Записки научных
семинаров ПОМИ
Том 451, 2016 г.

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CONNECTION OF THE DIFFERENT TYPES OF INVERSE DATA FOR THE ONE-DIMENSIONAL SCHRÖDINGER OPERATOR ON THE HALF-LINE

Abstract. We consider inverse dynamical, spectral, quantum and acoustical scattering problems for the Schrödinger operator on the half line. The goal of the paper is to establish the connections between different types of inverse data for these problems. The central objects which serve as a source for all formulaes are kernels of so-called connecting operators and Krein equations.

§1. Introduction

This paper is of methodological character, its primary goal is to show the connection of the different types of inverse data for the Shrödinger operator on the half line. The idea of using the connection of the inverse data in solving the inverse problems is not knew, to mention [1,3,4,15,21]. In our approach we exploit the central objects of the Boundary Control method – the connecting operators and corresponding Krein equations, and show that using just these two well-known objects leads to interesting results.

The central object we will be dealing with is a wave equation on the half-line with the potential $q \in L^1_{loc}(\mathbb{R}_+)$:

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + q(x)u(x,t) = 0, \quad x > 0,\ t > 0, \\ u(x,0) = u_t(x,0) = 0,\ u(0,t) = f(t). \end{cases} \tag{1.1}$$

Here f is an arbitrary $L^2_{loc}(\mathbb{R}_+)$ function referred to as a *boundary control*, by u^f we denote the solution to (1.1). Let $T > 0$ be fixed. The dynamical

Key words and phrases: inverse problem, Schrödinger operator, Boundary Control method, scattering matrix, Weyl function.

The research of Victor Mikhaylov was supported in part by NIR SPbGU 11.38.263.2014 and RFBR 14-01-00535. Alexandr Mikhaylov was supported by RFBR 14-01-00306; A. S. Mikhaylov and V. S. Mikhaylov were partly supported by VW Foundation program "Modeling, Analysis, and Approximation Theory toward application in tomography and inverse problems."

inverse data is given by the *response operator* (the dynamical Dirichlet-to-Neumann map) $\left(R^T f\right) := u_x^f(0,t)$, and the inverse problem associated with (1.1) is to recover $q(x)$, $0 < x < T$, by given R^{2T}. One of the efficient methods of solving this problem is the Boundary Control method [1,4,8]. The *control operator* and *connecting operator* are introduced by $W^T f := u^f(\,\cdot\,,T)$, $C^T := \left(W^T\right)^* W^T$. The fact that C^T is expressed in terms of the inverse data [4] plays an important role in BC method.

We also consider the spectral, quantum and acoustical scattering problems for the Schrödinger operator with the same potential q on the half-line $H = -\partial_x^2 + q$ on $L_2(0,\infty)$ with Dirichlet boundary condition $\phi(0) = 0$. For each problem we define corresponding data: spectral measure and Titchmarsh-Weil function for the spectral problem, discrete spectrum with norming coefficients and scattering matrix for the scattering problem (we need to assume that the potential satisfy some additional condition on growth at infinity); acoustical response operator and acoustical response function (for this problem we assume that potential is infinitely smooth and compactly supported). Our aim will be to show the connection of the dynamical data, which is the kernel of the response operator R^T with spectral and scattering data and connection of the acoustical response with the scattering data. Some of the results have been obtained in [1,3], we list them for the sake of completeness or give a different proof. The main objects which play the key role in our considerations is the kernels of the connecting operators and Krein equations. The central role of the connecting operators in different inverse problems have been pointed out in [2,4,7], in [11] the author studied the singular values of connecting operator for the observation problem.

In the second section we set up the forward and inverse problems: dynamical, spectral, quantum and acoustical scattering, and for each of them introduce the corresponding inverse data. In the third section we study in details the integral kernel of the connecting operator and reveal the links with the spectral function of Levitan [19]. In the last section we derive the spectral and scattering representation of the response function and explain the connection of the response function with the Titchmarsh-Weil function see also [1,3]; we also derive the scattering representation for the acoustical scattering response function and establish its connection with the scattering matrix.

§2. Inverse data

2.1. Dynamical inverse data. For the potential $q \in L^1_{loc}(\mathbb{R}_+)$ we consider the initial boundary value problem for the 1d wave equation with the potential (1.1) with f be an arbitrary $L^2_{loc}(\mathbb{R}_+)$ function referred to as a *boundary control*. It is known [1] that the solution $u^f(x,t)$ of the problem (1.1) can be written in terms of the integral kernel $w(x,s)$:

$$u^f(x,t) = \begin{cases} f(t-x) + \int\limits_x^t w(x,s)f(t-s)\,ds, & x \leqslant t, \\ 0, \quad x > t. \end{cases} \tag{2.1}$$

where $w(x,s)$ is the unique solution to certain Goursat problem. Fix $T > 0$ and introduce the outer space of the system (1.1), the space of controls: $\mathcal{F}^T := L_2(0,T)$. The dynamical inverse data is given by the *response operator* (the dynamical Dirichlet-to-Neumann map) $R^T : \mathcal{F}^T \mapsto \mathcal{F}^T$ with the domain $\{f \in C^2([0,T]) : \; f(0) = f'(0) = 0\}$, acting by the rule:

$$(R^T f)(t) = u^f_x(0,t), \; t \in (0,T).$$

According to (2.1) it has a representation

$$(R^T f)(t) = -f'(t) + \int\limits_0^t r(s)f(t-s)\,ds, \tag{2.2}$$

where $r(t) := w_x(0,t)$ is called the *response function*. The natural set up of a inverse problem [1,4,8] is to recover the potential $q(x)$, $x \in (0,T)$ from R^{2T}, or what is equivalent, from $r(t)$, $t \in (0,2T)$.

We introduce the inner space of system (1.1), the space of states: $\mathcal{H}^T := L_2(0,T)$, so for all $0 \leqslant t \leqslant T$, $u^f(\,\cdot\,,t) \in \mathcal{H}^T$. The *control operator* W^T is defined by

$$W^T : \mathcal{F}^T \mapsto \mathcal{H}^T, \; W^T f = u^f(\,\cdot\,,T),$$

is bounded. From (2.1) it follows that

$$(W^T f)(x) = f(T-x) + \int\limits_x^T w(x,\tau)f(T-\tau)\,d\tau.$$

It is not hard to show that W^T is in fact boundedly invertible. The *connecting operator* $C^T : \mathcal{F}^T \mapsto \mathcal{F}^T$, plays a central role in the BC method.

It connects the outer space of the dynamical system (1.1) with the inner space, and is defined by its bilinear product:

$$\big(C^T f, g\big)_{\mathcal{F}^T} = \big(u^f(\,\cdot\,,T), u^g(\,\cdot\,,T)\big)_{\mathcal{H}^T}\,, \quad C^T = (W^T)^* W^T. \tag{2.3}$$

The invertibility of W^T implies that C^T is positive definite, bounded and boundedly invertible in $\mathcal{F}^T$. The fact that C^T is expressed in terms of the response operator is widely used in BC-method. In [1] we have shown this for the case of nonsmooth potential:

Proposition 1. *For $q \in L^1_{loc}(0,T)$ and $T > 0$, operator C^T admits the representation*

$$(C^T f)(t) = f(t) + \int_0^T c^T(t,s) f(s)\, ds\,,\ 0 < t < T\,, \tag{2.4}$$

where

$$c^T(t,s) = [p(2T - t - s) - p(t - s)], \quad p(t) = \frac{1}{2}\int_0^t r(s)\, ds. \tag{2.5}$$

We fix a function y to be solution to the following Cauchy problem:

$$\begin{cases} -y'' + qy = \lambda y, & x > 0 \\ y(0) = 0, \quad y'(0) = 1. \end{cases}$$

Set up the *special control problem*: to find a control f^T that

$$(W^T f^T)(x) = \begin{cases} y(x),\ 0 < x < T, \\ 0,\ x > T. \end{cases}$$

Theorem 1. *The control $f^T = W^{-1} y$, which solves the special control problem, is the solution of the Krein equation $Cf = \frac{\sin\sqrt{\lambda}(T-t)}{\sqrt{\lambda}}$ in $\mathcal{F}$, i.e., satisfies*

$$f^T(\tau) + \int_0^T c^T(t,s) f^T(s)\, ds = \frac{\sin\sqrt{\lambda}(T-t)}{\sqrt{\lambda}}, \qquad 0 \leqslant t \leqslant T.$$

Notice that $\big(W^T\big)^*$ is a transformation operator: it maps the solution of the perturbed problem to the solution of the unperturbed (modulo shift by T):

$$\big(W^T\big)^* y(\,\cdot\,,\lambda) = \frac{\sin\sqrt{\lambda}(T-t)}{\sqrt{\lambda}}. \tag{2.6}$$

2.2. Spectral inverse data. we consider the Schrödinger operator

$$H=-\partial_x^2+q\left(x\right) \tag{2.7}$$

on $L^2\left(\mathbb{R}_+\right)$, $\mathbb{R}_+:=[0,\infty)$, with a real-valued locally integrable potential q and Dirichlet boundary condition at $x=0$. For $z\in\mathbb{C}$ consider the solution

$$\begin{cases} -\varphi''(x)+q(x)\varphi(x)=z\varphi(x), \\ \varphi(0,z)=0,\ \varphi'(0,z)=1. \end{cases} \tag{2.8}$$

It is known [18] that there exist a spectral measure $d\rho(\lambda)$, such that for all $f,g\in L^2(\mathbb{R}_+)$ the Parseval identity holds:

$$\int_0^\infty f(x)g(x)\,dx=\int_{-\infty}^\infty (Ff)(\lambda)(Fg)(\lambda)\,d\rho(\lambda), \tag{2.9}$$

where $F:L_2(\mathbb{R}_+)\mapsto L_{2,\,\rho}(\mathbb{R})$ is a Fourier transformation:

$$(Ff)(\lambda)=\int_0^\infty f(x)\varphi(x,\lambda)\,dx \tag{2.10}$$

$$f(x)=\int_{-\infty}^\infty (Ff)(\lambda)\varphi(x,\lambda)\,d\rho(\lambda).$$

The so-called transformation operator transforms the solutions of (2.8) to the solution of (2.8) with zero potential:

$$\left(I_s+L_s\right)\varphi(\,\cdot\,,\lambda)=\varphi(s,\lambda)+\int_0^s w(x,s)\varphi(x,\lambda)\,dx=\frac{\sin\sqrt{\lambda}s}{\sqrt{\lambda}}. \tag{2.11}$$

We assume that (2.7) is limit point case at ∞, that is, for each $z\in\mathbb{C}_+:=\{z\in\mathbb{C}:\operatorname{Im}z>0\}$ the equation

$$-u''+q\left(x\right)u=zu \tag{2.12}$$

has a unique, up to a multiplicative constant, solution in L_2 at ∞ , we denote this solution by u_+:

$$\int_{\mathbb{R}_+}\left|u_+\left(x,z\right)\right|^2dx<\infty,\quad z\in\mathbb{C}_+.$$

Then the *Titchmarsh-Weyl m-function*, $m(z)$, is defined for $z \in \mathbb{C}_+$ as

$$m(z) := \frac{u_+'(0,z)}{u_+(0,z)}.$$

The function $m(z)$ is analytic in $\mathbb{C}_+$ and satisfies the Herglotz property: $m : \mathbb{C}_+ \to \mathbb{C}_+$, so m satisfies a Herglotz representation theorem,

$$m(z) = c + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\rho(t),$$

where $c = \operatorname{Re} m(i)$ and ρ is spectral measure of H. The measure can be recovered from $m(z)$ by the rule:

$$d\rho(t) = \text{w-}\lim_{\varepsilon \to +0} \frac{1}{\pi} \operatorname{Im} m(t + i\varepsilon)\, dt.$$

On the problems of uniqueness and recovering the potential from the Weyl function we refer to to classical papers by Borg [10] and Marchenko [20], and to modern results by Simon [23] and Gesztesy and Simon [14]. The inverse problem on recovering the potential from the spectral measure $d\rho$ was solved by Krein in [16, 17] and Gelfand and Levitan in [13].

2.3. Quantum scattering inverse data. We consider the Schrödinger equation with the real-valued potential $q \in L_{1+|x|}(\mathbb{R}_+)$

$$-\phi'' + q(x)\phi = k^2\phi, \quad x > 0. \tag{2.13}$$

The solution $e(k,x)$ of the above equation is determined by the condition

$$\lim_{x \to \infty} e^{-ikx} e(k,x) = 1.$$

It admits the representation

$$e(k,x) = e^{ikx} + \int_x^\infty K(x,t) e^{ikt}\, dt,$$

where the kernel $K(x,t)$ solves certain Goursat problem. The pair $\{e(k,x), e(-k,x)\}$ forms a fundamental set of solutions when $k \in \mathbb{R}$. Another solution to (2.13) $\varphi(k,x)$ is defined by the the conditions

$$\varphi(k,0) = 0, \quad \varphi_x(k,0) = 1.$$

We set the notation $M(k)=e(0,k)$. Then e and φ when k is on real axis are connected by

$$-\frac{2ik\varphi(k,x)}{M(k)}=e(-k,x)-S(k)e(k,x), \tag{2.14}$$

where the *scattering matrix* is defined by

$$S(k)=\frac{M(-k)}{M(k)}=\frac{1+\widehat{K}(0,-k)}{1+\widehat{K}(0,k)},\quad k\in\mathbb{R}.$$

And on introducing the amplitude and phase of $M(k)$, we have:

$$M(k)=A(k)e^{i\eta(k)},\quad A(k)=|M(k)|,\ \eta(k)=\arg M(k), \tag{2.15}$$

$$A(k)=A(-k),\quad \eta(k)=-\eta(-k). \tag{2.16}$$

The problem (2.13) has a finite number of (negative) eigenvalues $-k_1^2,\dots,-k_n^2$, where ik_l are zeroes of the function $e(k,0)$, $l=1\dots,n$. By $(C_j)^{-1}$ we denote $(C_j)^{-1}=\int\limits_0^\infty|e(ik_j,x)|^2\,dx$. Then the set of functions

$$\{\varphi(k,x),\,k\in\mathbb{R}_+,\quad \varphi_j(x)=e(ik_j,x),\,j=1,\dots,n\}$$

is a complete orthonormal set of eigenfunctions of the problem (2.13). The Parseval identity has the form

$$\delta(x-y)=\sum_{j=1}^n C_j^2\varphi_j(x)\varphi_j(y)+\int\limits_0^\infty\varphi(x,k)\frac{1}{M(k)M(-k)}\varphi(y,k)k^2\,dk.$$

After we introduce notations (here $f\in L_2(\mathbb{R}_+)$)

$$U(k):=\frac{1}{M(-k)M(k)},$$

$$\left(F^sf\right)(k)=\int\limits_0^\infty f(x)\varphi(k,x)\,dx,\quad \left(F_j^sf\right)=\int\limits_0^\infty f(x)\varphi_j(x)\,dx, \tag{2.17}$$

the Parseval equality for arbitrary $f,g\in L_2(\mathbb{R}_+)$ reads

$$(f,g)_{L_2(\mathbb{R}_+)}=\sum_{j=1}^n C_j^2\left(F_j^sf\right)\left(F_j^sg\right)+\frac{2}{\pi}\int\limits_0^\infty\left(F^sf\right)(k)\left(F^sg\right)(k)U(k)k^2\,dk.$$

The set

$$S_D:=\left\{(k_j,C_j)_{j=1}^n,\,S(k),\,k\in\mathbb{R}\right\}$$

is called the *scattering data*. For the solution of the inverse problem from S_D see [12] and references therein. It is important that the set S_D determines the function $M(k)$ and $U(k)$.

2.4. Inverse acoustical scattering problem. We consider the dynamical system associated with the (forward) problem

$$\begin{cases} u_{tt} - u_{xx} + qu = 0, \quad x > 0, \ -\infty < t < x \\ u|_{t<-x} = 0 \\ \lim_{s\to\infty} u(s, \tau - s) = f(\tau), \quad \tau \geqslant 0, \\ u(0,t) = 0, \end{cases} \tag{2.18}$$

where $q \in C^\infty[0,\infty)$, $\operatorname{supp} q \subset [0,a]$, $a < \infty$ is a *potential*, f is a *control*, $u = u^f(x,t)$ is a solution (*wave*). Notice that due to the hyperbolicity of (2.18), the boundary condition at $x = 0$ does not influence on the solution to the inverse problem [8].

Since $q|_{x>a} = 0$, for large x's the solution satisfies $u_{tt} - u_{xx} = 0$ and, hence, is a sum of two D'Alembert waves:

$$u^f(x,t)|_{x>a} = f(x+t) + f^*(x-t), \tag{2.19}$$

where the second summand describes the wave reflected by the potential and outgoing to $x = \infty$. Taking $f = \delta(t)$, one can introduce a fundamental solution of the form $u^\delta(x,t) = \delta(t+x) + w(x,t)$, which satisfies

$$u^\delta(x,t)|_{x>a} = \delta(x+t) + p(x-t) \tag{2.20}$$

with $p \in C^\infty[0,\infty)$, $\operatorname{supp} p \subset (-\infty, 2a)$. The Duhamel representation $u^f = u^\delta * f$ holds for the classical solutions. Note that $\operatorname{supp} f^* \subset [0,2a]$, so that the reflected wave $f^*(x-t)$ in (2.19) is compactly supported on $t \leqslant x \leqslant \infty$ for any t.

An *outer space* of the system (2.18) is the space of controls $\mathcal{F} := L_2(0,\infty)$. An *inner space* is $\mathcal{H} := L_2(0,\infty)$ (of functions of x). A *control operator* $W : \mathcal{F} \to \mathcal{H}$ acts by the rule

$$(Wf)(x) := u^f(x,0), \qquad x \geqslant 0.$$

It maps $\mathcal{F}$ onto $\mathcal{H}$ isomorphically. These facts are derived from the representation

$$(Wf)(x) = f(x) + \int_0^x w(x,-s) f(s)\, ds, \qquad x \geqslant 0. \tag{2.21}$$

A *connecting operator* $C : \mathcal{F} \to \mathcal{F}$,

$$C := W^*W$$

is a positive definite isomorphism in $\mathcal{F}$. It connects the metrics of the outer and inner spaces:

$$(Cf, g)_{\mathcal{F}} = (Wf, Wg)_{\mathcal{H}} = \big(u^f(\,\cdot\,,0), u^g(\,\cdot\,,0)\big)_{\mathcal{H}}. \tag{2.22}$$

A **response operator** of the system (2.18) is $R : \mathcal{F} \to \mathcal{F}$,

$$(Rf)(\tau) := \lim_{s\to+\infty} u^f(s, s-\tau), \qquad \tau \geqslant 0.$$

For $f \in \mathcal{F}$ vanishing at ∞, by (2.19), this limit is $f^*(\tau)$. Hence we get

$$(Rf)(\tau) = \int_0^\infty p(\tau+s)f(s)\,ds, \qquad \tau \geqslant 0. \tag{2.23}$$

Here p, the *acoustical response function* could be determined as a response to delta function:

$$p(\tau) = \lim_{s\to+\infty} u^\delta(s, s-\tau), \quad \tau > 0.$$

The inverse acoustical scattering problem is to recover potential $q|_{x\geqslant 0}$ by given response operator R (or what is equivalent, from acoustical response $p|_{t\geqslant 0}$). Note that to recover $q|_{(0,a)}$ it is enough to know $p|_{(0,2a)}$ (see [8]).

Theorem 2. *The equality*

$$C = \mathbb{I} + R \tag{2.24}$$

holds, where $\mathbb{I}$ *is the identity operator in* $\mathcal{F}$.

A natural setup of a *control problem* for the system (2.18) is by given $y \in \mathcal{H}$ to find $f \in \mathcal{F}$ such that $u^f(\,\cdot\,,0) = y$. This problem is equivalent to the equation $Wf = y$, which has a unique solution $f = W^{-1}y \in \mathcal{F}$ due to (2.21).

Let us consider a *special control problem*: take y, which satisfies

$$\begin{cases} -y'' + qy = k^2 y, & x > 0, \\ y|_{x>a} = e^{ikx}. \end{cases} \tag{2.25}$$

Theorem 3. *The control $f = W^{-1}y$, which solves the special CP, is the solution of the equation $Cf = e^{ik(\,\cdot\,)}$ in $\mathcal{F}$, i.e., it satisfies*

$$f(\tau) + \int_0^\infty r(\tau + s) f(s)\, ds = e^{ik\tau}, \qquad \tau \geqslant 0. \tag{2.26}$$

Writing (2.26) in the form $W^*Wf = e^{ik(\,\cdot\,)}$, with regard to $Wf = y$, we have

$$W^* y(\,\cdot\,, k) = e^{ik(\,\cdot\,)}. \tag{2.27}$$

Hence, W^* is a *transformation operator*, which maps the solution $y(x,k)$ of (2.25) to the solution e^{ikx} of the unperturbed problem.

§3. Kernel of the connecting operator C^T

3.1. The spectral function of Levitan and the kernel of the connecting operator. Here we derive the spectral representation of the connecting operator (2.3), (2.4) following [1].

We take a Fourier transform (2.10) of $u^f(\,\cdot\,, T)$ and use the transformation property (2.6) of $\left(W^T\right)^*$:

$$\left(Fu^f(\,\cdot\,, T)\right)(\mu) = \int_{-\infty}^{\infty} u^f(x,T)\varphi(x,\mu)\, dx = (W^T f, \varphi(\,\cdot\,,\mu))_{\mathcal{H}^T} \tag{3.1}$$

$$= (f, \left(W^T\right)^* \varphi(\,\cdot\,,\mu))_{\mathcal{H}^T} = \int_0^T \frac{\sin\sqrt{\mu}s}{\sqrt{\mu}} f(T-s)\, ds.$$

Let $f, g \in \mathcal{F}^T$. Using (2.9) and (2.10), we rewrite $\left(C^T f, g\right)_{\mathcal{F}^T}$ as

$$\left(C^T f, g\right)_{\mathcal{F}^T} = \int_0^T u^f(x,T) u^g(x,T)\, dx = \int_{-\infty}^{\infty} (Fu^f)(\lambda, T)(Fu^g)(\lambda, T)\, d\rho(\lambda). \tag{3.2}$$

Making use of (3.1), we can rewrite (3.2) as

$$\left(C^T f, g\right)_{\mathcal{F}^T} = \int_{-\infty}^{\infty} \int_0^T \int_0^T \frac{\sin\sqrt{\lambda}(T-t) \sin\sqrt{\lambda}(T-s)}{\lambda} f(t) g(s)\, dt\, ds\, d\rho(\lambda). \tag{3.3}$$

Now we make use of the sin transform: for all $h, j \in L^2(\mathbb{R}_+)$

$$\widehat{h}(\lambda) = \int_0^\infty h(x)\frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}\,dx, \quad h(x) = \int_0^\infty \widehat{h}(\lambda)\sin(\sqrt{\lambda}x)\,d\left(\frac{2}{3\pi}\lambda^{\frac{3}{2}}\right),$$

$$\int_0^\infty h(x)j(x)\,dx = \int_0^\infty \widehat{h}(\lambda)\widehat{j}(\lambda)\,d\left(\frac{2}{3\pi}\lambda^{\frac{3}{2}}\right).$$

Let us extend the functions f and g to the whole real axis setting $f(t) = g(t) = 0$ for $t > T$ and $t < 0$ and use the notation $f_T(s) = f(T-s)$. Then we can rewrite

$$\int_0^T f(t)g(t)\,dt = \int_0^\infty f(T-s)g(T-s)\,ds = \int_0^\infty \widehat{f}_T(\lambda)\widehat{g}_T(\lambda)\,d\left(\frac{2}{3\pi}\lambda^{\frac{3}{2}}\right)$$

$$= \int_0^\infty\int_0^T\int_0^T \frac{\sin\sqrt{\lambda}(T-t)\sin\sqrt{\lambda}(T-s)}{\lambda}f(t)g(s)\,dt\,ds\,d\left(\frac{2}{3\pi}\lambda^{\frac{3}{2}}\right). \tag{3.4}$$

On introducing the function

$$\sigma(\lambda) = \begin{cases} \rho(\lambda) - \frac{2}{3\pi}\lambda^{\frac{3}{2}}, & \lambda \geqslant 0, \\ \rho(\lambda), & \lambda < 0, \end{cases}$$

we can rewrite (3.3) using (3.4) and counting that for fixed n we we can change the order of integration:

$$\lim_{n\to\infty}\int_0^T\int_0^T\int_{-\infty}^n \frac{\sin\sqrt{\lambda}(T-t)\sin\sqrt{\lambda}(T-s)}{\lambda}\,d\sigma(\lambda)f(t)g(s)\,dt\,ds \tag{3.5}$$

$$= \int_0^T\int_0^T c^T(s,t)f(t)g(s)\,dt\,ds.$$

Let us introduce the function

$$\Psi_n(t,s) := \int_{-\infty}^n \frac{\sin\sqrt{\lambda}(T-t)\sin\sqrt{\lambda}(T-s)}{\lambda}\,d\sigma(\lambda). \tag{3.6}$$

Since f, g are arbitrary functions from $\mathcal{F}^T$, we can deduce from (3.5) that

$$\Psi_n(t,s) \xrightarrow[n\to\infty]{} c^T(t,s), \quad \text{weekly in } L_2\left((0,T)^2\right).$$

To strengthen the result on the convergence we need the theorem of Levitan [19] on the convergence of spectral functions:

Theorem 4. *The sequence of functions*

$$\Phi_n(t,s)=\int_{-\infty}^{n}\varphi(t,\lambda)\varphi(s,\lambda)\,d\rho(\lambda)-\int_0^n\frac{\sin\sqrt{\lambda}t\sin\sqrt{\lambda}s}{\lambda}\,d\left(\frac{2}{3\pi}\lambda^{\frac{3}{2}}\right), \quad (3.7)$$

converges uniformly on every bounded set to a differentiable outside the diagonal function, as n tends to infinity.

Applying operators $(\mathbf{I}_s+\mathbf{L}_s)(\mathbf{I}_t+\mathbf{L}_t)$ (see (2.11)) to (3.7) we have:

$$\begin{aligned}(\mathbf{I}_s+\mathbf{L}_s)(\mathbf{I}_t+\mathbf{L}_t)\Phi_n(t,s)&=\Psi_n(s,t) \qquad (3.8)\\
&-\int_0^n\left(\int_0^t L(t,\tau)\frac{\sin\sqrt{\lambda}\tau}{\sqrt{\lambda}}\,d\tau\right)\frac{\sin\sqrt{\lambda}s}{\sqrt{\lambda}}\,d\left(\frac{2}{3\pi}\lambda^{\frac{3}{2}}\right)\\
&-\int_0^n\left(\int_0^s L(s,\tau)\frac{\sin\sqrt{\lambda}\tau}{\sqrt{\lambda}}\,d\tau\right)\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\,d\left(\frac{2}{3\pi}\lambda^{\frac{3}{2}}\right)\\
&-\int_0^n\left(\int_0^t L(t,\tau)\frac{\sin\sqrt{\lambda}\tau}{\sqrt{\lambda}}\,d\tau\right)\left(\int_0^s L(s,\tau)\frac{\sin\sqrt{\lambda}\tau}{\sqrt{\lambda}}\,d\tau\right)d\left(\frac{2}{3\pi}\lambda^{\frac{3}{2}}\right).\end{aligned}$$

The sum of the last three terms in the right hand side of the above expression converges to $-L(s,t)-L(t,s)-\int_0^{\min\{s,t\}}L(s,\tau)L(t,\tau)\,d\tau$. This fact and the convergence of the left hand side of (3.8) imply that

$$\Psi_n(t,s)\xrightarrow[n\to\infty]{}c^T(t,s)=\int_{-\infty}^{\infty}\frac{\sin\sqrt{\lambda}(T-t)\sin\sqrt{\lambda}(T-s)}{\lambda}\,d\sigma(\lambda), \quad (3.9)$$

uniformly on every compact set in $\mathbb{R}^2$.

The estimates on regularized spectral function $\Phi_n(t,s)$ receive a lot of attention, to mention [22] and literature cited therein. We believe that the connection of the regularized spectral function Φ_n with the kernel of the connecting operator C^T allows one to extends some of the results to different dynamical systems, for example to vector Schrödinger system, Dirac system, canonical systems.

§4. On the connection of the spectral, dynamical and scattering data

4.1. Weyl function and response function. We now demonstrate the connection between the response function $r(s)$ and the Titchmarsh-Weyl m-function. A connection between spectral and time-domain data is widely used in inverse problems, see, e.g., [6,7,15] where the equivalence of several types of boundary inverse problems is discussed.

Let $f \in C_0^\infty(0,\infty)$ and

$$\widehat{f}(k) := \int_0^\infty f(t)\, e^{-kt}\, dt$$

be its Laplace transform. Function $\widehat{f}(k)$ is well defined for $k \in \mathbb{C}$. Going in (1.1) over to the Laplace transforms, one has

$$\begin{cases} -\widehat{u}_{xx}(x,k) + q(x)\widehat{u}(x,k) = -k^2\widehat{u}(x,k), \\ \widehat{u}(0,k) = \widehat{f}(k), \end{cases}$$

and

$$\widehat{(Rf)}(k) = \widehat{u}_x(0,k),$$

respectively. The values of the function $\widehat{u}(0,k)$ and its first derivative at the origin, $\widehat{u}_x(0,k)$, are related through the Titchmarsh–Weyl m-function

$$\widehat{u}_x(0,k) = m(-k^2)\widehat{f}(k)\,.$$

Therefore,

$$\widehat{(Rf)}(k) = m(-k^2)\widehat{f}(k)\,, \tag{4.1}$$

and thus the spectral and dynamic Dirichlet-to-Neumann maps are in one-to-one correspondence. Taking the Laplace transform of (2.2) we get

$$\widehat{(Rf)}(k) = -k\widehat{f}(k) + \widehat{r}(k)\widehat{f}(k). \tag{4.2}$$

In [3] the authors have shown that, if the potential belongs to the class

$$q \in l^\infty\left(L^1\left(\mathbb{R}_+\right)\right) := \left\{ q : \int_n^{n+1} |q(x)|\, dx \in l^\infty \right\}$$

with the norm defined by $||q|| = \sup \int\limits_x^{x+1} |q\,(s)|\, ds < \infty$. Then (4.1) and (4.2) imply

$$m(-k^2) = -k + \int\limits_0^\infty e^{-k\alpha} r(\alpha)\, d\alpha\,, \tag{4.3}$$

with the integral in (4.3) is absolute convergent for $z = -k^2$ where $\operatorname{Re} k > 2\max\{\sqrt{2\,||q||}, e\,||q||\}$.

We notice that it was shown in [23] that there exists a unique real valued function $A \in L^1_{loc}\,(\mathbb{R}_+)$ (the $A-$*amplitude*) such that

$$m(-k^2) = -k - \int\limits_0^\infty A(t)e^{-2tk}\, dt\,. \tag{4.4}$$

The absolute convergence of the integral was proved for $q \in L^1\,(\mathbb{R}_+)$ and $q \in L^\infty\,(\mathbb{R}_+)$ in [14] for sufficiently large $\operatorname{Re} k$. Clearly, the $A-$amplitude and response function are connected by

$$A(t) = -2r(2t).$$

4.2. Response function and spectral measure. Using the representation for $c^T(t,s)$ (3.9), we can derive the formula for the response function:

Theorem 5. *The representation for the response function* r

$$r(t) = \int\limits_{-\infty}^{\infty} \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\, d\sigma(\lambda), \tag{4.5}$$

holds for almost all $t \in [0, +\infty)$.

Proof. Let us note that according to (3.9)

$$\int\limits_{-\infty}^{\infty} \frac{\sin\sqrt{\lambda}t \sin\sqrt{\lambda}s}{\lambda}\, d\sigma(\lambda) = c^T(T-t, T-s), \quad t,s \in [0,T]. \tag{4.6}$$

Using (2.5), we have

$$c^T(T-t, T-s) = \frac{1}{2} \int\limits_{|t-s|}^{t+s} r(\tau)\, d\tau, \quad t,s \in [0,T]. \tag{4.7}$$

The integral in (4.6) can be rewritten as

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{(\cos\sqrt{\lambda}(s+t)-1-(\cos\sqrt{\lambda}|s-t|-1)}{\lambda}\,d\sigma(\lambda) \tag{4.8}$$
$$=\frac{1}{2}\int_{-\infty}^{\infty}\int_{|t-s|}^{t+s}\frac{\sin\sqrt{\lambda}\theta}{\sqrt{\lambda}}\,d\theta\,d\sigma(\lambda),\quad t,s\in[0,T].$$

Equating the expressions in (4.7) and (4.8) for $t=s$ we get

$$2c^T(T-t,T-t)=\int_0^{2t}r(\tau)\,d\tau=\int_{-\infty}^{\infty}\int_0^{2t}\frac{\sin\sqrt{\lambda}\theta}{\sqrt{\lambda}}\,d\theta\,d\sigma(\lambda),\quad t\in[0,T]. \tag{4.9}$$

According to (2.2), $r\in L^1(0,T)$, so we can use the Lebesgue theorem and differentiate the last equation. We obtain the following equality almost everywhere on $[0,2T]$:

$$r(t)=\int_{-\infty}^{\infty}\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\,d\sigma(\lambda).$$

Since the parameter T can be chosen arbitrary large, the last formula proves the statement of the proposition. $\square$

A different proof is given in [1], see also [23], where the regularized version of (4.5) was derived.

The finite speed of the wave propagation in (1.1) implies the local nature of the response function $r(t)$: the values of $r(t)$, $t\in[0,2T]$ are determined by the potential $q(x)$, $x\in[0,T]$. That is why if we are interested in the representation of $r(t)$ on the interval $t\in[0,2T]$, we can replace in (4.5) the regularized spectral function $\sigma(\lambda)$ by any of the following functions:

$$\sigma_{\mathrm{tr}}(\lambda)=\begin{cases} l\rho_{\mathrm{tr}}(\lambda)-\frac{2}{3\pi}\lambda^{\frac{3}{2}}, & \lambda\geqslant 0,\\ \rho_{\mathrm{tr}}(\lambda), & \lambda<0,\end{cases},\quad \sigma_d(\lambda)=\begin{cases}\rho_d(\lambda)-\rho_0(\lambda), & \lambda\geqslant 0,\\ \rho_d(\lambda), & \lambda<0.\end{cases}$$

Here ρ_{tr} is the spectral function corresponding to the truncated potential: $q_T(x)=q(x)$ for $0\leqslant x\leqslant T$ and $q_T(x)=\tilde{q}(x)$ for $x>T$ with arbitrary locally integrable $\tilde{q}$; $\rho_d(\lambda)$ is the spectral function associated to the discrete problem on the interval $(0,T)$ with the potential $q_d(x)=q(x)$, $x\in[0,T]$ and $\rho_0(\lambda)$ is the spectral function associated to the discrete problem on

$[0,T]$ with zero potential. (Any self-adjoint boundary condition can be prescribed at $x = T$).

We notice that the function $\Phi(t) = \int\limits_0^t r(\tau)\,d\tau$, in accordance with (4.5) is given by

$$\Phi(t) = \int\limits_{-\infty}^{\infty} \frac{1 - \cos\sqrt{\lambda}\tau}{\lambda}\,d\sigma(\lambda), \quad 0 < t < 2T$$

has been used by Krein in [16, 17] as an inverse data.

4.3. Quantum scattering data and response function. Using the representation for $c^T(t,s)$ obtained in (3.9), we can derive the formula for the response function:

Theorem 6. *The representation for the response function r in terms of scattering data:*

$$r(t) = \sum_{j=1}^{n} C_j^2 \frac{\sin k_j t}{k_j} + \frac{2}{\pi}\int\limits_0^{\infty} \sin kt\,(U(k) - 1)\,k\,dk \tag{4.10}$$

holds for almost all $t \in [0, +\infty)$.

Proof. Let us take arbitrary $f, g \in \mathcal{F}^T$ and consider the connecting operator C^T (2.3)

$$\left(C^T f, g\right)_{\mathcal{F}^T} = \left(u^f(\,\cdot\,,T), u^g(\,\cdot\,,T)\right)_{\mathcal{H}^T}. \tag{4.11}$$

Where u^f is solutions to the wave equation (1.1) with the control f. Rewriting (4.11) using the Parseval identity, we obtain

$$\begin{aligned}\left(C^T f, g\right)_{\mathcal{F}^T} &= \sum_{j=1}^{n} C_j^2 F_j^s(u^f(\,\cdot\,,T)) F_j^s(u^g(\,\cdot\,,T)) \\ &\quad + \frac{2}{\pi}\int\limits_0^{\infty} \left(F^s u^f(\,\cdot\,,T)\right)(k)\left(F^s u^g(\,\cdot\,,T)\right)(k) U(k) k^2\,dk.\end{aligned}$$

Using the transformation property (2.6) of $\left(W^T\right)^*$ yields

$$\left(F^s u^f(\,\cdot\,,T)\right)(k) = \int_0^T \varphi(x,k)u^f(x,T)\,dx = \left(\varphi(x,k), W^T f\right)_{H^T}$$
$$= \left(\left(W^T\right)^* \varphi(x,k), f\right)_{H^T} = \int_0^T f(T-s)\frac{\sin ks}{k}\,ds.$$

Similarly,

$$\left(F_j^s u^f(\,\cdot\,,T)\right)(k) = \int_0^T f(T-s)\frac{\sin k_j s}{k_j}\,ds.$$

Using these observations we get an equivalent expression for (4.11):

$$\left(C^T f, g\right)_{\mathcal{F}^T} = \sum_{j=1}^n C_j^2 \int_0^T\int_0^T \frac{\sin k_j(T-s)}{k_j}\frac{\sin k_j(T-\tau)}{k_j} f(s)g(\tau)\,ds\,d\tau$$
$$+ \frac{2}{\pi}\int_0^\infty\int_0^T\int_0^T \frac{\sin k(T-s)}{k}\frac{\sin k(T-\tau)}{k} U(k)k^2 f(s)g(\tau)\,ds\,d\tau\,dk. \quad (4.12)$$

Using the representation for C^T (2.4), (2.5) and

$$\int_0^T f(t)g(t)\,dt = \frac{2}{\pi}\int_0^\infty\int_0^T\int_0^T \frac{\sin k(T-s)}{k}\frac{\sin k(T-\tau)}{k} k^2 f(s)g(\tau)\,ds\,d\tau\,dk,$$

we can rewrite (4.12) as

$$\int_0^T\int_0^T c^T(t,s)f(t)g(s)\,dt\,ds \quad (4.13)$$
$$= \int_0^T\int_0^T \sum_{j=1}^n C_j^2 \frac{\sin k_j(T-s)}{k_j}\frac{\sin k_j(T-\tau)}{k_j} f(s)g(\tau)\,ds\,d\tau$$
$$+ \frac{2}{\pi}\int_0^\infty\int_0^T\int_0^T \frac{\sin k(T-s)}{k}\frac{\sin k(T-\tau)}{k}(U(k)-1)k^2 f(s)g(\tau)\,ds\,d\tau\,dk.$$

We notice that it is possible to change the order of integration in the last integral in (4.13) due to results on convergence from [12]. The latter observation leads to the representation for the kernel $c(t,x)$:

$$c(t,x) = \sum_{j=1}^{n} C_j^2 \frac{\sin k_j(T-x)}{k_j} \frac{\sin k_j(T-t)}{k_j} + \frac{2}{\pi} \int_0^\infty \frac{\sin k(T-t)}{k} \frac{\sin k(T-x)}{k} (U(k)-1)\, k^2\, dk.$$

We have that on the one hand (4.7), and on the other hand

$$c(T-t,T-x) = \sum_{j=1}^{n} C_j^2 \frac{\sin k_j x}{k_j} \frac{\sin k_j t}{k_j} + \frac{2}{\pi} \int_0^\infty \frac{\sin kt}{k} \frac{\sin kx}{k} (U(k)-1)\, k^2\, dk. \tag{4.14}$$

Note that

$$\frac{\sin kt}{k} \frac{\sin kx}{k} = \frac{1}{2} \int_{|t-x|}^{t+x} \frac{\sin k\theta}{k}\, d\theta. \tag{4.15}$$

Using (4.7), (4.14) and (4.15) we arrive at (4.10). This formula but in weaker form was derived in [14]. □

4.4. Scattering matrix and acoustical response function. Let $f \in C_0^\infty(0,\infty)$ and

$$(Ff)(k) := \int_{-\infty}^{\infty} f(t)\, e^{ikt}\, dt$$

be its Fourier transform. Function $\widehat{f}(k)$ is well defined for $k \in \mathbb{C}$. Going in (2.18) with control $f = \delta$ over to the Fourier transforms, one has

$$\begin{cases} -\widehat{u}^\delta_{xx}(x,k) + q(x)\widehat{u}^\delta(x,k) = k^2 \widehat{u}^\delta(x,k), \\ \widehat{u}(0,k) = 0. \end{cases}$$

On the other hand, for $x > a$ one has the representation (2.20) for u^δ, applying Fourier transform to it, we get

$$\widehat{u}^\delta(x,k) = e^{-ikx} + \int\limits_{-\infty}^{\infty} p(s)e^{-iks}\,ds\,e^{ikx}, \quad x > a. \tag{4.16}$$

Comparing (4.16) with (2.14), we conclude that in the case of absence of the discrete spectrum the scattering matrix and acoustical response are connected by

$$S(k) = -\int\limits_{-\infty}^{\infty} p(s)e^{-iks}\,ds = -2\pi\left(F^{-1}p\right)(k), \tag{4.17}$$

where the inverse Fourier transform is understood in the sence of generalized functions. We study the connecting operator for the acoustical problem (2.22). Using the transformation F^s (2.17), we rewrite (2.22) as

$$(Cf,g)_{\mathcal{F}} = (Wf,Wg)_{\mathcal{H}} = \sum_{j=1}^{n} C_j^2\left(F_j^sWf\right)\left(F_j^sWg\right) \tag{4.18}$$
$$+\frac{2}{\pi}\int\limits_0^{\infty}\left(F^sWf\right)(k)\left(F^sWg\right)(k)U(k)k^2\,dk.$$

Let us evaluate using (2.14) and transformation property of W^* (2.27):

$$\left(F^sWf\right)(k) = \int\limits_0^{\infty}(Wf)(x)\varphi(x,k)\,dx \tag{4.19}$$
$$= \int\limits_0^{\infty}(Wf)(x)\frac{i}{2k}\left(M(k)e(x,-k) - M(-k)e(x,k)\right)dx$$
$$= \frac{i}{2k}\left(f, M(k)e^{-ik\,\cdot} - M(-k)e^{ik\,\cdot}\right);$$
$$\left(F_j^sWf\right)(k) = \int\limits_0^{\infty}(Wf)(x)\varphi_j(x)\,dx = (Wf, e(ik_j,\,\cdot)) \tag{4.20}$$
$$= (f, W^*e(ik_j,\,\cdot)) = \left(f, e^{-k_j\,\cdot}\right) =: f_j.$$

We continue evaluate (4.18) using (4.19), (4.19) and (2.15), (2.16):

$$(Cf,g)_{\mathcal{F}} = \sum_{j=1}^{n} C_j^2 f_j g_j$$
$$+ \frac{2}{\pi}\int\limits_0^\infty \left(f, \frac{e^{i(\eta(k)-k\,\cdot\,)} - e^{i(k\,\cdot\,-\eta(k))}}{2i}\right)\left(g, \frac{e^{i(\eta(k)-k\,\cdot\,)} - e^{i(k\,\cdot\,-\eta(k))}}{2i}\right) dk$$
$$= \sum_{j=1}^{n} C_j^2 f_j g_j + \frac{2}{\pi}\int\limits_0^\infty (f, \sin(k\,\cdot\,-\eta(k)))\,(g, \sin(k\,\cdot\,-\eta(k)))\; dk. \quad (4.21)$$

In the dynamical representation (2.22), (2.26):

$$(Cf,g)_{\mathcal{F}} = \frac{2}{\pi}\int\limits_0^\infty (f, \sin(k\,\cdot))\,(g, \sin(k\,\cdot))\; dk + \int\limits_0^\infty\int\limits_0^\infty p(t+s)f(t)g(s)\,dt\,ds.$$

Then we can write:

$$\int\limits_0^\infty\int\limits_0^\infty p(t+s)f(t)g(s)\,dt\,ds = \int\limits_0^\infty\int\limits_0^\infty \sum_{j=1}^{n} C_j^2 e^{-k_j(x+y)} f(x)g(y)\,dx\,dy$$
$$+ \int\limits_0^\infty\int\limits_0^\infty f(x)g(y)\frac{2}{\pi}\int\limits_0^\infty \sin(kx-\eta(k))\sin(ky-\eta(k))\,dk$$
$$- \int\limits_0^\infty\int\limits_0^\infty f(x)g(y)\frac{2}{\pi}\int\limits_0^\infty \sin(kx)\sin(ky)\,dk$$

by the trigonometry,

$$\int\limits_0^\infty\int\limits_0^\infty p(t+s)f(t)g(s)\,dt\,ds = \int\limits_0^\infty\int\limits_0^\infty \sum_{j=1}^{n} C_j^2 e^{-k_j(x+y)} f(x)g(y)\,dx\,dy$$
$$+ \int\limits_0^\infty\int\limits_0^\infty f(x)g(y)\frac{1}{\pi}\int\limits_0^\infty (\cos k(x+y) - \cos(k(x+y)-2\eta(k)))\; dk$$

from where we deduce the representation for p on the interval $(0, 2a)$:

$$p(t) = \sum_{j=1}^{n} C_j e^{-k_j t} + \frac{1}{\pi} \int\limits_0^\infty \left(\cos kt - \cos\left(kt - 2\eta(k)\right)\right) dk, \tag{4.22}$$

$$p(t) = \sum_{j=1}^{n} C_j e^{-k_j t} + \frac{2}{\pi} \int\limits_0^\infty \left(\sin\eta(k)\sin\left(\eta(k) - kt\right)\right) dk, \tag{4.23}$$

where the right hand sides are understood as generalized functions.

Notice that the acoustical response in (4.17) plays the same role for the scattering matrix as response function (or $A-$amplitude) plays for Weyl function in (4.3) and (4.4).

We think that the result of the convergence of integrals in (4.22), (4.23) can be improved, we are planing to return to this question elsewhere in the framework of studying the inverse acoustical scattering problem for the system (2.18) with a potential with non compact support.

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Поступило 2 октября 2016 г.