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G. Panina

DIAGONAL COMPLEXES FOR PUNCTURED POLYGONS

Abstract. It is known that taken together, all collections of nonintersecting diagonals in a convex planar n-gon give rise to a (combinatorial type of a) convex $(n-3)$-dimensional polytope As_n called the *Stasheff polytope*, or *associahedron*. In the paper, we act in a similar way by taking a convex planar n-gon with k labeled punctures. All collections of mutually nonintersecting and mutually nonhomotopic topological diagonals yield a complex $\mathrm{As}_{n,k}$. We prove that it is a topological ball. We also show a natural cellular fibration $\mathrm{As}_{n,k} \to \mathrm{As}_{n,k-1}$. A special example is delivered by the case $k=1$. Here the vertices of the complex are labeled by all possible permutations together with all possible bracketings on n distinct entries. This hints to a relationship with M. Kapranov's *permutoassociahedron.*

§1. Introduction

Although the combinatorics of the associahedron was first described by Dov Tamari, a more common reference is James Stasheff's paper [4]. It reads as follows. Assume that $n > 2$ is fixed. We say that two diagonals in a convex n-gon are nonintersecting if they intersect only at their endpoints (or do not intersect at all). Consider all possible collections of mutually nonintersecting diagonals[1] in a convex polygon. This set is partially ordered by reverse inclusion, and it was shown by John Milnor that the poset is isomorphic to the face poset of some convex $(n-3)$-dimensional polytope, called the *associahedron*, which is denoted here[2] by As_n.

In particular, the vertices of the associahedron As_n correspond to the triangulations of an n-gon, and the edges correspond to the edge flips in

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[1] It is important that the vertices of the polygon are labeled and, therefore, we do not identify collections of diagonals that differ by a rotation.

[2] In the literature, it is sometimes denoted by As_{n-3}. This indicates the dimension of the associahedron. However, in the present paper, we keep the notation where n refers to the number of vertices of the n-gon.

which one of the diagonals is removed and replaced by a (uniquely defined) different diagonal. The single diagonals are in a bijection with the facets of As_n, and the empty set corresponds to the entire associahedron As_n.

There exist many explicit constructions of the associahedron: as a special instance of a secondary polytope, a truncation of a simplex, etc.

There also exist many ways to meaningfully generalize the associahedron. In the present paper, we propose one more way of generalizing. Assume that $n \geqslant 1$ and $k \geqslant 0$ with $n-3+2k \geqslant 1$ are fixed. Take a disk D^2 with n (also labeled) marked points on its boundary. If $n > 2$, one may think of a (convex) n-gon, so we call these n points *vertices*. The vertices divide the boundary of the disk into *edges*. Assume also that k labeled *punctures* (that is, k points in the interior of the disk) are fixed.

A *diagonal* is a simple (that is, not self-intersecting) smooth curve c in the disk whose endpoints are some of the (possibly one and the same) vertices such that

(1) c avoids punctures,
(2) c lies in the interior of the disk (except for its endpoints),
(3) c is homotopic (in the punctured disk) to no edge, and
(4) c is noncontractible in the punctured disk.

A *diagonal arrangement* is a collection of diagonals with the following properties:

(1) no two diagonals intersect (except for endpoints),
(2) no two diagonals are homotopic,
(3) the collection is nonempty.

We identify two arrangements A_1 and A_2 whenever there exists a diffeomorphism of the disk that maps A_1 to A_2, preserves the orientation (this is important if $n < 3$), maps vertices to vertices and punctures to punctures, keeping the numbering.

Diagonal arrangements are partially ordered by reverse inclusion. Euler's formula implies that arrangements with maximum number of diagonals contain exactly $n-3+2k$ diagonals.

A cell complex K is *nice* if each k-dimensional cell C is attached to some subcomplex of the $(k-1)$-skeleton of K via a bijective mapping on ∂C.

Theorem.

(1) *The poset described above is combinatorially isomorphic to some nice cell complex* $\mathrm{As}_{n,k}$ *whose support is the ball* B^{n-2+2k}.

(2) *There exists a natural cellular mapping* $\mathrm{As}_{n,k} \to \mathrm{As}_{n,k-1}$ *that is a fibration with combinatorially explicitly describable fibers. Each fiber is a topological disk* D^2.

The cell complex $\mathrm{As}_{n,k}$ is a close relative of the complexes introduced in [1, 2]. The idea of the paper is applicable to the case when an n-gon is replaced by some surface with a higher genus; details are left beyond the paper.

§2. Proof of the theorem and explicit constructions

The following example gives a base for the further inductive construction.

Example 1.

(1) $\mathrm{As}_{n,o}$ is combinatorially isomorphic to the associahedron As_n.
(2) $\mathrm{As}_{1,2}$ is D^2 with two vertices and two edges.
(3) $\mathrm{As}_{2,1}$ is a segment.
(4) $\mathrm{As}_{3,1}$ is a hexagon.

Example 2. The vertices of $\mathrm{As}_{1,k}$ are labeled by the same labels as the vertices of M. Kapranov's *permutoassociahedron* [3]: in Fig. 1 one sees a permutation and a complete bracketing on the set $\{1, ..., k\}$. However, here we do not have a complete combinatorial isomorphism because of different incidence relations.

Now comes the **proof of the theorem**.

Due to the above examples, the theorem holds true for small n and k.

Let us first prove that $\mathrm{As}_{n,k}$ is a nice cell complex. We need to prove that it is possible to attach cells (= balls) starting from small dimensions according to the desired combinatorics. The vertices of the complex (= diagonal arrangements maximal by inclusion) are already well defined. Assume that the m-skeleton is already constructed. We have to attach an $(m+1)$-dimensional ball which encodes some arrangement A of $n-3+2k-m$ diagonals. Let us prove that the complex of all arrangements strictly contained in A is a topological sphere. Cut the disk along the diagonals of A. We get a collection of punctured polygons and can apply the inductive assumption for each of them. Altogether, we have a product of balls minus the interior, which gives us a sphere. So the attachment of a new cell is well defined.

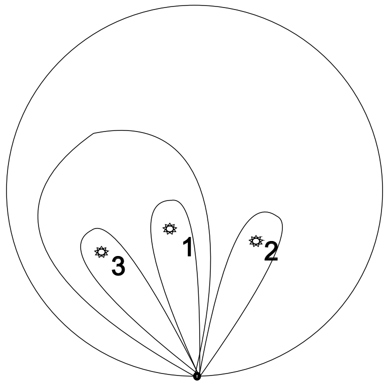

Fig. 1. Here we have a one-gon with three punctures. This configuration gives the permutation and bracketing $(3 \cdot 1) \cdot 2$.

Consider now the following forgetful projection

$$\pi : \mathrm{As}_{n,k+1} \to \mathrm{As}_{n,k}.$$

By definition, it is a cellular mapping. Given a cell corresponding to an arrangement A in an n-gon with $k+1$ punctures, we first eliminate the puncture number $k+1$ and keep all the diagonals. Some diagonals become mutually homotopic; we leave exactly one representative in each of the classes. Some may become homotopic to edges of the polygon; we eliminate them. Some may become contractible; we eliminate them as well. We get either an arrangement $\pi(A)$ in an n-gon with k punctures, or an empty set.

Let us examine the preimage of an inner point of a cell, corresponding to an arrangement A in an n-gon with k punctures. We will show that it is a cell complex homeomorphic to the disk D^2. Here is its explicit construction: the arrangement A cuts the n-gon into cells, so we have a representation of D^2 as a cell complex $K(A)$. Its vertices are the vertices of the n-gon, its edges are either edges of the n-gon or the diagonals. A *corner* is a vertex with two germs g_1 and g_2 of incident edges such that there are no other germs between g_1 and g_2. For each of the corners, we blow up its vertex, that is, replace it by an extra edge, as shown in Fig. 2. We get another cell complex $K'(A)$ which encodes the combinatorics of $\pi^{-1}(A)$.

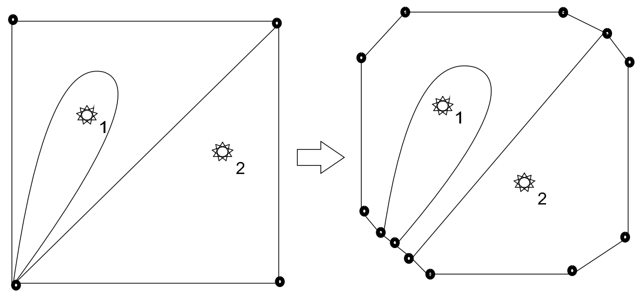

Fig. 2. Blowing up all the corners.

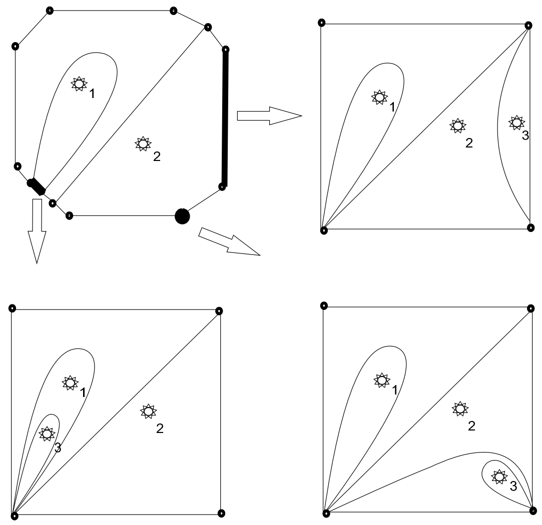

Fig. 3. Here we depict arrangements that correspond to a vertex and two edges (an initial edge of the n-gon and the blow-up of a corner).

Indeed, each cell C of $K'(A)$ gives rise to an arrangement $A(C) \in \pi^{-1}(A)$ in an n-gon with $k+1$ punctures by the following rule:

(1) If C is a 2-dimensional cell, we keep A and add the puncture number $k+1$ in the cell C.

(2) If the cell is an edge e of the polygon, we keep A, add the puncture number $k+1$ in the (uniquely defined) cell C which is adjacent to e, and also add one more diagonal embracing the puncture which is parallel to e.

(3) If the cell is the blow-up of one of the corners, we keep A, add the puncture number $k+1$ in the (uniquely defined) cell C which is adjacent to e, and also add one loop diagonal embracing the puncture which starts and ends at the corner.

(4) If the cell is one of the diagonals c, we keep A, duplicate c, and put the puncture number $k+1$ between c and its copy.

(5) If the cell is one of the new vertices, that is, corresponds to a corner and a vertex of the n-gon, we combine either (3) and (2), or (3) and (4), that is, add both a loop and a diagonal parallel to the edge.

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St.Petersburg State University,
Universitetsky pr., 28,
Stary Peterhof, 198504,
St.Petersburg, Russia
E-mail: gaiane-panina@rambler.ru

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