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INTERACTION OF HECKE–SHIMURA RINGS AND ZETA FUNCTIONS

Abstract. An automorphic structure on a Lie group consists of Hecke–Shimura ring of an arithmetical discrete subgroup and a linear representation of the ring on an invariant space of automorphic forms given by Hecke operators. The paper is devoted to *interactions* (transfer homomorphisms) of Hecke–Shimura rings of integral symplectic groups and integral orthogonal groups of integral positive definite quadratic forms.

Introduction

There is a hope that further progress in diophantine number theory is closely related to investigation of representations of Hecke–Shimura rings (HS-rings) for arithmetical discrete subgroups of Lie groups on suitable spaces of automorphic forms and their interaction with each other. Important examples of, say, "vertical" interaction are given by lifts of automorphic structures to similar groups of higher orders (see, e.g., [7]). Not less, if not more, interesting are cases of "horizontal" interaction arising from consideration of HS-rings of different Lie groups, say, symplectic and orthogonal (see, e.g., [8]).

In general, an automorphic structure on a Lie group is a HS–ring of an arithmetical discrete subgroup of the group together with a linear representation of the ring on a space of automorphic forms by Hecke operators. An *interaction* from one automorphic structure to automorphic structure on another group consists of an interaction mapping of HS–rings of discrete subgroups compatible with action of corresponding Hecke operators on suitable spaces of automorphic forms.

In this paper we consider interaction mappings for HS–rings of certain subgroups of integral symplectic groups and groups of units of integral positive definite quadratic forms in even number of variables. For the action of Hecke operators on theta-series see, e.g., [3, 4, 9].

Key words and phrases: Hecke operators, Hecke–Shimura rings, interaction mappings, interaction sums, theta functions of integral quadratic forms.

Notation. We fix the letters $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$,and $\mathbb{C}$ for the ring of rational integers, the field of rational numbers, the field of real numbers, and the field of complex numbers, respectively.

If $\mathbb{A}$ is a set, $\mathbb{A}_n^m$ denotes the set of all $m \times n$-matrices with elements in $\mathbb{A}$. If $\mathbb{A}$ is a ring with the identity element, 1_n denote the identity element of the ring $\mathbb{A}_n^n$ and 0_n is the zero element of the ring.

The transpose of a matrix M is denoted by ${}^tM$. For two matrices Q and N of appropriate size we set

$$Q[N] = {}^tNQN.$$

For a complex square matrix A we write

$$\mathbf{e}\{A\} = \exp(\pi\sqrt{-1}\sigma(A)),$$

where $\sigma(A)$ is the sum of diagonal entries of A.

§1 Theta-series, symplectic HS–rings, Hecke operators

Let Q be an even positive definite matrix of order m. For $n = 1, 2, \ldots,$ we define the *theta-series of genus n of Q* as a function in variable

$$Z \in \mathbb{H}^n = \{Z = X + \sqrt{-1}Y \in \mathbb{C}_n^n \mid {}^t0Z = Z, \, Y > 0\}$$

(*the Siegel upper half-plane of genus n*) given by the series

$$\Theta(Z;\, Q) = \Theta^n(Z;\, Q) = \sum_{N \in \mathbb{Z}_n^m} \mathbf{e}\{0\}Q[N]Z. \tag{1.1}$$

The series converges absolutely and uniformly on any subset of $\mathbb{H}^n$ of the form $\{Z = X + \sqrt{-1}Y \in \mathbb{H}^n \mid Y \geqslant \varepsilon 1_n\}$ with $\varepsilon > 0$. Therefore, the series defines a real analytic function on $\mathbb{H}^n$.

According to [5, Theorems 4.1–4.3], theta-series has the following automorphic properties: if Q is an even positive definite matrix of even order $m = 2k$ and level q, then for each matrix M of the group

$$\Gamma_0^n(q) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{Z}) \,\Big|\, C \equiv 0 \pmod q \right\}$$

the theta-function (1.1) of Q of genus $n \geqslant 1$ satisfies the functional equation

$$\Theta\left(M\langle Z\rangle;\, Q\right) = j_Q(M,\, Z)\Theta(Z;\, Q), \tag{1.2}$$

where $M\langle Z\rangle = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)\langle Z\rangle = (AZ + B)(CZ + D)^{-1}$,

$$j_Q(M,\, Z) = \chi_Q(\det D)\det(CZ + D)^{m/2}$$

is automorphic factor, and χ_Q – the character of quadratic form with matrix Q.

On the other hand, the theta–series has symmetries corresponding to the action of the group $GL_m(\mathbb{Z})$ of integral unimodular matrices of order m. Namely,

$$\Theta^n(Z;\, Q[M]) = \Theta^n(Z;\, Q) \quad \text{for each } M \in GL_m(\mathbb{Z}). \tag{1.3}$$

Further, for $q \geqslant 1$, let us introduce the multiplicative semigroup

$$\Sigma_0^n(q) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{Z}_{2n}^{2n} \;\middle|\; {}^t0M \begin{pmatrix} 0_n & 1_n \\ -1_n, 0_n & \end{pmatrix} M \right.$$
$$\left. = \mu(M) \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix},\ C \equiv 0_n \pmod{q},\ \mu(M) > 0,\ \gcd(\mu(M),\, q) = 1 \right\}.$$

The group $\Gamma_0^n(q)$ can be considered as a subgroup of $\Sigma_0^n(q)$ consisting of matrices $M \in \Sigma_0^n(q)$ with *multiplier* $\mu(M) = 1$. Let

$$\mathcal{H}_0^n(q) = \mathcal{H}(\Gamma_0^n(q),\, \Sigma_0^n(q))$$

be the Hecke–Shimura ring (over $\mathbb{C}$) of the semigroup $\Sigma_0^n(q)$ relative to the subgroup $\Gamma_0^n(q)$. Here we only note that this ring consists of all those finite formal linear combinations T with complex coefficients of symbols $(\Gamma_0^n(q)M)$, corresponding in one-to-one way to different left cosets

$$\Gamma_0^n(q)M \subset \Sigma_0^n(q),$$

which are invariant with respect to all right multiplication by elements of $\Gamma_0^n(q)$:

$$T = \sum_\alpha c_\alpha(\Gamma_0^n(q)\, M_\alpha),\ T\gamma = \sum_\alpha c_\alpha(\Gamma_0^n(q)\, M_\alpha \gamma) = T \quad (\forall \gamma \in \Gamma_0^n(q)). \tag{1.4}$$

The semigroup $\Sigma_0^n(q)$ operates on the space $\mathcal{F}^n$ of all complex-valued real-analytic functions $F = F(Z) : \mathbb{H}^n \mapsto \mathbb{C}$ by the *Petersson operators*

$$\Sigma_0^n(q) \ni M : F = F(Z) \mapsto F|M = F|_{\mathbf{j}}M = j_Q(M,\, Z)^{-1} F(M\langle Z\rangle), \tag{1.5}$$

where $j_Q(M,\, Z)$ is the automorphic factor. Petersson operators map the space $\mathfrak{F}^n$ into itself and satisfy the rule $F|M|M' = F|MM'$. It allows us to define the standard representation $T \mapsto |T = |_{\mathbf{j}}T$ of the HS–ring $\mathcal{H}_0^n(q)$ on the subspace

$$\mathfrak{F}_0^n = \{F \in \mathfrak{F}^n \mid F|M = F \ \text{ for all } \ M \in \Gamma_0^n(q)\} \tag{1.6}$$

of all $\Gamma_0^n(q)-$invariant functions of $\mathfrak{F}^n$ by *Hecke operators*: the Hecke operator $|T = |_{\mathbf{j}}T$ corresponding to an element of $\mathcal{H}_0^n(q)$ of the form (1.4) is defined by

$$F|T = \sum_{\alpha} c_\alpha F|M_\alpha \quad \bigl(F = F(Z) \in \mathfrak{F}_0^n\bigr), \tag{1.7}$$

where $|M_\alpha$ are the Petersson operators (1.5). The Hecke operators are independent of the choice of representatives $M_\alpha \in \Gamma_0^n(q)M_\alpha$ and map the space $\mathfrak{F}_0^n$ into itself. The map $T \mapsto |T$ is a linear representation of the ring $\mathcal{H}_0^n(q)$ on the space $\mathfrak{F}_0^n$.

Theta-functions of different genuses n of a fixed quadratic form are related by *Siegel operators* $\Phi^{n,r} : \mathfrak{F}^n \mapsto \mathfrak{F}^r$, where $0 \leqslant r \leqslant n$, whereas the action of Hecke operators on the spaces are related by *Zharkovskaya homomorphisms* $\Psi_Q^{n,r} : \mathcal{H}_0^n(q) \mapsto \mathcal{H}_0^r(q)$ of HS-rings. For definition and properties of the mappings see [9, §4]. Here we shall only note that the Zharkovskaya homomorphism $\Psi_Q^{n,r}$ is not always epimorphic, it is epimorphic if $n, r \geqslant m/2$ [6, Proposition 3.3].

The functional equations (1.2) show that theta-functions $\Theta^n(Z;\, Q)$, viewed as functions of Z, belong to the space $\mathfrak{F}_0^n$. Explicit formulas for the action of Hecke operators on theta-functions show that images of the theta-functions under Hecke operators can be often written as finite linear combinations with constant coefficients of similar theta-functions. According to [1, Theorem 1] and [9, Propositions 5.1, 5.2(2)], for each homogeneous element $T \in \mathcal{H}_0^n(q)$ of a multiplier μ (i.e., a linear combination of left cosets $(\Gamma_0^n(q)\, M_\alpha)$ with a fixed multiplier $\mu(\Gamma_0^n(q)\, M_\alpha) = \mu(M_\alpha) = \mu$), which in the case $n < m$ belongs to the image of the ring $\mathcal{H}_0^m(q)$ under the Zharkovskaya map $\Psi_Q^{m,n} : \mathcal{H}_0^m(q) \mapsto \mathcal{H}_0^n(q)$, the image of the theta-function (1.1) under the Hecke operator $|T$ can be written as a linear combination with constant coefficients of similar theta-functions in the form

$$\Theta^n(Z;\, Q)|T = \begin{cases} \sum_{D \in A(Q,\,\mu)/GL_m(\mathbb{Z})} I(D,\, Q,\, \Psi_Q^{n,m}T)\Theta^n(Z;\, \mu^{-1}Q[D]), \\ 0, \end{cases} \tag{1.8}$$

depending on whether the set

$$A(Q, \mu) = \left\{ D \in \mathbb{Z}_m^m \;\middle|\; \mu^{-1}Q[D] \in \mathbb{E}^m,\, \det \mu^{-1}Q[D] = \det Q \right\}, \tag{1.9}$$

of all *automorphes of* Q *with multiplier* μ, is not empty or empty, where the element $\Psi_Q^{n,m}T = T' \in \mathcal{H}_0^m(q)$ is the image of T under the Zharkovskaya map if $n \geqslant m$ and an inverse image if $n < m$, and where for the element $T' = \sum_\alpha c_\alpha\bigl(\Gamma_0^m(q)N_\alpha\bigr)$ written with "riangular" representatives $N_\alpha = \begin{smallmatrix} A_\alpha & B_\alpha \\ 0_m & D_\alpha \end{smallmatrix}$, the coefficients on the right are the *interaction sums*

$$I(D, Q, T') = \sum_{\alpha,\, D\, {}^t0D_\alpha \equiv 0 \pmod{\mu}} c_\alpha \chi_Q(|\det D_\alpha|)|\det D_\alpha|^{-m/2}\mathbf{e}\{0\}\mu^{-2}Q[D]\cdot {}^t0D_\alpha B_\alpha. \tag{1.10}$$

Note that the interaction sums satisfy the relations

$$I(MDM', Q, T') = I(D, Q[M], T') \quad (T' \in \mathcal{H}_0^m(q),\ M, M' \in GL_m(\mathbb{Z})). \tag{1.11}$$

§2. Orthogonal HS–rings

In this section we briefly recall definition of orthogonal Hecke–Shimura rings. For details see [3].

Suppose that for an even positive definite matrix Q of even order m and level q, a system of representatives $\langle Q\rangle$ of all different classes with respect to integral equivalence of even positive definite matrices of the same order, divisor, level, and determinant as those of Q consists of the single matrix Q,

$$\langle Q\rangle = \{Q\}. \tag{2.1}$$

Given such a matrix, we define the groups

$$\mathbf{E} = \mathbf{E}(Q) = \{D \in GL_m(\mathbb{Z}) \mid Q[D] = Q\}$$

of *units of matrix* Q and the set

$$\mathbf{A} = \mathbf{A}(Q) = \{D \in \mathbb{Z}_m^m \mid Q[D] = \mu(D)Q, \quad \mu(D) > 0, \quad \gcd(\mu(D), q) = 1\} \tag{2.2}$$

of (*regular*) *automorphes of* Q. It can be verified that the groups $\mathbf{E}$ and sets $\mathbf{A}$ satisfy the following three condition: $\mathbf{AA} \subset \mathbf{A}$, $\mathbf{E} \subset \mathbf{A}$, and each double coset $\mathbf{E}M\mathbf{E}$ with $M \in \mathbf{A}$ is a finite union of left cosets modulo $\mathbf{E}$. Let us denote by

$$\mathbf{D} = \mathbf{D}(\mathbf{E}, \mathbf{A}) \tag{2.3}$$

the set of those finite formal linear combinations with integral coefficients of symbols $(\mathbf{E}D)$, corresponding in one-to-one way to different left cosets

$\mathbf{E}D$ contained in the set $\mathbf{A}$, which are invariant with respect to all right multiplication by elements of $\mathbf{E}$:

$$\mathbf{D} \ni t = \sum_{\alpha} a_\alpha(\mathbf{E}\, D_\alpha), \quad tA = \sum_{\alpha} a_\alpha(\mathbf{E}\, D_\alpha A) = t \qquad (\forall A \in \mathbf{E}).$$

The set $\mathbf{D}$ is an associative ring, called the (*regular*) *Hecke–Shimura ring of the matrix* Q. The matrices Q satisfying condition (2.1) will be called *one-class matrices.*

Finally, we define the linear representation $\circ$ of the ring $\mathbf{D}(Q)$ on functions $f : Q \to \mathbb{C}$ defined by

$$f \circ t = f \circ \sum_{\alpha} a_\alpha(\mathbf{E}\, D_\alpha) = \sum_{\alpha} a_\alpha f(\mu_\alpha^{-1} Q[D_\alpha]) \qquad (t = \sum_{\alpha} a_\alpha(\mathbf{E}\, D_\alpha) \in \mathbf{D}). \tag{2.4}$$

§3. Interaction mappings

Let $T \in \mathcal{H}_0^n(q)$ be an homogeneous element of a multiplier $\mu(T) = \mu$. Let us suppose that an even positive definite matrix Q of even order m and level q satisfies the condition (2.1), i.e., it is an one-class matrix. If set $A(Q, \mu)$ of the form (1.9) is not empty, then for each $D \in A(Q, \mu)$, the matrix $\mu^{-1}Q[D]$ is integrally equivalent to the matrix Q. By choosing appropriate representative in the coset $A \cdot GL_m(\mathbb{Z})$, one can assume that $\mu^{-1}Q[D] = Q$, i.e., $Q[D] = \mu Q$, and the coset $D \cdot GL_m(\mathbb{Z})$ for such D reduces to the coset $D \cdot \mathbf{E}$ of the group $\mathbf{E} = \mathbf{E}(Q)$ of units of Q. It shows that one can take

$$A(Q, \mu)/GL_m(\mathbb{Z}) = \mathbf{A}(\mu)/\mathbf{E}, \tag{3.1}$$

where

$$\mathbf{A}(\mu) = \mathbf{A}(Q, \mu) = \{D \in \mathbf{A}(Q) \mid \mu(D) = \mu\} = \{D \in \mathbb{Z}_m^m \mid Q[D] = \mu Q\}.$$

Then the relation (1.8) for Q takes the form

$$\begin{aligned}(\Theta^n|T)(Z;\, Q) &= \begin{cases} \sum_{D \in \mathbf{A}(\mu)/\mathbf{E}} I(D,\, Q,\, \Psi^{n,m}T)\Theta^n(Z;\, \mu^{-1}Q[D]), \\ 0, \end{cases} \\ &\qquad = \begin{cases} \left(\sum_{D \in \mathbf{A}(\mu)/\mathbf{E}} I(D,\, Q,\, \Psi^{n,m}T)\right)\Theta^n(Z;\, Q), \\ 0, \end{cases}\end{aligned}$$

depending on whether the set $\mathbf{A}(\mu)$ is not empty or empty, where $\Psi^{n,m} = \Psi^{n,m}_Q$ is the Zharkovskaya mapping for the matrix Q. Since μ is prime to the level q of Q, it follows that the condition $D \in \mathbf{A}(\mu)/\mathbf{E}$ is equivalent to the condition $\mu D^{-1} \in \mathbf{E}\backslash\mathbf{A}(\mu)$. Therefore, by replacing $M \mapsto \mu D^{-1}$, the last relations can be rewritten in the form

$$(\Theta^n|T)(Z;\, Q) = \begin{cases} \bigl(\sum_{M\in\mathbf{E}\backslash\mathbf{A}(\mu)} I(\mu M^{-1},\, Q,\, \Psi^{n,m}T)\bigr)\Theta^n(Z;\, Q), \\ 0, \end{cases} \tag{3.2}$$

depending on whether the set $A(Q,\, \mu)$ is not empty or empty.

On the other hand, for $n \geqslant 1$ let us set

$$\tau^n(T) = \begin{cases} \sum_{D\in\mathbf{E}\backslash\mathbf{A}(\mu)} I(\mu D^{-1},\, Q,\, \Psi^{n,m}T)\,(\mathbf{E}D) & \text{if } \mathbf{A}(\mu) \neq \varnothing, \\ 0 & \text{if } \mathbf{A}(\mu) = \varnothing. \end{cases} \tag{3.3}$$

It follows from (1.11) that for each $M \in \mathbf{E}$ linear combinations (3.3) satisfy relations

$$\begin{aligned} \tau^n(T)M &= \sum_{D\in\mathbf{E}\backslash\mathbf{A}(\mu)} I(\mu M(DM^{-1})^{-1},\, Q,\, \Psi^{n,m}T)\,(\mathbf{E}DM) \\ &= \sum_{D\in\mathbf{E}\backslash\mathbf{A}(\mu)} I(\mu D^{-1},\, Q[M],\, \Psi^{n,m}T)\,(\mathbf{E}D) = \tau^n(T). \end{aligned}$$

Thus, $\tau^n(T) \in \mathbf{D}$, and so the element $\tau^n(T) = \tau^n_{<Q>}(T) = (\tau^n(T))$ belongs to the HS-ring (2.4). Extending the mapping by linearity to arbitrary $T \in \mathcal{H}^n_0(q)$, we obtain a linear mapping of HS-rings

$$\mathcal{H}^n_0(q) \ni T \mapsto \tau^n(T) \in \mathbf{D} = D(\mathbf{E};\, \mathbf{A}). \tag{3.4}$$

Finally, let us define the action of (orthogonal) operator $\circ\tau^n(T)$ on theta-series by

$$(\Theta^n \circ \tau^n(T))(Z,\, Q) = \sum_{D\in\mathbf{E}\backslash\mathbf{A}(\mu)} I(\mu D^{-1},\, Q,\, \Psi^{n,m}T)\Theta^n(Z,\, \mu Q[D^{-1}]). \tag{3.5}$$

The following theorem expresses images of the theta-series under action of symplectic Hecke operators in terms of action of orthogonal Hecke operators by means of interaction mapping (3.4).

Theorem 1. *Let Q be an even positive definite one-class matrix of even order m. Then for each $n \geqslant m/2$ the action of a Hecke operator $|T$ with*

$T \in \mathcal{H}_0^n(q)$ on the theta-series $\Theta^n(Z, Q)$ can be written in the terms of action of the operator $\circ\tau^n(T)$ defined by (2.4) *in the form*

$$\Theta^n|T = \Theta^n \circ \tau^n(T). \tag{3.6}$$

Proof. Suppose first that $T \in \mathcal{H}_0^n(q)$ is an homogeneous element of a multiplier $\mu(T) = \mu$, and that the set $A(Q, \mu)$ of the form (1.9) is not empty. Then, according to definition, the action of operators $\circ\tau^n(T)$ on the theta-series can be written in the form (3.5). Hence the formulas (3.2) can be rewritten as (3.6) This formula together with the formula (1.8) is true for all homogeneous elements $T \in \mathcal{H}_0^n(q)$ of a multiplier μ and such that the set $A(Q, \mu)$ of the form (1.9) is not empty for the matrix Q. By linearity, the formulas remain true for all $T \in \mathcal{H}_0^n(q)$. If the set $A(Q, \mu)$ are empty, then $(\Theta^n)|T = \mathbf{0}$, and the formula remains true with the convention $\tau^n(T) = \mathbf{0}_m^m$. The theorem is proved. □

Note that when $m/2 \leqslant n < m$, inverse image $\Psi^{n,m}T \in \mathcal{H}_0^m(q)$ is not unique, which cause an indeterminacy of the definition of the mapping (3.4), but in view of the theorem it does not affect the action of operators $\circ\tau^n(T)$ on theta-vectors. We call the mapping $T \mapsto \tau^n(T)$ the *interaction mapping of the HS-rings.*

Theorem 2. *Let Q be an even positive definite one-class matrix of even order m. Then for every $n \geqslant m$ the mapping* (3.4) *is a linear ring-homomorphism of the Hecke-Shimura rings.*

Proof. The mapping (3.4) is linear by definition. We consider first the case $n = m$. By linearity, it is sufficient to prove in this case that

$$\tau^m(T\,T') = \tau^m(T)\tau^m(T') \tag{3.7}$$

for every homogeneous elements $T, T' \in \mathcal{H}_0^m(q)$. If $\mu(T) = \mu$ and $\mu(T') = \mu'$, then $\mu(TT') = \mu\mu'$ and by (3.3) we have

$$\tau^m(TT') = \begin{cases} \sum_{D'' \in \mathbf{E}\backslash\mathbf{A}(\mu\mu')} I(\mu\mu'(D'')^{-1}, Q, TT')\,(\mathbf{E}D'') & \text{if } A(Q, \mu\mu') \neq \varnothing, \\ 0 & \text{if } A(Q, \mu\mu') = \varnothing, \end{cases}$$

By definitions we can write

$$\tau^m(T)\tau^m(T') = \sum_{D \in \mathbf{E}\backslash\mathbf{A}(\mu)} \sum_{D' \in \mathbf{E}\backslash\mathbf{A}(\mu')} I(\mu D^{-1}, Q, T) I(\mu'(D')^{-1}, Q, T')(\mathbf{E}DD'),$$

if $A(Q,\,\mu)\neq\varnothing$ and $A(Q,\,\mu')\neq\varnothing$. Otherwise, $\tau^m(T)\tau^m(T')=0$. Therefore, in order to prove (3.7) it is sufficient to show that

$$I(\mu\mu'(D'')^{-1},\,Q_j,\,TT') \sum_{\substack{(D,D')\in(\mathbf{E}\backslash\mathbf{A}(\mu),\mathbf{E}\backslash\mathbf{A}(\mu')),\\ DD'\in\mathbf{E}D''}} I(\mu D^{-1},\,Q,\,T)I(\mu'(D')^{-1},\,Q,\,T') \quad (3.8)$$

for each $D''\in\mathbf{A}(\mu\mu')$, unless the left or the right sides are both zero.

On the other hand, by [6, Proposition 3.8], for every $\tilde{D}\in A(Q,\mu\mu')$ interaction sums satisfy relations

$$I(\tilde{D},\,Q_j,\,TT') = \sum_{\substack{(D_1,D_2)\in(A(Q,\mu)/\Lambda,A(\mu^{-1}Q[D_1],\mu')/\Lambda),\\ D_1D_2\in\tilde{D}\Lambda}} I(D_1,\,Q,\,T)I(D_2,\,\mu^{-1}Q[D_1],\,T'),$$

where $\Lambda=GL_m(\mathbb{Z})$. By (3.1), the inclusion $D_1\in A(Q,\mu)/\Lambda$ can be replaced by the inclusion $D_1\in\mathbf{A}(\mu)/\mathbf{E}$ and vice versa. If $D_1\in\mathbf{A}(\mu)/\mathbf{E}$, then $\mu^{-1}Q[D_1]=Q$. Hence, again by (3.1), the condition

$$D_2\in\mathbf{A}(\mu^{-1}Q[D_1],\mu')/\Lambda$$

means that $D_2\in\mathbf{A}(\mu')/\mathbf{E}$. Then $D_1D_2\in\tilde{D}\mathbf{E}$, and the last relation turns into relation

$$I(\tilde{D},\,Q,\,TT') = \sum_{\substack{(D_1,D_2)\in(\mathbf{A}(\mu)/\mathbf{E},\mathbf{A}(\mu')/\mathbf{E}),\\ D_1D_2\in\tilde{D}\mathbf{E}}} I(D_1,\,Q,\,T)I(D_2,\,Q,\,T'),$$

Since μ and μ' are prime to the level q of the matrix of Q, it follows that the conditions $D_1\in\mathbf{A}(\mu)/\mathbf{E}$ and $D_2\in\mathbf{A}(\mu')/\mathbf{E}$ are equivalent to the conditions $\mu D_1^{-1}\in\mathbf{E}\backslash\mathbf{A}(\mu)$ and $\mu' D_2^{-1}\in\mathbf{E}\backslash\mathbf{A}(\mu')$, respectively. Therefore, after the substitution $\mu D_1^{-1}=D$, $\mu' D_2^{-1}=D'$ and $\mu\mu'\tilde{D}^{-1}=D''$, i.e.,

$D_1 = \mu D^{-1}$, $D_2 = \mu'(D')^{-1}$ and $\tilde{D} = \mu\mu'(D'')^{-1}$, we came to the relation

$$I(\mu\mu'(D'')^{-1}, Q_j, TT') = \sum_{\substack{(D,D')\in(\mathbf{E}\backslash\mathbf{A}(\mu)/,\mathbf{E}\backslash\mathbf{A}(\mu')),\\ D'D\in\mathbf{E}D''}} I(\mu D^{-1}, Q, T)I(\mu'(D')^{-1}, Q, T') \sum_{\substack{(D',D)\in(\mathbf{E}\backslash\mathbf{A}(\mu'),\mathbf{E}\backslash\mathbf{A}(\mu)),\\ D'D\in\mathbf{E}D''}} I(\mu'(D')^{-1}, Q, T')I(\mu D^{-1}, Q, T),$$

which is actually the relation (3.6) for elements $T', T \in \mathcal{H}^m(q)$, since $TT' = T'T$, by commutativity of the ring $\mathcal{H}^m(q)$.

The case when $n \geqslant m$ follows from the case $n = m$, since by definition the mapping $T \mapsto \tau^n(T) = \tau^m(\Psi^{n,m}(T))$ is the composition of the Zharkovskaya homomorphism $\Psi^{n,m} = \Psi_Q^{n,m}$ and the homomorphism τ^m. □

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