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THE SPECTRUM AND SEPARABILITY OF MIXED TWO-QUBIT X-STATES

Abstract. The separable mixed two-qubit X-states are classified in accordance with the degeneracies in the spectrum of density matrices. It is shown that there are four classes of separable X-states, among them: one four-dimensional family, a pair of two-dimensional families, and a single zero-dimensional maximally mixed state.

Introduction

Consider the space $\mathfrak{P}_X$ of 4×4 Hermitian matrices of the form

$$\varrho_X := \begin{pmatrix} \varrho_{11} & 0 & 0 & \varrho_{14} \\ 0 & \varrho_{22} & \varrho_{23} & 0 \\ 0 & \varrho_{32} & \varrho_{33} & 0 \\ \varrho_{41} & 0 & 0 & \varrho_{44} \end{pmatrix}. \tag{1}$$

Due to the Hermiticity, the diagonal entries in (1) are real numbers, while the elements of the minor diagonal are pairwise complex conjugate numbers, $\varrho_{14} = \overline{\varrho}_{14}$ and $\varrho_{23} = \overline{\varrho}_{32}$. Assuming that the matrix ϱ_X is semipositive definite,

$$\varrho_X \geqslant 0, \tag{2}$$

and has unit trace,

$$\mathrm{tr}\varrho_X = 1, \tag{3}$$

ϱ_X can be regarded as the density matrix of a 4-level quantum system. Since the nonzero elements in (1) lie in a shape similar to the Latin letter "X," the corresponding quantum states are called X-states.

The 7-dimensional space $\mathfrak{P}_X$ is a subspace of the 15-dimensional state space $\mathfrak{P}$ of a generic 4-level quantum system, $\mathfrak{P}_X \subset \mathfrak{P}$. Since the introduction of X-states [1], various subfamilies of $\mathfrak{P}_X$ have been attracting special attention. There are at least two reasons for that interest. First of all, it was discovered that microscopic systems being in certain X-states

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show a highly nontrivial quantum behavior.[1] Second, due to the simple algebraic structure of X-states, many computational difficulties common for generic states can be resolved when dealing with this special subclass of states.[2]

The aforementioned simplification turned out to be very important in describing such a complicated phenomenon as entanglement in composite quantum systems. In particular, it is well known that the famous entanglement measure – concurrence – can be reduced to a simple analytic expression for X-states. In the present note, we will move towards a detailed entanglement classification of mixed two-qubit X-states. Namely, a parametrization of the separable mixed X-states of two qubits with an arbitrary spectrum of the density matrix will be described. Our analysis in the subsequent sections includes the following steps:

(1) Two unitary groups, both acting adjointly on the 7-dimensional space of two-qubit X-states, will be introduced.
   (a) The first one is the so-called "*global group*," $G_X \in SU(4)$, defined as the invariance group of the subspace $\mathfrak{P}_X$:
$$G_X \varrho_X G_X^\dagger \in \mathfrak{P}_X \quad \text{for every} \quad \varrho_X \in \mathfrak{P}_X .$$
   (b) The second one is a subgroup of G_X, the so-called "*local group*," $LG_{\text{x}} \in G_X$. Its elements have a tensor product form corresponding to the decomposition of the state space $\mathfrak{P}_X$ into two qubit subspaces, $LG_{\text{x}} \in SU(2) \times SU(2)$.

(2) The "global orbits" $\mathcal{O}_\varrho$ of the group G_X will be identified and classified into families/types according to the degeneracies in the spectrum of the density matrices.

(3) Considering the equivalence classes induced by the action of the local group LG_{x} on $\mathcal{O}_\varrho$, one can divide the latter into different

[1]Well-known entangled states, such as Bell states [2], Werner states [3], isotropic states [4], and maximally entangled mixed states [5,6], are particular subsets of X-states. For further references on X-states, see [7,8].

[2]Such simplifications take place owing to a discrete symmetry X-states possess. Namely, it can easily be verified that every X-state (1) is equivalent to a block-diagonal matrix

$$\varrho_X = P_\pi \begin{pmatrix} \varrho_{11} & \varrho_{14} & 0 & 0 \\ \varrho_{41} & \varrho_{44} & 0 & 0 \\ 0 & 0 & \varrho_{33} & \varrho_{32} \\ 0 & 0 & \varrho_{23} & \varrho_{22}. \end{pmatrix} P_\pi, \quad \text{with} \quad P_\pi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{4}$$

subfamilies according to their entanglement characteristics. Having in mind this grouping, the separable density X-matrices will be categorized within the global classification of orbits.

§1. The global and local invariance groups of X-states

In order to prove the properties of two-qubit X-states announced above, let us start with a few definitions.

• **The invariance subalgebra of X-states** • A basis for the algebra $\mathfrak{su}(4)$ is constructed as follows. Let $\sigma_\mu = (\sigma_0, \boldsymbol{\sigma})$ denote the set of 2×2 matrices, where $\sigma_0 = I$ is the unit matrix and $\boldsymbol{\sigma} := (\sigma_x, \sigma_y, \sigma_z)$ are the three Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The set of all possible tensor products of two copies of matrices σ_μ,

$$\sigma_{\mu\nu} := \sigma_\mu \otimes \sigma_\nu, \qquad \mu, \nu = 0, x, y, z,$$

forms a basis of the algebra $\mathfrak{su}(4)$. For our aims, it is useful to write the latter as the direct sum, $\mathfrak{su}(4) = \mathfrak{l} \oplus \mathfrak{p}$, where the 6-dimensional vector space $\mathfrak{l}$ is

$$\mathfrak{l} = \operatorname{span} \frac{i}{2}\{\sigma_{x0}, \sigma_{y0}, \sigma_{z0}, \sigma_{0x}, \sigma_{0y}, \sigma_{0z}\}, \tag{5}$$

while the 9-dimensional space $\mathfrak{p}$ is[3]

$$\mathfrak{p} = \operatorname{span} \frac{i}{2}\{\sigma_{xx}, \sigma_{xy}, \sigma_{xz}, \sigma_{yx}, \sigma_{yy}, \sigma_{yz}, \sigma_{zx}, \sigma_{zy}, \sigma_{zz}\}. \tag{6}$$

From now on, to denote the matrices in (5) and (6), we use the notation λ_k, where k runs from 1 to 15:

$$\mathfrak{l} = \operatorname{span}\{\lambda_1, \lambda_2, \ldots, \lambda_6\}, \qquad \mathfrak{p} = \operatorname{span}\{\lambda_7, \lambda_8, \ldots, \lambda_{15}\}. \tag{7}$$

X-states (1) expand over the subset $\alpha_X = \{\lambda_{15}, \lambda_{10}, \lambda_6, -\lambda_{11}, \lambda_8, \lambda_3, \lambda_7\}$ of the introduced basis of $\mathfrak{su}(4)$:

$$\varrho_X = \frac{1}{4}\Big(I + 2i \sum_{\lambda_k \in \alpha_X} h_k \lambda_k\Big). \tag{8}$$

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[3]Since the commutators between elements of two subspaces $\mathfrak{l}$ and $\mathfrak{p}$ are such that

$$[\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{l}, \qquad [\mathfrak{p}, \mathfrak{l}] \subset \mathfrak{p}, \qquad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{l},$$

the direct sum $\mathfrak{l} \oplus \mathfrak{p}$ is nothing else than the Cartan decomposition of $\mathfrak{su}(4)$.

The real coefficients h_k in (8) are given by linear combinations of the density matrix elements:

$$h_3 = -\varrho_{11} - \varrho_{22} + \varrho_{33} + \varrho_{44}, \qquad h_6 = -\varrho_{11} + \varrho_{22} - \varrho_{33} + \varrho_{44}, \tag{9}$$

$$h_7 = -\varrho_{14} - \varrho_{23} - \varrho_{32} - \varrho_{41}, \qquad h_{11} = -\varrho_{14} + \varrho_{23} + \varrho_{32} - \varrho_{41}, \tag{10}$$

$$h_8 = i(-\varrho_{14} + \varrho_{23} - \varrho_{32} + \varrho_{41}), \quad h_{10} = i(-\varrho_{14} - \varrho_{23} + \varrho_{32} + \varrho_{41}), \tag{11}$$

$$h_{15} = -\varrho_{11} + \varrho_{22} + \varrho_{33} - \varrho_{44}. \tag{12}$$

The subset α_X possesses the following properties:

(i) The subset is closed under the matrix commutator operation, i.e., its elements span a subalgebra of $\mathfrak{su}(4)$.

(ii) From the commutators collected in Table 1 it follows that the element λ_{15} commutes with all other elements of α_X.

(iii) The remaining six elements, $\{\lambda_3, \lambda_6, \lambda_7, \lambda_8, \lambda_{10}, \lambda_{11}\}$, span the algebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

To check the last property, one can construct the linear combinations

$$S_z = i(\lambda_3 + \lambda_6), \qquad S_\pm = \pm(\lambda_8 + \lambda_{10}) + i(\lambda_7 - \lambda_{11}), \tag{13}$$

$$T_z = i(\lambda_3 - \lambda_6), \qquad T_\pm = \mp(\lambda_8 - \lambda_{10}) + i(\lambda_7 + \lambda_{11}) \tag{14}$$

and verify that their commutator relations are

$$[S_z, S_\pm] = \pm 2S_\pm, \qquad [S_+, S_-] = 4S_z, \tag{15}$$

$$[T_z, T_\pm] = \pm 2T_\pm, \qquad [T_+, T_-] = 4T_z. \tag{16}$$

Thus, two sets of elements

$$\boldsymbol{S} = \{\frac{1}{2}(S_+ + S_-),\ \frac{i}{2}(S_+ - S_-),\ S_z\}, \tag{17}$$

$$\boldsymbol{T} = \{\frac{1}{2}(T_+ + T_-),\ \frac{i}{2}(T_+ - T_-),\ T_z\} \tag{18}$$

generate two copies of $\mathfrak{su}(2)$.[4] Gathering all together, we conclude that the set α_X generates the subalgebra $\mathfrak{g}_X := \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \in \mathfrak{su}(4)$.

• **The global unitary group of X-states** • The exponentiation of the algebra $\mathfrak{g}_X$ results in the 7-parameter subgroup of $SU(4)$,

$$G_X := \exp(\mathfrak{g}_X) \in SU(4),$$

[4] In the terminology of [9], such operators describe "pseudospins" for a two-spin system.

whose action preserves the space of X-states $\mathfrak{P}_X$, i.e., $G_X \varrho_X G_X^\dagger \in \mathfrak{P}_X$. Using the expansion $\mathfrak{g}_X = \sum_i \omega_i \lambda_i$ over the 7-tuple $\lambda_i \in \alpha_X$ and formulas (43)–(46) from Sec. 5, one can verify that the group G_X has the following representation:

$$G_X = P_\pi \left(\begin{array}{c|c} e^{-i\omega_{15}} SU(2) & 0 \\ \hline 0 & e^{i\omega_{15}} SU(2)' \end{array} \right) P_\pi, \tag{19}$$

where the two copies of SU(2) are parametrized as follows:

$$SU(2) = \exp\left[i\left(\omega_4 + \omega_7\right)\sigma_1 + i\left(\omega_2 + \omega_5\right)\sigma_2 + i\left(\omega_3 + \omega_6\right)\sigma_3\right],$$

$$SU(2)' = \exp\left[i\left(-\omega_4 + \omega_7\right)\sigma_1 + i\left(-\omega_2 + \omega_5\right)\sigma_2 + i\left(\omega_3 - \omega_6\right)\sigma_3\right].$$

• **The local subgroup of G_X** • Now assume that our 4-level system is composed of 2-level subsystems, i.e., two qubits. In this case, the Hilbert space $\mathcal{H}$ is the tensor product of 2-dimensional Hilbert spaces, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, and one can consider the tensor product of operators acting independently on the subspaces of individual qubits, $\mathcal{H}_1$ and $\mathcal{H}_2$. In particular, having in mind the intuitive idea of the mutual independence of isolated qubits, we define the group LG_X as the subgroup of the global invariance group of X-states G_X such that each its element $g \in LG_X$ has the tensor product form $g = g_1 \times g_2$, with $g_1, g_2 \in SU(2)$. From (19) it follows that the local unitary group can be written as

$$LG_X = P_\pi \exp(\imath \frac{\varphi_1}{2}\sigma_3) \times \exp(\imath \frac{\varphi_2}{2}\sigma_3) P_\pi. \tag{20}$$

§2. Global G_X-orbits

Now it will be shown that every X-state density matrix can be diagonalized using some subgroup of the global group G_X. Therefore, the adjoint structure of G_X-orbits is completely determined by the coset G_X/H_ϱ, where H_ϱ stands for the isotropy group of a density matrix ϱ. This isotropy group, in turn, depends on the degeneracies occurring in the spectrum of density matrices. Thus, the latter determines all possible types of G_X-orbits, and the corresponding classification can be carried out as follows.

2.1. The dimension of the tangent space of G_X-orbits. Consider the adjoint action of the global unitary group G_X on the 7-dimensional space $\mathfrak{P}_X$ and introduce the following vectors at each point $\varrho \in \mathfrak{P}_X$:

$$t_k = \frac{\partial}{\partial v_k} \left(g(\boldsymbol{v}) \varrho_X g^\dagger(\boldsymbol{v}) \right) \bigg|_{v_k=0} = [\lambda_k, \varrho_X], \quad k = 3, 6, 7, 8, 10, 11, 15. \tag{21}$$

In Eq. (21), group elements $g(\boldsymbol{v}) \in G_X$ are parametrized by 7-tuples $\boldsymbol{v} = \{v_3, v_6, v_7, v_8, v_{10}, v_{11}, v_{15}\}$:

$$g(\boldsymbol{v}) = \exp\left(\sum_{\lambda_k \in \alpha_X} v_k \lambda_k\right). \tag{22}$$

These vectors belong to the tangent space of G_X-orbits. The dimension of this tangent space is given by the rank of the 7×7 Gram matrix

$$G = \|G_{kl}\| = \frac{1}{2}\|\mathrm{Tr}\,(t_k t_l)\|. \tag{23}$$

A straightforward evaluation of the spectrum $\sigma(G)$ of the Gram matrix G shows that it comprises two eigenvalues of multiplicity 2 and three identically vanishing eigenvalues,

$$\sigma(G) = \{\mu_1, \mu_1, \mu_2, \mu_2, 0, 0, 0\}, \tag{24}$$

where the multiplicity 2 eigenvalues are

$$\mu_1 = (h_3 + h_6)^2 + (h_8 + h_{10})^2 + (h_7 + h_{11})^2, \tag{25}$$

$$\mu_2 = (h_3 - h_6)^2 + (h_8 - h_{10})^2 + (h_7 - h_{11})^2. \tag{26}$$

Formulas (25) and (26) ensure that there exist 4 types of G_X-orbits:

- **dim** $\mathcal{O}$ **=4**, the generic orbits;
- **dim** $\mathcal{O}$ **=2**, the degenerate orbits defined by the equations

$$h_6 = h_3,\ h_{10} = h_8,\quad h_{11} = h_7\,; \tag{27}$$

- **dim** $\mathcal{O}$ **=2**, the degenerate orbits defined by the equations

$$h_6 = -h_3,\ h_{10} = -h_8,\quad h_{11} = -h_7\,; \tag{28}$$

- **dim** $\mathcal{O}$ **=0**, the single orbit $\varrho_X = \frac{1}{4}I$ – the maximally mixed state.

The four-dimensional orbits comprise all matrices with a generic spectrum, while the two-dimensional orbits are generated by X-matrices with double multiplicity eigenvalues of the following form:

$$P_\pi \begin{pmatrix} \varrho_{11} & \varrho_{14} & 0 & 0 \\ \varrho_{41} & \varrho_{44} & 0 & 0 \\ 0 & 0 & \varrho_{22} & 0 \\ 0 & 0 & 0 & \varrho_{22}. \end{pmatrix} P_\pi \text{ and } P_\pi \begin{pmatrix} \varrho_{11} & 0 & 0 & 0 \\ 0 & \varrho_{11} & 0 & 0 \\ 0 & 0 & \varrho_{33} & \varrho_{32} \\ 0 & 0 & \varrho_{23} & \varrho_{22}. \end{pmatrix} P_\pi. \tag{29}$$

2.2. Parametrization of G_X-orbits. Here, a detailed representation for each type of G_X-orbits will be given, starting from the orbit of the highest dimension.

2.2.1. *Generic orbits,* $\dim(\mathcal{O}) = 4$. Let us assume that the spectrum of ϱ_X is generic, i.e., all eigenvalues $\sigma(\varrho) := \{r_1, r_2, r_3, r_4\}$ are different positive real numbers. Furthermore, in the block-diagonal representation (4) of the density matrix, $\{r_1, r_2\}$ denote the eigenvalues of the upper block and $\{r_3, r_4\}$ are the eigenvalues of the lower block.

The 4×4 density matrix ϱ_X can be block-diagonalized,

$$\varrho_X = W \left(\begin{array}{c|c} \operatorname{diag}(r_1, r_2) & 0 \\ \hline 0 & \operatorname{diag}(r_3, r_4) \end{array} \right) W^\dagger, \tag{30}$$

using the special unitary matrix

$$W = P_\pi \left(\begin{array}{c|c} e^{i\omega} U & 0 \\ \hline 0 & e^{-i\omega} V \end{array} \right) P_\pi, \tag{31}$$

where U and V are 2×2 special unitary matrices diagonalizing the upper and lower subblocks in (4). Since we have assumed a generic spectrum, the matrices U and V belong to the coset $SU(2)/U(1) \times S_2$, where the group S_2 interchanges the eigenvalues inside the pairs $\{r_1, r_2\}$ and $\{r_3, r_4\}$. In order to have uniqueness in (30), one can fix a certain order in the spectrum $\sigma(\varrho_X)$. Namely, we assume that the elements of the spectrum form a partially ordered simplex $\underline{\Delta}_3$, i.e.,

$$\underline{\Delta}_3 : \quad \sum_{i=1}^{4} r_i = 1, \qquad 0 \leqslant r_2 \leqslant r_1 \leqslant 1, \quad 0 \leqslant r_4 \leqslant r_3 \leqslant 1; \tag{32}$$

this simplex is depicted in Fig. 1.[5]

Comparing expression (31) with (19), we see that the diagonalizing matrix is an element of the global group G_X with 2×2 special unitary matrices U and V from the coset $SU(2)/U(1)$ parametrized by angles $\phi_1, \phi_2 \in [0, \pi]$, $\psi_1, \psi_2 \in [0, 2\pi]$:

$$U = e^{i\frac{\psi_1}{2}\sigma_3} e^{i\frac{\phi_1}{2}\sigma_2}, \qquad V = e^{i\frac{\psi_2}{2}\sigma_3} e^{i\frac{\phi_2}{2}\sigma_2}. \tag{33}$$

[5]Note that the case of general position considered here consists of points inside $\underline{\Delta}_3$ and satisfies the inequalities $r_2 < r_1$ and $r_4 < r_3$.

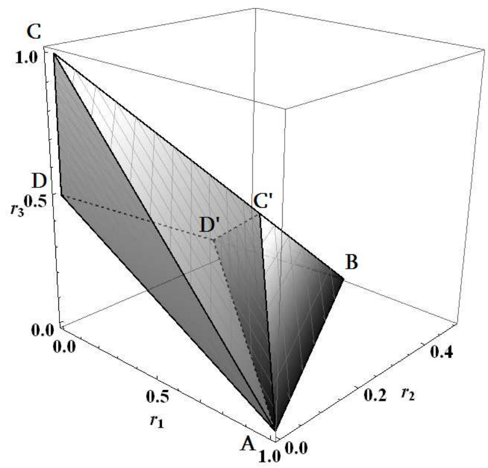

Fig. 1. The tetrahedron $ABCD$ is the image of the partially ordered simplex $\underline{\Delta}_3$, while the tetrahedron $ABC'D'$ inside it corresponds to a three-dimensional simplex with the following complete order of eigenvalues: $\{\ \sum_{i=1}^{4} r_i = 1,$ $1 \geqslant r_1 \geqslant r_2 \geqslant r_3 \geqslant r_4 \geqslant 0\ \}$.

The three-dimensional isotropy group H_{Generic} of generic orbits is

$$H_{\text{Generic}} = P_\pi \left(\begin{array}{c|c} e^{i\omega} \exp \frac{\gamma_1}{2}\sigma_3 & 0 \\ \hline 0 & e^{-i\omega} \exp \frac{\gamma_2}{2}\sigma_3 \end{array} \right) P_\pi . \qquad (34)$$

This is in accordance with the maximum dimension of G_X-orbits:

$$\dim(\mathcal{O})_{\text{Generic}} = \dim(G_X) - \dim H_{\text{Generic}} = 7 - 3 = 4.$$

Summarizing, the adjoint action of the global group G_X determines the generic orbits, which are locally given by the product of 2-spheres.

2.2.2. *Degenerate orbits,* $\dim(\mathcal{O}) = 2$. According to the representation (29), two types of two-dimensional degenerate G_X-orbits are generated by matrices with degenerate 2×2 subblocks, either upper or lower. In the first case, the isotropy group $H_{\text{Degenerate}}$ is

$$H_{\text{Degenerate}} = P_\pi \left(\begin{array}{c|c} e^{i\omega} SU(2) & 0 \\ \hline 0 & e^{-i\omega} \exp \dfrac{\gamma_2}{2}\sigma_3 \end{array} \right) P_\pi , \qquad (35)$$

while for the second case, $H_{\text{Degenerate}}$ is

$$H'_{\text{Degenerate}} = P_\pi \left(\begin{array}{c|c} e^{i\omega} \exp \dfrac{\gamma_1}{2}\sigma_3 & 0 \\ \hline 0 & e^{-i\omega} SU(2)' \end{array} \right) P_\pi . \qquad (36)$$

In both cases, $\dim H_{\text{Degenerate}} = \dim H'_{\text{Degenerate}} = 5$, and the dimension of these degenerate G_X-orbits is

$$\dim(\mathcal{O})_{\text{Degenerate}} = \dim(G_X) - \dim H_{\text{Degenerate}} = 7 - 5 = 2.$$

2.2.3. *Degenerate orbit,* $\dim(\mathcal{O}) = 0$. Finally, there is one point in the state space $\mathfrak{P}_X$ whose isotropy group coincides with the invariance group G_X. This point corresponds to the maximally mixed state, $\varrho_X = \frac{1}{4}I$.

§3. Separable states

Now we are in a position to prove that every type of G_X-orbits includes separable states.[6]

3.1. Separable states on generic G_X-orbits. The separability of states as a function of the spectrum $\sigma(\varrho_X)$ of the density matrix can be analyzed using the representation (30) for generic G_X-orbits.

According to the Peres–Horodecki criterion [10], which is a necessary and sufficient condition for the separability of 2×2 and 2×3 systems, a state ϱ is separable if its partial transposition, i.e., $\varrho^{T_2} = I \otimes T\varrho$, is semi-positive as well.[7] A straightforward computation with ϱ_X in the form

[6] A density matrix ϱ describing a mixed state of a composite system $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is *separable* if it allows a convex decomposition

$$\varrho = \sum_k \omega_k \varrho_1^k \otimes \varrho_2^k, \qquad \sum_k \omega_k = 1, \quad \omega_k > 0, \qquad (37)$$

where ϱ_1^k and ϱ_2^k are density matrices acting on the factors $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Otherwise, it is *entangled*, see [3].

[7] Here we consider the partial transposition with respect to the ordinary transposition T in the second subsystem; similarly, one can use the alternative action $\varrho^{T_1} = T \otimes I\varrho$.

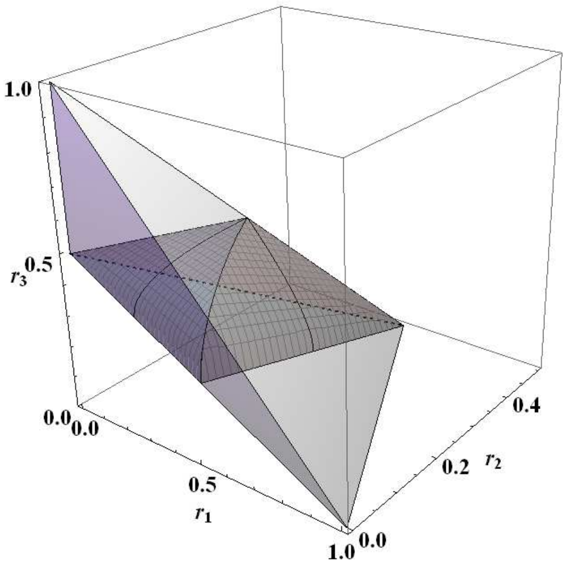

Fig. 2. The set of absolutely separable states inside the tetrahedron of X-states.

(30) shows that the semi-positivity of the partially transposed matrix $\varrho_X^{T_2}$ requires the fulfilment of the following inequalities:

$$(r_1 - r_2)^2 \cos^2 \phi_1 + (r_3 - r_4)^2 \sin^2 \phi_2 \leqslant (r_1 + r_2)^2, \qquad (38)$$

$$(r_3 - r_4)^2 \cos^2 \phi_2 + (r_1 - r_2)^2 \sin^2 \phi_1 \leqslant (r_3 + r_4)^2. \qquad (39)$$

Note that inequalities (38) and (39) do not constraint two angles ψ_1 and ψ_2 in (33) that parametrize the local group $K = \exp(i\frac{\psi_1}{2}\sigma_3) \times \exp(i\frac{\psi_2}{2}\sigma_3)$. This conforms with the general observation that the separability property is independent from the local characteristics of a composite system. This local group is a factor of the global group $G_X = KG'_X$, and the corresponding factor in the matrix W diagonalizing ϱ_X is irrelevant for the separability of X-states.

Analyzing inequalities (38) and (39), one can conclude the following.

(i) There are separable states for any values of eigenvalues from the partially ordered simplex $\underline{\Delta}_3$. In other words, inequalities (38) and

(39) determine a nonempty domain of definition for the angles ϕ_1 and ϕ_2 in (33) for every nondegenerate spectrum $\sigma(\varrho_X)$.

(ii) There is a special family of so-called "absolutely separable" X-states, such that the angles ϕ_1 and ϕ_2 can be arbitrary. The absolutely separable X-states are generated by the subset of the partially ordered simplex (32) defined by the inequalities

$$(r_1 - r_2)^2 \leqslant 4r_3r_4, \tag{40}$$

$$(r_3 - r_4)^2 \leqslant 4r_1r_2. \tag{41}$$

Figure 2 illustrates the location of the subset of absolutely separable states inside the partially ordered simplex $\underline{\Delta}_3$.

3.2. Separable states on degenerate G_x-orbits. Testing the degenerate density matrices of the form (29) by the Peres–Horodecki criterion, we reveal the following picture. The positivity requirement for the partially transposed density matrix with double multiplicity of eigenvalues gives inequalities similar to (38) and (39). However, owing to the larger isotropy group $H_{\text{Degenerate}}$ of states, the new inequalities depend solely on a single coordinate of the coset $G_x/H_{\text{Degenerate}}$. More precisely, if $r_1 = r_2$, i.e., a degeneracy occurs in the upper subblock, then the angle ϕ_2 that parametrizes the matrix V in (33) plays the role of such a coordinate. In this case, the Peres–Horodecki criterion asserts that the degenerate X-state is separable if and only if

$$\cos^2 \phi_2 \leqslant \frac{4\zeta}{(1-\zeta)^2}, \tag{42}$$

where $\zeta = r_4/r_3 < 1$. This inequality points out the critical value $\zeta_* = 3 - 2\sqrt{2}$, such that for $\zeta \leqslant \zeta_*$ the angle ϕ_2 is constrained, while for the interval $\zeta_* < \zeta < 1$ the state is separable for an arbitrary angle ϕ_2. The analogous results for the angle ϕ_1 (see the matrix U in (33)) hold true if the lower subblock in (29) is degenerate, i.e., $r_3 = r_4$. Therefore, in both classes of degenerate two-dimensional global orbits one can point out a two-dimensional family of separable degenerate states. Furthermore, among them there are "degenerate absolutely separable" states, i.e., degenerate global two-dimensional orbits consisting only of separable states.

§4. Concluding remarks

The present article is devoted to a discussion of an interplay between local and global characteristics of a pair of qubits in mixed X-states. With this aim, the orbits of the action of the global unitary group G_X were described and classified according to the degeneracies occurring in the spectrum of density matrices. Using this analysis, the dependence of the separability of X-states on the spectrum was studied. In particular, the separable X-states were collected into the following families:

- the four-dimensional family of separable states with spectrum in general position;
- two classes of two-dimensional separable states with doubly degenerate spectrum;
- the maximally mixed state.

In conclusion, it is worth mentioning that, according to the aforementioned classification, the entangled states, being complementary to the separable states, are likewise partitioned into three types. However, this classification is not complete. A further, more subtle, grouping of the entangled states located at a given G_X-orbit into subclasses is necessary. The latter subclasses are determined not by invariants of the global group G_X, but by the values of LG_X-invariants. In forthcoming publications, we plan to discuss this issue in more detail. Apart from that, following the approach elaborated in [11] and [12], a generalization of the derived results to a generic case of 15-dimensional two-qubit states will be considered.

§5. Supplementary material

Here we collect a technical material useful for performing computations described in the main text. It includes a basis of the Lie algebra $\mathfrak{su}(4)$, commutators of its elements, and a block-diagonal representation for the subalgebra α_X.

•A basis for the Lie algebra $\mathfrak{su}(4)$ • The anti-Hermitian matrices

$$\{\lambda_1, \lambda_2, \ldots, \lambda_6\} = \frac{i}{2}\{\sigma_{x0}, \sigma_{y0}, \sigma_{z0}, \sigma_{0x}, \sigma_{0y}, \sigma_{0z}\}$$

and

$$\{\lambda_7, \lambda_8, \ldots, \lambda_{15}\} = \frac{i}{2}\{\sigma_{xx}, \sigma_{xy}, \sigma_{xz}, \sigma_{yx}, \sigma_{yy}, \sigma_{yz}, \sigma_{zx}, \sigma_{zy}, \sigma_{zz}\}$$

are

$$\lambda_1=\frac{i}{2}\begin{Vmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{Vmatrix},\ \lambda_2=\frac{i}{2}\begin{Vmatrix}0&0&-i&0\\0&0&0&-i\\i&0&0&0\\0&i&0&0\end{Vmatrix},\ \lambda_3=\frac{i}{2}\begin{Vmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{Vmatrix},$$

$$\lambda_4=\frac{i}{2}\begin{Vmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{Vmatrix},\ \lambda_5=\frac{i}{2}\begin{Vmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&-i\\0&0&i&0\end{Vmatrix},\ \lambda_6=\frac{i}{2}\begin{Vmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{Vmatrix},$$

$$\lambda_7=\frac{i}{2}\begin{Vmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{Vmatrix},\ \lambda_8=\frac{i}{2}\begin{Vmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{Vmatrix},\ \lambda_9=\frac{i}{2}\begin{Vmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{Vmatrix},$$

$$\lambda_{10}=\frac{i}{2}\begin{Vmatrix}0&0&0&-i\\0&0&-i&0\\0&i&0&0\\i&0&0&0\end{Vmatrix},\quad \lambda_{11}=\frac{i}{2}\begin{Vmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{Vmatrix},$$

$$\lambda_{12}=\frac{i}{2}\begin{Vmatrix}0&0&-i&0\\0&0&0&i\\i&0&0&0\\0&-i&0&0\end{Vmatrix},\quad \lambda_{13}=\frac{i}{2}\begin{Vmatrix}0&1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0\end{Vmatrix},$$

$$\lambda_{14}=\frac{i}{2}\begin{Vmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&i\\0&0&-i&0\end{Vmatrix},\quad \lambda_{15}=\frac{i}{2}\begin{Vmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{Vmatrix}.$$

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}	λ_{12}	λ_{13}	λ_{14}	λ_{15}
λ_1	0	$-\lambda_3$	λ_2	0	0	0	0	0	0	$-\lambda_{13}$	$-\lambda_{14}$	$-\lambda_{15}$	λ_{10}	λ_{11}	λ_{12}
λ_2	λ_3	0	$-\lambda_1$	0	0	0	λ_{13}	λ_{14}	λ_{15}	0	0	0	$-\lambda_7$	$-\lambda_8$	$-\lambda_9$
λ_3	$-\lambda_2$	λ_1	0	0	0	0	$-\lambda_{10}$	$-\lambda_{11}$	$-\lambda_{12}$	λ_7	λ_8	λ_9	0	0	0
λ_4	0	0	0	0	$-\lambda_6$	λ_5	0	$-\lambda_9$	λ_8	0	$-\lambda_{12}$	λ_{11}	0	$-\lambda_{15}$	λ_{14}
λ_5	0	0	0	λ_6	0	$-\lambda_4$	λ_9	0	$-\lambda_7$	λ_{12}	0	$-\lambda_{10}$	λ_{15}	0	λ_{13}
λ_6	0	0	0	$-\lambda_5$	λ_4	0	$-\lambda_8$	λ_7	0	$-\lambda_{11}$	λ_{10}	0	$-\lambda_{14}$	λ_{13}	0
λ_7	0	$-\lambda_{13}$	λ_{10}	0	$-\lambda_9$	λ_8	0	$-\lambda_6$	λ_5	$-\lambda_3$	0	0	λ_2	0	0
λ_8	0	$-\lambda_{14}$	λ_{11}	λ_9	0	$-\lambda_7$	λ_6	0	$-\lambda_4$	0	$-\lambda_3$	0	0	λ_2	0
λ_9	0	$-\lambda_{15}$	λ_{12}	$-\lambda_8$	λ_7	0	$-\lambda_5$	λ_4	0	0	0	$-\lambda_3$	0	0	λ_2
λ_{10}	λ_{13}	0	$-\lambda_7$	0	$-\lambda_{12}$	λ_{11}	λ_3	0	0	0	$-\lambda_6$	λ_5	$-\lambda_1$	0	0
λ_{11}	λ_{14}	0	$-\lambda_8$	λ_{12}	0	$-\lambda_{10}$	0	λ_3	0	λ_6	0	$-\lambda_4$	0	$-\lambda_1$	0
λ_{12}	λ_{15}	0	$-\lambda_9$	$-\lambda_{11}$	λ_{10}	0	0	0	λ_3	$-\lambda_5$	λ_4	0	0	0	$-\lambda_1$
λ_{13}	$-\lambda_{10}$	λ_7	0	0	$-\lambda_{15}$	λ_{14}	$-\lambda_2$	0	0	λ_1	0	0	0	$-\lambda_6$	λ_5
λ_{14}	$-\lambda_{11}$	λ_8	0	λ_{15}	0	$-\lambda_{13}$	0	$-\lambda_2$	0	0	λ_1	0	λ_6	0	$-\lambda_4$
λ_{15}	$-\lambda_{12}$	λ_9	0	$-\lambda_{14}$	λ_{13}	0	0	0	$-\lambda_2$	0	0	λ_1	$-\lambda_5$	λ_4	0

Table 1. The commutator relations for $\mathfrak{su}(4)$.

The block-diagonal form of the basis elements of the subalgebra α_X resulting from applying the transposition P_π:

$$P_\pi \lambda_3 P_\pi = \frac{i}{2}\left(\begin{array}{c|c} \sigma_3 & 0 \\ \hline 0 & -\sigma_3 \end{array}\right), \quad P_\pi \lambda_6 P_\pi = \frac{i}{2}\left(\begin{array}{c|c} \sigma_3 & 0 \\ \hline 0 & \sigma_3 \end{array}\right), \tag{43}$$

$$P_\pi \lambda_7 P_\pi = \frac{i}{2}\left(\begin{array}{c|c} \sigma_1 & 0 \\ \hline 0 & \sigma_1 \end{array}\right), \quad P_\pi \lambda_8 P_\pi = \frac{i}{2}\left(\begin{array}{c|c} \sigma_2 & 0 \\ \hline 0 & \sigma_2 \end{array}\right), \tag{44}$$

$$P_\pi \lambda_{10} P_\pi = \frac{i}{2}\left(\begin{array}{c|c} \sigma_2 & 0 \\ \hline 0 & -\sigma_2 \end{array}\right), \quad P_\pi \lambda_{11} P_\pi = \frac{i}{2}\left(\begin{array}{c|c} \sigma_1 & 0 \\ \hline 0 & -\sigma_1 \end{array}\right), \tag{45}$$

$$P_\pi \lambda_{15} P_\pi = \frac{i}{2}\left(\begin{array}{c|c} I & 0 \\ \hline 0 & -I \end{array}\right). \tag{46}$$

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