. Finally, applying $hl_{nn}(1)$ and then $hr_{nn}(-1)$ we get the required vector $(0, \ldots, 0, 1)$. This completes the proof. □

Theorem 7.10. *Let k be a commutative ring with identity and R an associative k-algebra such that R is finite as a left k-module. Then the following are equivalent for $n \geqslant 3$ in the quadratic case and $n \geqslant r + 3$ in the Hermitian case:*

(1) **(Normality)** $\mathrm{E}(2n, R, \Lambda)$ *is a normal subgroup of* $\mathrm{G}(2n, R, \Lambda)$.
(2) **(L-G Principle)** *If* $\alpha(X) \in \mathrm{G}(2n, R[X], \Lambda[X])$, $\alpha(0) = \mathrm{I}_n$ *and*

$$\alpha_{\mathfrak{m}}(X) \in \mathrm{E}(2n, R_{\mathfrak{m}}[X], \Lambda_{\mathfrak{m}}[X])$$

for every maximal ideal $\mathfrak{m} \in \mathrm{Max}(k)$, *then*

$$\alpha(X) \in \mathrm{E}(2n, R[X], \Lambda[X]).$$

(*Note that* $R_{\mathfrak{m}}$ *denotes* $S^{-1}R$, *where* $S = k \setminus \mathfrak{m}$.)

Proof. In Section 6 we have proved Lemma 6.3 for any form ring with identity and shown that the local-global principle is a consequence of Lemma 6.3. So, the result is true in particular if $\mathrm{E}(2n, R, \Lambda)$ is a normal subgroup of $\mathrm{G}(2n, R, \Lambda)$.

To prove the converse we need R to be finite as k-module, where k is a commutative ring with identity (*i.e.*, a ring with trivial involution).

Let $\alpha \in \mathrm{E}(2n, R, \Lambda)$ and $\beta \in \mathrm{G}(2n, R, \Lambda)$. Then α can be expressed as a product of matrices of the form ϑ_{ij}(ring element) and ϑ_i(column vector). Hence we can write $\beta\alpha\beta^{-1}$ as a product of the matrices of the form $(\mathrm{I}_{2n} + \beta\, \mathrm{M}(\star_1, \star_2)\beta^{-1})$, with $\langle \star_1, \star_2 \rangle = 0$, where $\star_1$ and $\star_2$ are suitably chosen standard basis vectors. Now let $v = \beta\star_1$. Then we can write $\beta\alpha\beta^{-1}$ as a product of the matrices of the form $(\mathrm{I}_{2n} + \beta\, \mathrm{M}(v, w)\beta^{-1})$, with $\langle v, w \rangle = 0$ for some row vector w in R^{2n}. We show that each $(\mathrm{I}_{2n} + \mathrm{M}(v, w)) \in \mathrm{E}(2n, R, \Lambda)$.

Let $\gamma(X) = \mathrm{I}_{2n} + X\mathrm{M}(v, w)$. Then $\gamma(0) = \mathrm{I}_{2n}$. By Lemma 7.4 it follows that $S^{-1}R$ is a semilocal ring, where $S = k - \mathfrak{m}$, $\mathfrak{m} \in \mathrm{Max}(k)$. Since

$v \in \mathrm{Um}(2n, R)$, using Theorem 7.2 we get

$$v \in \mathrm{E}(2n, S^{-1}R, S^{-1}\Lambda)e_1,$$

hence $Xv \in \mathrm{E}(2n, S^{-1}R[X], S^{-1}\Lambda[X])e_1$. Therefore, applying Lemma 6.3 over $S^{-1}(A[X], \Lambda[X])$ it follows that

$$\gamma_{\mathfrak{m}}(X) \in \mathrm{E}(2n, S^{-1}R[X], S^{-1}\Lambda[X]).$$

Now applying Theorem 6.7, it follows that $\gamma(X) \in \mathrm{E}(2n, R[X], \Lambda[X])$. Finally, putting $X = 1$ we get the result. □

§8. Nilpotent property for K_1 of Hermitian groups

We devote this section to discuss the study of nilpotent property of unstable K_1-groups. The literature in this direction can be found in the work of A. Bak, N. Vavilov and R. Hazrat. Throughout this section we assume R is a commutative ring with identity, *i.e.*, we are considering trivial involution and $n \geqslant r+3$. Following is the statement of the theorem.

Theorem 8.1. *The quotient group* $\frac{\mathrm{SH}(2n,R,a_1,\dots,a_r)}{\mathrm{EH}(2n,R,a_1,\dots,a_r)}$ *is nilpotent for* $n \geqslant r+3$. *The class of nilpotency is at the most* $\max\,(1, d+3-n)$, *where* $d = \dim\,(R)$.

The proof follows by imitating the proof of Theorem 4.1 in [6].

Lemma 8.2. *Let I be an ideal contained in the Jacobson radical $J(R)$ of R, and $\beta \in \mathrm{SH}(2n, R, \Lambda)$, with $\beta \equiv \mathrm{I}_n$ modulo I. Then there exists $\theta \in \mathrm{EH}(2n, R, a_1, \dots, a_r)$ such that $\beta\theta =$ the diagonal matrix $[d_1, d_2, \dots, d_{2n}]$, where each d_i is a unit in R with $d_i \equiv 1$ modulo I, and θ a product of elementary generators with each congruent to identity modulo I.*

Proof. The diagonal elements of β are units. Let $\beta = (\beta_{ij})$, where $d_i = \beta_{ii} = 1 + s_{ii}$ with $s_{ii} \in I \subset J(R)$, for $i = 1, \dots, 2n$, and $\beta_{ij} \in I \subset J(R)$ for $i \neq j$. First we make all the $(2n, j)$th, and $(i, 2n)$th entries zero, for $i = 2, \dots, n$, $j = 2, \dots, n$. Then repeating the above process we can reduce the size of β. Since we are considering trivial involution, we take

$$\begin{aligned}\alpha = {} & \prod_{j=1}^{n} hl_{nj}(-\beta_{2nj}d_j^{-1}) \\ & \times \prod_{\substack{n+r+1\leqslant i\leqslant 2n-1\\ n+1\leqslant j\leqslant n+r}} hm_i(-\zeta_j d_j^{-1}) \prod_{\substack{r+1\leqslant i\leqslant n-1\\ n+r+1\leqslant j\leqslant 2n-1}} h\varepsilon_{in}(\beta_{\rho(n)\rho(i)} d_j^{-1}),\end{aligned}$$

where $j = i - r$ and $\zeta_j = (0, \ldots, 0, \beta_{2nj})$, and

$$\gamma = \prod_{\substack{r+1 \leqslant j \leqslant 2n-1 \\ r+1 \leqslant i \leqslant 2n-1}} h\varepsilon_{nj}(a_{i-r}(\star) d_{2n}^{-1}) hr_n(\eta),$$

where $a_t = 0$ for $t > r$, and $\eta = (\beta_{12n} d_{2n}^{-1}, \beta_{22n} d_{2n}^{-1}, \ldots, \beta_{n2n} d_{2n}^{-1})$. Then the last column and last row of $\gamma\beta\alpha$ become $(0, \ldots, 0, d_{2n})^t$, where d_{2n} is a unit in R and $d_{2n} \equiv 1$ modulo I. Repeating the process we can modify β to the required form. $\square$

Proposition 8.3. (*cf.* [30, Lemma 7]) *Let* (R, Λ) *be a commutative form ring, i.e., with trivial involution, and* s *be a non-nilpotent element in* R *and* $a \in R$. *Then for* $l \geqslant 2$

$$\left[\vartheta_{ij}\left(\frac{a}{s}\right), \mathrm{SH}(2n, s^l R)\right] \subset \mathrm{EH}(2n, R).$$

More generally,

$$[\varepsilon, \mathrm{SH}(2n, s^l R)] \subset \mathrm{EH}(2n, R), \ \textit{for } l \gg 0 \textit{ and } \varepsilon \in \mathrm{EH}(2n, R_s).$$

Proof of Theorem 8.1. Recall

Let G be a group. Define $Z^0 = G$, $Z^1 = [G, G]$ and $Z^i = [G, Z^{i-1}]$. Then G is said to be nilpotent if $Z^r = \{e\}$ for some $r > 0$, where e denotes the identity element of G.

Since the map $\mathrm{EH}(2n, R, a_1, \ldots, a_r) \to \mathrm{EH}(2n, R/I, \overline{a}_1, \ldots, \overline{a}_r)$ is surjective we may and do assume that R is a reduced ring. Note that if $n \geqslant d + 3$, then the group $\mathrm{SH}(2n, R, a_1, \ldots, a_r)/\mathrm{EH}(2n, R, a_1, \ldots, a_r) = \mathrm{KH}_1(R, a_1, \ldots, a_r)$, which is abelian and hence nilpotent. (For details see [4]). So we consider the case $n \leqslant d + 3$. Let us first fix a n. We prove the theorem by induction on $d = \dim R$. Let

$$G = \mathrm{SH}(2n, R, a_1, \ldots, a_r)/\mathrm{EH}(2n, R, a_1, \ldots, a_r).$$

Let $m = d + 3 - n$ and $\alpha = [\beta, \gamma]$, for some $\beta \in G$ and $\gamma \in Z^{m-1}$. Clearly, the result is true for $d = 0$. Let $\widetilde{\beta}$ be the pre-image of β under the map

$$\mathrm{SH}(2n, R, a_1, \ldots, a_r) \to \mathrm{SH}(2n, R, a_1, \ldots, a_r)/\mathrm{EH}(2n, R, a_1, \ldots, a_r).$$

If R is reduced then arguing as Lemma 8.2 it follows that we can choose a non-zero-divisor s in R such that $\widetilde{\beta}_s \in \mathrm{EH}(2n, R_s, a_1, \ldots, a_r)$.

Consider $\overline{G}$, where bar denote reduction modulo s^l, for some $l \gg 0$. By the induction hypothesis $\overline{\gamma} = \{1\}$ in $\overline{\mathrm{SH}(2n, R)}$, where bar denote the reduction modulo the subgroup $\mathrm{EH}(2n, R)$. Since $\mathrm{EH}(2n, R)$ is a normal subgroup of $\mathrm{SH}(2n, R)$, for $n \geqslant r + 3$, by modifying γ we may assume that $\widetilde{\gamma} \in \mathrm{SH}(2n, R, s^l R, a_1, \dots, a_r)$, where $\widetilde{\gamma}$ is the pre-image of γ in $\mathrm{SH}(2n, R, a_1, \dots, a_r)$. Now by Proposition 8.3 it follows that $[\widetilde{\beta}, \widetilde{\gamma}] \in \mathrm{EH}(2n, R, a_1, \dots, a_r)$. Hence $\alpha = \{1\}$ in G. □

Remark 8.4. In ([12, Theorem 3.1]) it has been proved that the question of normality of the elementary subgroup and the local-global principle are equivalent for the elementary subgroups of the linear, symplectic and orthogonal groups over an almost commutative ring with identity. There is a gap in the proof of the statement (3) $\Rightarrow$ (2) of Theorem 3.1 in [12] (for an almost commutative ring). The fact that over a non-commutative semilocal ring the elementary subgroups of the classical groups acts transitively on the set of unimodular and isotropic (*i.e.*, $\langle v, v\rangle = 0$) vectors of length $n \geqslant 3$ in the linear case, and $n = 2r \geqslant 6$ in the non-linear cases has been used in the proof, but it is not mentioned anywhere in the article. This was pointed by Professor R. G. Swan and he provided us a proof for the above result.

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