Записки научных
семинаров ПОМИ
Том 448, 2016 г.

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ON LOCAL COMBINATORIAL FORMULAS FOR CHERN CLASSES OF A TRIANGULATED CIRCLE BUNDLE

Abstract. A principal circle bundle over a PL polyhedron can be triangulated and thus obtains combinatorics. The triangulation is assembled from triangulated circle bundles over simplices. To every triangulated circle bundle over a simplex we associate a necklace (in the combinatorial sense). We express rational local formulas for all powers of the first Chern class in terms of expectations of the parities of the associated necklaces. This rational parity is a combinatorial isomorphism invariant of a triangulated circle bundle over a simplex, measuring the mixing by the triangulation of the circular graphs over vertices of the simplex. The goal of this note is to sketch the logic of deducing these formulas from Kontsevitch's cyclic invariant connection form on metric polygons.

§1. Introduction

0. After submitting this note we discovered that the main computation in Secs. 5, 6 is equivalent to the computation in [13, Secs. 1, 2] of universal combinatorial cochains on the cyclic category from the universal cyclic connection form. Still, we have an accent on the geometry and combinatorics of triangulations.

1. A circle bundle $\mathbb{T} \to E \xrightarrow{p} B$ (see [2]) is a principal fiber bundle with a commutative Lie structure group $\mathbb{T} = \mathbb{R}/\mathbb{Z} \approx \mathrm{U}(1)$. There is a classical chain of homotopy equivalences

$$B\mathbb{T} \approx B\mathrm{U}(1) \approx B\mathrm{O}(2) \approx \mathbb{C}P^{\infty} \approx K(\mathbb{Z}, 2). \qquad (1)$$

Thus the isomorphism classes of circle bundles over B are in a one-to one correspondence with the classes of complex line bundles, classes of oriented two-dimensional real linear vector bundles, and elements of the

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Key words and phrases: circle bundle, Chern class, local formula.

The main result of the paper (Theorem 4.1) was supported by the Russian Science Foundation grant 14-21-00035. G. Sharygin was additionally supported by the RFBR grant 14-01-00007.

two-dimensional integer cohomology group $H^2(B;\mathbb{Z})$. The class of a bundle $c_1(p) \in H^2(B;\mathbb{Z})$ is called its first Chern, or Chern–Euler, class. All characteristic classes of p are powers $c_1^h(p) \in H^{2h}(B;\mathbb{Z})$.

If the base B can be triangulated, then p can be triangulated over a subdivision of the given triangulation. It is not true that the base B can always be triangulated (see [23]), but this happens in most interesting cases. See [9, 30] for the general triangulation and bundle triangulation theory. We are interested in the combinatorics of triangulations in connection with integer and rational local combinatorial formulas for the Chern class and its powers. The triangulated bundle obtains a structure of an oriented PL bundle with fiber S^1. The old folklore theorem

$$\mathrm{PL}(\overrightarrow{\mathrm{S}})/\mathrm{U}(1) \approx * \tag{2}$$

completes the sequence (1) by "$B\,\mathrm{PL}(\overrightarrow{\mathrm{S}}) \approx \ldots$." Therefore, the combinatorics of a triangulation provides full information about characteristic classes.

2. Our initial intention is to find some more hints for correct combinatorics in the classical problem of detecting local combinatorial formulas for characteristic classes ([6–8]). It is well known that the problem faces many troubles in a fruitful way. In particular, the Chern–Simons theory was a byproduct, as mentioned in the first lines of [28]. The point of view of configuration spaces on the problem ([7, 8]) faces the algebro-geometric universality of moduli spaces of configurations in a way especially interesting for the first author. The rational Euler [27] and Stiefel–Whitney [11] classes of the tangent bundle have a clear local combinatoric nature (but in the latter case, up to now there is only a not very enlightening proof). From the combinatorial point of view, local formulas are very interesting combinatorial functions, universal cocycles associated to elementary families of cell complex reconstructions; such as chains of abstract subdivisions or chains of simple maps, abstract mixed subdivisions (multi-simplicial complexes), chains in the MacPhersonian, etc.

The line of simple examples, obviously, should contain the Chern classes of triangulated (or locally combinatorially encoded in some other way) circle bundles. The setup for the combinatorics of a circle bundle and an outline of a construction for rational simplicial local formulas were presented by I. Gelfand and R. MacPherson in 14 lines around Proposition 2 in [8, p. 306]. There are deep formulas in the differential situation, when the bundle is encoded by the pattern of fiberwise singularities of the Morse function on the total space [14, 17, 18]. These formulas naturally connect the problem to a higher Franz–Reidemeister torsion [12] and cyclic homology. The role of line bundles in the geometry of characteristic classes and related mathematical physics is special (see [1]), so perhaps a reasonable idea is to understand the case of circle bundles slightly better. One day, to our surprise, we discovered that canonically looking local formulas for all characteristic cocycles of a triangulated circle bundle trivially pop out from associating to a triangulated circle bundle Kontsevich's connection [19, p. 8] on metric polygons. The answer is expressed through mathematical expectations of the parities of necklaces associated with the triangulation. Below we will present this "parity local formulas" and a sketch of a construction.

3. **Plan.** In Sec. 2, we introduce an abstract simplicial circle bundle (s.c. bundle) and describe the Gelfand–MacPherson setup for simplicial local formulas in the case of circle bundles. In Sec. 3, we associate to a s.c. bundle a cyclic diagram of words on the base. In Sec. 4, we present a rational parity formula for all powers of the first Chern class and formulate the main Theorem 4.1. In Sec. 5, on a geometric realization of a s.c. bundle, we introduce a canonical metric gp of "geometric proportions." With this special metric gp, the bundle becomes a piecewise differential principal circle bundle. It has gauge transition transformations described by functions encoded by matrices of words as linear operators (analogs of permutation matrices). The circle bundle canonically appears as the pullback of a formal universal circle bundle over Connes' cyclic simplex by a very special classifying map. The classifying map is described by the same matrices of words. In Sec. 6, we describe Kontsevich's connection form on metric polygons as a universal cyclic invariant connection on a universal circle bundle over Connes' cyclic cosimplex. The pullback of the connection is a PD connection on the geometric realization of the s.c. bundle with the metric gp. We compute the pullbacks of universal cyclic characteristic forms using matrix maps and, using Okuda's sum of minors Pfaffian identity, obtain rational parity coefficients. In Sec. 7, we assemble the proof of Theorem 4.1. In Sec. 8, we comment on omissions and possible outputs.

4. The first author is deeply grateful to Peter Zograf for pointing to Kontsevich's connection form. The authors thank for hospitality the Oberwolfach Mathematical Institute and IHES, where they had an opportunity to work together.

§2. Local simplicial formulas for circle bundles

5. Triangulation of a circle bundle. If a map of finite abstract simplicial complexes $\mathfrak{E} \xrightarrow{\mathfrak{p}} \mathfrak{B}$ triangulates a circle bundle p, then we can suppose that the geometric realization $|\mathfrak{B}|$ is B and the triangulation is the set of data $(\mathfrak{p}, p, h)$:

$$\begin{tikzcd} \mathfrak{E} \arrow[d, "\mathfrak{p}"] \\ \mathfrak{B} \end{tikzcd} \qquad \begin{tikzcd} E \arrow[d, "p"] \\ |\mathfrak{B}| \end{tikzcd} \qquad \begin{tikzcd} |\mathfrak{E}| \arrow[rr, "h"] \arrow[dr, "|\mathfrak{p}|"'] & & E \arrow[dl, "p"] \\ & |\mathfrak{B}|, & \end{tikzcd} \tag{3}$$

where $|\mathfrak{E}| \xrightarrow{h} E$ is a fiberwise homeomorphism commuting with $|\mathfrak{p}|$ and p. The map $|\mathfrak{p}|$ is a PL fiber bundle. From the point of view of $\mathfrak{p}$, the triangulation homeomorphism h is equivalent to introducing on the PL bundle $|\mathfrak{p}|$ with fiber S^1 an orientation and a continuous metric such that for any $x \in |\mathfrak{B}|$ the fiber over x, the oriented PL circle $|\mathfrak{p}|^{-1}(x)$, has perimeter equal to one. On the other hand, the triangulation homeomorphism h provides p with a PL structure related to a very special system of local sections. We suppose that the simplicial complex $\mathfrak{B}$ is *locally ordered*, i.e., its simplices have total orders on the vertices and the face maps are monotone injections. That is to say, $\mathfrak{B}$ is a finite semi-simplcial set with the extra property that each simplex is determined by its vertices. A local order makes simplicial chain and cochain complexes available.

The simplicial bundle $\mathfrak{p}$ in (3) can be assembled as a colimit from subbundles over base ordered simplices using simplicial face transition maps. The disassembly on bundles over simplices commutes with the geometric realization. Therefore, a subbundle of $\mathfrak{p}$ over a simplex triangulates the circle bundle over the geometric simplex, which is a trivial bundle. An orientation of a trivial circle bundle selects a preferred generator in the 1-homology of the total space. Hence subbundles of $\mathfrak{p}$ over simplices are equipped with an orientation class in simplicial homology in such a way that boundary transition maps of subbundles send a generator to a generator. This assembles to a constant fiber orientation local system on $\mathfrak{B}$.

6. Basic simplicial notations. We denote by $\mathbf{\Delta}$ the category of finite ordinals $[0], [1] \ldots$. Nonstrictly monotone maps between them are called operators, injections are called face operators, surjections are called degeneracy operators. We denote by $\underline{\mathbf{\Delta}}$ the subcategory of $\mathbf{\Delta}$ having the face operators only. The category of finite ordinals can be presented by generators and relations. The standard generators of $\mathbf{\Delta}$ are the elementary boundary operators $[k-1] \xrightarrow{\delta_j} [k]$, $j = 0, \dots, k$, where δ_j is the monotone injection omitting the element j in the image; and the elemetary degeneracy operators $[k+1] \xrightarrow{\sigma_j} [k]$, $j = 0, \dots, k$, where σ_j is the monotone surjection sending the elements j, $j+1$ in the domain to the element j in the codomain. The category $\underline{\mathbf{\Delta}}$ is generated only by the boundaries. We denote by $\langle k \rangle$ an ordered combinatorial simplex: the simplicial complex formed by all subsets of $[k]$. For an operator $[m] \xrightarrow{\mu} [k]$, we denote by $\langle m \rangle \xrightarrow{\langle \mu \rangle} \langle k \rangle$ the induced simplicial map of combinatorial simplices. The standard geometric k-simplex with ordered vertices is denoted by Δ^k. The geometric realization of $\langle \mu \rangle$ is the map $\Delta^m \xrightarrow{|\mu|} \Delta^k$ given in barycentric coordinates as follows:

$$|\mu_i|(t_0, \dots, t_m) = \begin{cases} 0 & \text{if } i \notin \operatorname{im} \mu, \\ \sum\limits_{j \in \mu^{-1}(i)} t_j & \text{if } i \in \operatorname{im} \mu \end{cases} \qquad \text{for } i = 0, \dots, k.$$

7. **Simplicial circle bundles.** By an *elementary simplicial circle bundle* (elementary s.c. bundle) we will mean a map $\mathfrak{R} \xrightarrow{e} \langle k \rangle$ of a simplicial complex $\mathfrak{R}$ onto an ordered simplex $\langle k \rangle$ whose geometric realization $|\mathfrak{e}|$ is a trivial PL fiber bundle over a geometric simplex with fiber S^1, equipped with a fixed orientation 1-dimensional homology class of $\mathfrak{R}$. Here is a picture of an elementary s.c. bundle (Fig. 1). We define the *boundary map* $\delta^* \mathfrak{e} \xrightarrow{\delta_* e} \mathfrak{e}$ of elementary s.c. bundles over the simplicial boundary $[m] \xrightarrow{\delta} [k]$ to be the pullback diagram

$$\begin{tikzcd} \delta^* \mathfrak{R} \arrow[r, "\delta_* \mathfrak{e}"'] \arrow[d, "\delta^* \mathfrak{e}"] & \mathfrak{R} \arrow[d, "\mathfrak{e}"] \\ \langle m \rangle \arrow[r, "\langle \delta \rangle"] & \langle k \rangle. \end{tikzcd}$$

We suppose that $\delta_* \mathfrak{e}$ sends the orientation class of $\delta^* \mathfrak{R}$ to the orientation class of $\mathfrak{R}$. Let $\mathfrak{B}$ be a locally ordered finite simplicial complex. A *simplicial circle bundle* (*s.c. bundle*) $\mathfrak{E} \xrightarrow{\mathfrak{p}} \mathfrak{B}$ is a map p of finite simplicial complexes that has a geometric PL fiber bundle with fiber S^1 as its geometric realization and is equipped with a fixed constant local system that gives a fiber orientation on elementary subbundles.

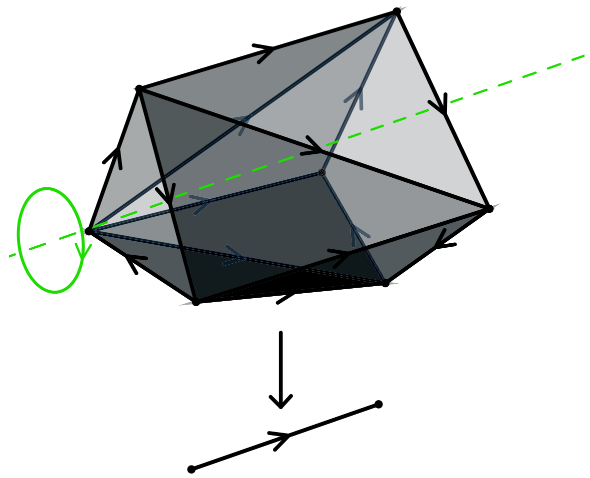

Fig. 1. An elementary simplicial circle bundle.

Simplicial circle bundles always triangulate some principal circle bundles. To build a circle bundle triangulated by $\mathfrak{p}$, it is sufficient to choose a continuous metric on the geometric realization $|\mathfrak{E}|$ such that all fibers of $|\mathfrak{p}|$ are circles with unit perimeter. Transition maps become orientation-preserving isometries, and hence they will be $\mathbb{T}$-transition maps. One possible choice of such a metric is to normalize fiberwise the standard flat metric on the simplices in $|\mathfrak{E}|$. Taking in consideration the simple fact (2), we see that all circle bundles triangulated by $\mathfrak{p}$ are isomorphic.

8. The Gelfand–MacPherson rational simplicial local formulas. Denote by $\mathfrak{N}^c(\overrightarrow{\mathrm{S}})$ the semi-simplicial set of isomorphism classes of all elementary simplicial circle bundles. Elements of $\mathfrak{N}^c_k(\overrightarrow{\mathrm{S}})$ are combinatorial isomorphism classes of elementary s.c. bundles over the k-dimensional oriented combinatorial simplex. The boundary map is generated by the elementary s.c. bundle boundary over the base simplex boundary. Boundaries are well defined, sending isomorphism classes to isomorphism classes. The superscript "c" stands for the fact that we are considering triangulations by classical simplicial complexes. With the bundle $\mathfrak{p}$, a map of semi-simplicial sets $\mathfrak{B} \xrightarrow{\mathfrak{G}_{\mathfrak{p}}} \mathfrak{N}^c(\overrightarrow{\mathrm{S}})$ is associated, sending the base simplex

$U \in \mathfrak{B}$ to the combinatorial isomorphism class of the elementary subbundle $\mathfrak{p}_U$ over that simplex. The map $\mathfrak{G}_\mathfrak{p}$ forgets a part of the information about the bundle. Since an elementary s.c. bundle can have nontrivial automorphisms, one cannot recover all boundary transition functions from $\mathfrak{G}_\mathfrak{p}$, and hence generally one cannot recover from $\mathfrak{G}_\mathfrak{p}$ the entire bundle $\mathfrak{p}$ up to isomorphism.

A *rational simplicial local formula for the Chern class* C_1^h is a $2h$-cocycle on $\mathfrak{R}^c(\overrightarrow{\mathrm{S}})$ represented as a rational combinatorial function of elementary bundles over the $2h$-simplex such that the pullback of this cocycle under the map $\mathfrak{B} \xrightarrow{\mathfrak{G}_\mathfrak{p}} \mathfrak{R}^c(\overrightarrow{\mathrm{S}})$ is a rational simplicial $2h$-Chern cocycle of $\mathfrak{p}$. To put it simple, the value of the cocycle on the ordered simplex U^{2h} of $\mathfrak{B}$ should be an automorphism invariant $C_1^{2h}(\mathfrak{p}_u) \in \mathbb{Q}$ of the subbundle $\mathfrak{p}_U$ over that simplex.

Due to the above-mentioned forgetful nature of $\mathfrak{G}_\mathfrak{p}$, it is not obvious that such universal cocycles exist; however, rational universal cocycles do indeed exist by Proposition 2 of [8]: it is speculated there that the transgression of a rational fiber coorientation class in the Serre spectral sequence of a s.c. bundle can be expressed as a local rational formula involving the combinatorial Laplacian. The result was never calculated, but clearly it ends up in a slightly more complicated automorphism invariant function of the associated 3-necklace for the first Chern class than our rational parity. Local formulas for powers can always be expressed automatically by using the Čech–Whitney formula for the cup product on simplicial cochains. This will result in certain not very transparent, but purely combinatorial concrete rational functions ${}^{GM}C_1^{2h}(\mathfrak{e})$ of elementary s.c. bundles.

§3. Simplicial circle bundles and cyclic words

3.1. Half of Connes' cyclic category. Connes' cyclic category $\mathbf{\Delta}C$ (see [21, Chap. 6.1]) has finite ordinals as objects, and morphisms are generated by all operators from $\mathbf{\Delta}$ and all cyclic permutations of finite ordinals. By this definition, $\mathbf{\Delta}C$ contains $\mathbf{\Delta}$ as a subcategory. For the cyclic category, we use new notation for the standard simplicial generators, since we will need a two-parameter cyclic-simplicial structures on words. The simplicial generators are

$$[n-1] \underset{s_i^n}{\overset{\partial_j^{n-1}}{\rightleftarrows}} [n]\,, \quad j=0,\dots,n, \quad i=0,\dots,n-1. \tag{4}$$

Besides the simplicial generators, $\mathbf{\Delta}C$ has the generator $[n] \xrightarrow{\tau_n} [n]$ acting on $[n]$ according to the rule $\tau_n(i) = (i-1) \bmod (n+1)$, i.e., as the permutation which is the left cyclic shift by one; $\tau_n^j(i) = (i-j) \bmod (n+1)$. In $\mathbf{\Delta}C$ there is an "extra degeneracy," the surjection $[n] \xrightarrow{s_n^n} [n-1]$, which does not exist in $\mathbf{\Delta}$ but exists in $\mathbf{\Delta}C$. It is defined as $s_n^n = s_0^n \tau_n^{-1}$.

9. **Duality.** There are two subcategories in $\mathbf{\Delta}C$. One is $\underline{\mathbf{\Delta}}C$, the subcategory generated by all boundaries ∂_i^n and cyclic shifts τ_n. Another category $\overline{\mathbf{\Delta}}C$ is generated by all standard monotone degeneracies s_i^n, the *extra degeneracy* s_n^n, and τ_n.

The category $\mathbf{\Delta}C$ is self-dual, i.e., there is a categorial isomorphism $\mathbf{\Delta}C \xrightarrow{\bullet^{\mathrm{op}}} \mathbf{\Delta}C^{\mathrm{op}}$. Since $\mathbf{\Delta}C$ has automorphisms, a duality involution is not unique. With the extra degeneracy, a duality can be presented on the generators as follows: for $i \in [k]$ and $[k-1] \xrightarrow{\partial_i^{k-1}} [k]$, the dual map is

$$\partial_i^{\mathrm{op}} = ([k] \xrightarrow{s_i^k} [k-1]); \quad \tau_k^{\mathrm{op}} = \tau_k^{-1}.$$

The duality interchanges $\underline{\mathbf{\Delta}}C$ and $\overline{\mathbf{\Delta}}C$.

10. The duality has a remarkable graphical interpretation in terms of the cylinder of a "simple map" of oriented graphs, oriented cycles (it can be a loop) having a vertex fixed.

A map of oriented graphs is a map sending vertices to vertices and arcs to vertices or arcs in such a way that an incident vertex and arc pair goes either to the same vertex or to an incident vertex and arc pair. Oriented graphs can be identified with 1-dimensional semi-simplicial sets, and maps of graphs, with singular maps of semi-simplicial sets. A graph g having a geometric realization $|g|$, a map of graphs $g_0 \xrightarrow{f} g$ also obtains a geometric realization $|g_0| \xrightarrow{|f|} |g|$, and we can consider the cylinder $\mathrm{Cyl}\,(|f|) \to [0,1]$ of this map. The cylinder has a natural structure of a 2-dimensional cell complex composed of solid triangles and quadrangles and having a cellular projection to the interval $[0,1]$. A map f is called *simple* [31] if for any point $u \in |g|$ the preimage $|f|^{-1}u \subseteq |g_0|$ is contractible. Let both graphs g_0, g_1 be oriented cycles. In this case, a map is simple if the preimage of every vertex is an embedded interval, or, dually, any arc has a single preimage. The fundamental observation is that a map is simple if and only if the projection $\mathrm{Cyl}\,(f) \to [0,1]$ is a trivial PL $\overrightarrow{\mathrm{S}}$-fiber bundle. In large generality, this is Cohen's theory of the cylinder of a PL map [3]. Let us have a fixed vertex s_0 on the oriented cycle g_0. Then the orientation creates

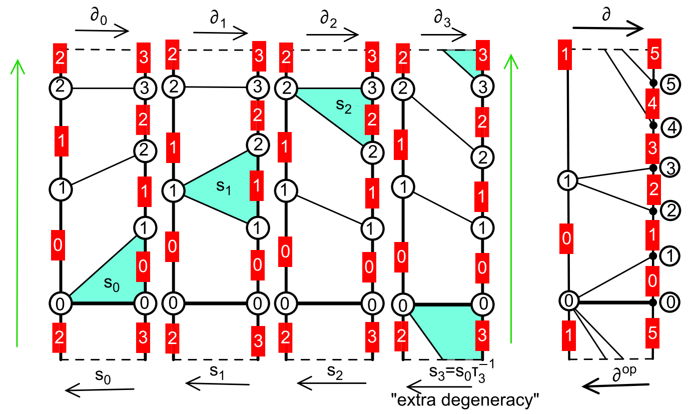

Fig. 2.

a linear order on the vertices and arcs of g_0, g. We suppose that in this order $s_0, f(s_0)$ are minimal. Also we suppose that the position of an arc in this order is equal to the position of its tail. In terms of the orders, a map f is simple if and only if f is a $\overline{\mathbf{\Delta}}C$-morphism $\overline{f}$ on ordered vertices. If this is the case, then the dual $\underline{\mathbf{\Delta}}C$-morphism $\overline{f}^{\mathrm{op}}$ is a monotone injection on ordered arcs sending an arc to its unique preimage. In Fig. 2 we depicted this duality on the cone for the generators and for the general case.

For a boundary $([k] \xrightarrow{\partial} [m]) \in \operatorname{Mor} \underline{\mathbf{\Delta}}C$, the general formula for the dual cyclic degeneracy $([m] \xrightarrow{\partial^{\mathrm{op}}} [k]) \in \operatorname{Mor} \overline{\mathbf{\Delta}}C$ is as follows:

$$\partial^{\mathrm{op}}(i) = \begin{cases} 0 & \text{if } \partial(j) < i \text{ for every } j, \\ \min_{\partial(j) \geqslant i} j & \text{if there exists } j \text{ such that } \partial(j) \geqslant i. \end{cases} \tag{5}$$

It can be computed graphically using the cylinder of a simple map or inductively on the generators.

11. Cyclic-mono factorization. Any morphism $[k] \xrightarrow{\varrho} [m]$ in $\mathbf{\Delta}C$ has a unique factorization into a cyclic permutation followed by a monotone $\mathbf{\Delta}$-operator. The same is true for $\underline{\mathbf{\Delta}}C$ and $\overline{\mathbf{\Delta}}C$. This is Connes' theorem ([21, Theorem 6.1.3]). We will fix a simplified formula for this decomposition for the case of $\underline{\mathbf{\Delta}}C$.

Lemma 3.1. *For any boundary map* $[k] \xrightarrow{\partial} [m]$ *and a cyclic permutation* $[m] \xrightarrow{\tau_m^i} [m]$ *there are a unique boundary* $[k] \xrightarrow{(\tau_m^i)^*\partial} [m]$ *and a cyclic permutation* $[k] \xrightarrow{\partial^*\tau_m^i} [k]$ *such that the following diagram is commutative:*

$$\begin{tikzcd} {[k]} \arrow[r, "\partial"] \arrow[d, "\partial^*\tau_m^i"'] & {[m]} \arrow[d, "\tau_m^i"] \\ {[k]} \arrow[r, "(\tau_m^i)^*\partial"'] & {[m].} \end{tikzcd}$$

The following formula holds:

$$\partial^*\tau_m^i = \tau_k^{\partial^{\mathrm{op}}(i)}, \tag{6}$$

where ∂^{op} *is defined in* (5).

Proof. This is a part of Connes' theorem, and it can be proved by a part of Connes' proof, which in our case is the consideration of the circular mapping cylinder of the map ∂^{op} with a fixed section, which we discussed earlier. The map τ_m^i acts by changing the zero section of the cylinder. □

3.2. Simplicial circle bundles and cyclic diagrams of words on the base. To a simplicial circle bundle defined in p. **7** with fixed combinatorial sections $\mathbf{S}_0$ over simplices of the base, we associate a cyclic diagram of words on the base.

12. Words and necklaces. *A word of length* $n+1$ *in an ordered alphabet with* $k+1$ *elements*, $k \leqslant n$, is any surjective map $[n] \xrightarrow{w} [k]$ (it is not required to be monotone). The cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$ acts by cyclic permutations on $[n]$. We can extend this cyclic action to words: just put $\tau_n w(i) = w(\tau_n(i))$. The orbit of a word under the cyclic permutations is called a "circular permutation," or an *"oriented necklace"* with $n+1$ beads colored by $[k]$. We regard a cyclic shift of a word as a morphism between words:

$$\begin{tikzcd} {[n]} \arrow[rr, "\tau_n^i"] \arrow[dr, "(\tau_n^i)^*w"'] & & {[n]} \arrow[dl, "w"] \\ & {[k].} & \end{tikzcd} \tag{7}$$

If we have a word w and a boundary operator, i.e., a monotone injection $[k_0] \xrightarrow{\delta} [k]$, then the unique pullback is defined by

$$\begin{CD} [n_0] @>{\partial_\delta}>> [n] \\ @V{\delta^* w}VV @VV{w}V \\ [k_0] @>{\delta}>> [k], \end{CD} \tag{8}$$

where ∂_δ is the boundary operator on finite ordinals induced by the boundary δ on the codomain of words. To say it differently, $\delta^* w$ is the word obtained from w by deleting all letters not in the image of δ, and ∂_δ is the embedding of $\delta^* w$ into w as a subword.

We define the *cyclic category of words* $C\mathcal{W}$ with words as objects and morphisms being *cyclic morphisms of words*, i.e., $\underline{\mathbf{\Delta}}C$-morphisms of words generated by cyclic shifts (7) and boundaries (8) (that is, $w \mapsto \delta^* w$) over boundaries on the alphabet. The category $C\mathcal{W}$ has two projection functors

$$\underline{\mathbf{\Delta}}C \xleftarrow{\text{Dom}} C\mathcal{W} \xrightarrow{\text{Codom}} \underline{\mathbf{\Delta}},$$

the "domain" projection to $\underline{\mathbf{\Delta}}C$ and the "codomain" projection to $\underline{\mathbf{\Delta}}$. The codomain projection makes $C\mathcal{W}$ a category fibered in commutative groupoids over $\underline{\mathbf{\Delta}}$. As in the category $\underline{\mathbf{\Delta}}C$, any cyclic morphism of words has a unique decomposition into a cyclic shift followed by a boundary.

13. Words as matrices. Words have associated matrices, and we may regard these matrices as linear operators. This is our key trick.

Let $\mathcal{L}(w)$ be the $[n] \times [k]$ matrix

$$\mathcal{L}_i^j(w) = \begin{cases} 1 & \text{if } w(i) = j, \\ 0 & \text{if } w(i) \neq j. \end{cases} \tag{9}$$

Put $m_j = \# w^{-1}(j)$, the number of times the letter $j \in [k]$ appears in the word w. The following is the definition of the matrix of a word w normalized by columns:

$$\overline{\mathcal{L}}_i^j(w) = \begin{cases} \frac{1}{m_j} & \text{if } w(i) = j, \\ 0 & \text{if } w(i) \neq j. \end{cases} \tag{10}$$

The sums of elements in columns of $\overline{\mathcal{L}}$ are all equal to 1, different columns of $\mathcal{L}$ and $\overline{\mathcal{L}}$ are orthogonal. A cyclic shift $(\tau_n^i)^*$ of words goes to a cyclic permutation of rows of matrices, a boundary operation δ^* on words corresponds to the deletion of columns in matrices with numbers not in the image of δ followed by the deletion of the zero rows in what remains. The opposite insertion of a matrix as a submatrix with adding zero rows is described by ∂_δ.

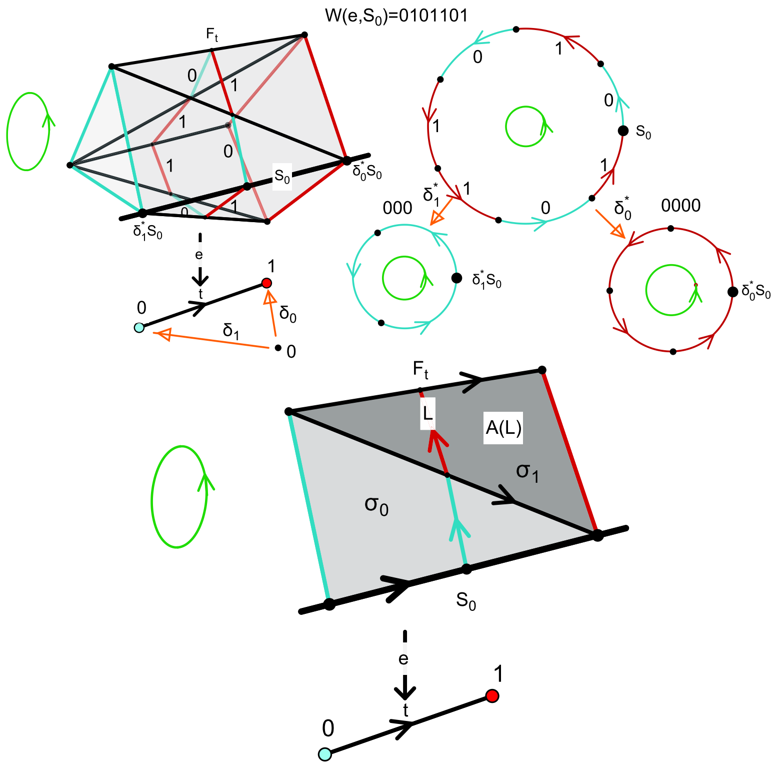

Fig. 3.

14. The geometric fiber of an elementary s.c. bundle; 0- and 1-sections; order out of orientation. Let $\mathfrak{R} \xrightarrow{\mathfrak{e}} \langle k \rangle$ be an elementary s.c. bundle over an ordered abstract k-simplex with $k+1$ vertices, $|\mathfrak{R}| \xrightarrow{|\mathfrak{e}|} \Delta^k$ be its geometric realization, which is, by definition, an oriented trivial PL S^1-bundle over Δ^k. Geometric simplices of $|\mathfrak{R}|$ that project epimorphically onto Δ^k can have dimensions only k and $k+1$. Let us call such a k-simplex a 0-*section* of $|\mathfrak{e}|$, and a $(k+1)$-simplex, a 1-*section* of $|\mathfrak{e}|$.

There are corresponding *combinatorial* 0- *and* 1-*sections*, simplices in $\mathfrak{e}$ that project epimorphically onto $\langle k \rangle$.

Take the geometric fiber F_t of $|\mathfrak{e}|$ over an interior point $t \in \operatorname{int} \Delta^k$. The fiber F_t is a broken PL circle. Each arc L of F_t is the intersection of the fiber with some $(k+1)$-simplex, a 1-section $A(L)$ of $|\mathfrak{e}|$. The end vertices of the arc L are intersections with geometric 0-sections, two faces of $A(L)$. The broken circle F_t has the same number of arcs and vertices; let this number be equal to $n+1$. This means that there are $n+1$ 0- and 1-sections of $|\mathfrak{e}|$ and $\mathfrak{e}$. Fix some 0-section, denote it by S_0, and denote by $S_0(t)$ the corresponding vertex in F_t. The orientation of the bundles $\mathfrak{e}$, $|\mathfrak{e}|$ creates an orientation of the circle F_t. Therefore, all arcs L obtain heads and tails, each vertex of F_t obtains its input and output arcs. This generates a linear order on the vertices of F_t depending on the choice of the fixed vertex $S_0(t)$. In this order, on any arc its tail precedes its head, and $S_0(t)$ is the minimal vertex. The order on vertices creates a linear order on the arcs in which an arc obtains the number of its tail. Therefore, on F_t we obtain $n+1$ ordered vertices $S_0(t), \ldots, S_n(t)$ and $n+1$ ordered intervals $L_0(t), \ldots, L_n(t)$;

$$L_i(t) = [S_i(t), S_{i+1 \bmod (n+1)}(t)], \quad i = 0, \ldots, n.$$

Therefore, the 1-section $A(L_i)$ also obtains the number i in the order, and we put $A_i = A(L_i)$.

15. The word associated to an elementary s.c. bundle with a fixed section. Every 1-section A_i of $|\mathfrak{e}|$ is the domain simplex of a subbundle of the type $\Delta^{k+1} \xrightarrow{|\sigma_j|} \Delta^k$ which is the geometric realization of an elementary simplicial degeneracy σ_j of ordered simplices $\langle k+1 \rangle \xrightarrow{\langle \sigma_j \rangle} \langle k \rangle$, corresponding to an elementary degeneracy operator $[k+1] \xrightarrow{\sigma_j} [k]$ (see p. **6**). The simplicial map $|\sigma_j|$ (Fig. 4) shrinks the edge of $A_i = \Delta^{k+1}$ with vertices v_j, v_{j+1} to the single vertex j of the base Δ^k. Here j is a function of A_i and, therefore, a function of the number i. We denote this function by $j = \mathcal{W}(\mathfrak{e}, S_0)(i)$. The function $\mathcal{W}(\mathfrak{e}, S_0)$ is surjective and independent of $t \in \operatorname{int} \Delta^k$; therefore, it is a *word* (see p. **12**) *associated to the elementary s.c. bundle* $\mathfrak{e}$ *with the fixed* 0*-section* S_0.

16. Changing the section of an elementary s.c. bundle vs. a cyclic shift of a word, the associated necklace. Let us have, besides the 0-section S_0, another 0-section ${S_0}'$ of $\mathfrak{e}$. The 0-section ${S_0}'$ corresponds to a vertex of F_t having the number $\imath(S_0, {S_0}')$ in the vertex order determined by the orientation and the section S_0. Then the word $\mathcal{W}(\mathfrak{e}, {S_0}')$ is obtained from $\mathcal{W}(\mathfrak{e}, S_0)$ by the cyclic shift $\tau_n^{\imath(S_0,{S_0}')}\mathcal{W}(\mathfrak{e}, S_0) = \mathcal{W}(\mathfrak{e}, {S_0}')$.

Thus the elementary s.c. bundle $\mathfrak{e}$ obtains the *associated oriented necklace* $\mathcal{N}(e)$ in the sense of p. **12** as the orbit of the words $\mathcal{W}(\mathfrak{e}, S_0)$ under the cyclic shifts.

17. **The boundary of an elementary s.c. bundle with a section vs. the boundary of a word.** Take a face of the base simplex $\langle k-1\rangle \xrightarrow{\langle\delta_j\rangle} \langle k\rangle$ and consider the corresponding boundary $\delta_j^*\mathfrak{e} \xrightarrow{(\delta_j)_*} \mathfrak{e}$ of the elementary s.c. bundle over that face. If a 0-section S_0 is fixed for $\mathfrak{e}$, then $\delta_i^* e$ has the induced 0-section $\delta_i^* S_0$. So we have a morphism $(\delta_j^*\mathfrak{e}, \delta_j^* S_0) \xrightarrow{(\delta_j)_*} (\mathfrak{e}, S_0)$. We are interested in the relation between $\mathcal{W} = \mathcal{W}(\mathfrak{e}, S_0)$ and $\mathcal{W}(\delta_j^*\mathfrak{e}, \delta_j^* S_0)$. All 1-sections B of $\delta_i^*\mathfrak{e}$ are faces of 1-sections A_i of $\mathfrak{e}$ such that $\mathcal{W}(i) \neq j$.

For every 1-section B of $\delta_j^*\mathfrak{e}$ there is a unique 1-section $\partial_{\delta_j} B$ of $\mathfrak{e}$ such that B is a face of $\partial_{\delta_j} B$ and this map is monotone with respect to the induced order corresponding to the induced section $\delta_i^* S_0$. This statement is correct but requires some simplicial work. One should use the 1-dimensional version of Cohen's theorem on the cylinder of a simple map for PL manifolds mentioned in p. **10**.

Combining the previous observations, we conclude that the word

$$\mathcal{W}(\delta_j^*\mathfrak{e}, \delta_j^* S_0) \equiv \delta^*\mathcal{W}(\mathfrak{e}, S_0)$$

and the induced map on 1-sections ∂_{δ_i} is the domain boundary operator (8) of words induced by the codomain boundary δ_i.

18. **The cyclic diagram of words associated to a s.c. bundle with fixed sections on elementary subbundles.** We denote by $\int^{\underline{\mathbf{\Delta}}}\mathfrak{B}$ the category of simplices of a locally ordered simplicial finite complex $\mathfrak{B}$. The objects are simplices, and the only morphism between an m-simplex U^m and a k-simplex U^k is the face map $U^m \xrightarrow{\delta} V^k$ corresponding to the face operator $[m] \xrightarrow{\delta} [k]$. This category has the canonical functor $\int^{\underline{\mathbf{\Delta}}}\mathfrak{B} \to \underline{\mathbf{\Delta}}$ making it a category fibered in finite sets.

Let us have a s.c. bundle $\mathfrak{E} \xrightarrow{\mathfrak{p}} \mathfrak{B}$ over $\mathfrak{B}$. Let any elementary s.c. subbundle $\mathfrak{p}_U$ over a simplex $U \in \mathfrak{B}$ have a fixed individual 0-section S_0^U. We denote this system of 0-sections by $\mathbf{S}_0$; so we fix the pair $(\mathfrak{p}, \mathbf{S}_0)$. Combining the constructions of pp. **15**, **16**, **17**, we see that any boundary

$U \xrightarrow{\delta} V$ between simplices of $\mathfrak{B}$ gives a cyclic morphism of words

$$\begin{aligned} &\mathcal{W}(\mathfrak{p}, \mathbf{S}_0)(U \xrightarrow{\delta} V) \\ &= \left(\mathcal{W}(\mathfrak{p}_U, S_0^U) \xrightarrow{\tau^{\imath}(S_0^U, \delta^* S_0^V)} \mathcal{W}(\mathfrak{p}_U, \delta^* S_0^V) \xrightarrow{\partial_\delta} \mathcal{W}(\mathfrak{p}_V, S_0^V) \right). \end{aligned} \tag{11}$$

These data, obviously, commute with the composition. Therefore, we obtain a functor $\int^{\underline{\mathbf{\Delta}}} \mathfrak{B} \xrightarrow{\mathcal{W}(\mathfrak{p}, \mathbf{S}^0)} C\mathcal{W}$ fibered over $\underline{\mathbf{\Delta}}$. We can imagine that this functor is a coloring of the base simplices by words in the alphabet consisting of the vertices of the simplex, so that boundary morphisms on the base simplices correspond to cyclic morphisms of words in such a way that the diagram is commutative. Changing the 0-sections causes an equivalence of functors, i.e., a system of cyclic permutations of words commuting with all the structure morphisms. Thus the isomorphism class of bundles goes to the equivalence class of fibered functors $\mathrm{Funct}(\int^{\underline{\mathbf{\Delta}}} \mathfrak{B}, C\mathcal{W})$. The inverse statement is also true with important comments (p. **29**), but we do not fully develop it here.

§4. Rational parities of words and local formulas for powers of Chern classes

4.1. The rational parity of a word and of an odd necklace. Consider a word $[n] \xrightarrow{w} [k]$. Call a "proper subword of w" any subword consisting of $k+1$ different letters (thus, it is a section of the map w). A proper subword defines a permutation of $k+1$ elements, and this permutation has a parity, even or odd. We define the rational parity of w to be the expectation of the parities of all its proper subwords. Namely, put

$$P(w) = \frac{\#(\text{even proper subwords}) - \#(\text{odd proper subwords})}{\#(\text{all proper subwords})}.$$

The parity of a permutation of an odd number of elements is invariant under cyclic shifts of the permutation. Therefore, if k is even, then $k+1$ is odd, and in this situation $P(w)$ is an invariant of the oriented necklace (p. **12**). Words are ordered and have no nontrivial automorphisms. Necklaces can have remarkable groups of automorphisms, see [4]. The parity of a proper subword survives an orientation-preserving automorphism. Therefore, if $k+1$ is odd, then $P(w)$ is an isomorphism invariant of the necklace defined by w.

The parity of a permutation coincides with the determinant of the matrix of the permutation. A similar interpretation exists for the rational parity $P(w)$. The rational parity of w is equal to the sum of the maximal minors of the normalized matrix of the word $\overline{\mathcal{L}}$ defined in p. **13**:

$$P(w) = \sum_{0 \leqslant i_0 < i_1 < \cdots < i_k \leqslant n} \det \overline{\mathcal{L}}_{(i_0,\dots,i_k)}(w), \tag{12}$$

where $\overline{\mathcal{L}}_{(i_0,\dots,i_k)}(w)$ denotes the square submatrix of $\overline{\mathcal{L}}(w)$ with row numbers $i_0, \dots, i_k$.

4.2. Local formulas for powers of the Chern class in terms of the rational parities of odd necklaces. Rational simplicial local formulas for Chern classes of circle bundles in the sense of p. **8** can be expressed through the rational parities of necklaces associated with elementary s.c. bundles (p. **16**).

Theorem 4.1. *The rational function*

$${}^{p}C_1^h(\mathfrak{e}) = (-1)^h \frac{h!}{(2h)!} P(\mathcal{N}(\mathfrak{e})) \tag{13}$$

of an elementary s.c. bundle $\mathfrak{R} \xrightarrow{\mathfrak{e}} \langle 2h \rangle$ over a $2h$-simplex is a rational simplicial local formula for the hth power of the first Chern class of the circle bundle.

Here the suffix p stays for "parity."

§5. Geometric simplicial, circle, and cyclic bundles

5.1. The metric of geometric proportions and the associated circle bundle.

19. **An elementary degeneracy and geometric proportions.** Let us introduce the simplest possible metric on the geometric total complex $|\mathfrak{R}|$ of an elementary s.c. bundle $\mathfrak{R} \xrightarrow{\mathfrak{e}} \langle k \rangle$ satisfying the properties described in p. **7**. Here we essentially rely on the facts about geometric proportions between elements of similar triangles from book VI of Euclid's *Elements*, where he presumably explicates the achievements of the Pythagorean or Athens school. Actually, geometric proportions is a geometric background of Milnor's geometric realization functor. The fact is as follows (see Fig. 4 for similar triangles).

Consider a standard geometric simplicial degeneracy $A = \Delta^{k+1} \xrightarrow{|\sigma_j|} \Delta^k = B$. It projects the edge α_j of A with vertices v'_j, v'_{j+1} to the vertex v_j. Let us fix a flat Euclidean metric ρ on the total simplex A in which α_j has positive length $a_j \in \mathbb{R}_{>0}$. Take a point with barycentric coordinates $t = (t_0, \dots, t_k) \in B$ and consider the interval $L_j(t) = |\sigma_i|^{-1}(t) \subset A$, the fiber of the projection $|\sigma_j|$. Then the length $l_j^\rho(t)$ of the interval $L_j(t)$ in the metric ρ is equal to $a_j \cdot t_j$.

The domain simplex A of the projection $|\sigma_j|$ obtains bundle coordinates if we declare one of the two 0-sections in it (see Fig. 4) as the "bottom," or "tail," section and the other one as the "top," or "head," section, respectively. A point $u \in A$ is given the bundle coordinates $u = (x^{st}(u); t_0(u), \dots, t_k(0))$, where $t(u)$ are the barycentric coordinates of the projection to the base, $x^{st}(u)$ is the distance in the fiber interval $L_j(t(u))$ from the tail point to u in the standard flat metric st on the simplex, $0 \leqslant x^{st}(u) \leqslant t_j(u)$. If we change the flat metric by assigning to the edge α_j the length a_j, then the coordinate $x^{st}(u)$ will change linearly, to $a_j \cdot x^{st}(u)$.

20. The matrix of a word and the global bundle coordinates. Geometric proportions impart a geometric meaning to the matrix $\mathcal{L}(\mathcal{W})$, given by formula (9) in p. **13**, of the word $\mathcal{W} = \mathcal{W}(\mathfrak{e}, S^0)$ constructed in p. **15** for an elementary s.c. bundle $\mathfrak{R} \xrightarrow{\mathfrak{e}} \langle k \rangle$ with fixed 0-section S^0. By the geometric proportion identities of p. **19**,

$$\sum_{a=0}^{k} \mathcal{L}_i^a t_a = t_{\mathcal{W}(i)} = l_i^{st}(t), \tag{14}$$

where $l_i^{st}(t)$ is the length of the interval $L_i(t) = |e|^{-1}(t) \cap A_i$ in the standard flat metric on $|\mathfrak{R}|$. So, viewed as a linear operator, the matrix $\mathcal{L}$ transforms the vector of the barycentric coordinates of a point t in the base to the vector of the lengths of the intersections with 1-sections of the fiber over t ordered by the bundle orientation:

$$\mathcal{L}(t_0, \dots, t_k) = (l_0^{st}(t), \dots, l_n^{st}(t)).$$

The bundle space $|\mathfrak{R}|$ also obtains global bundle coordinates relative to the chosen 0-section S_0. A point u lying in a 1-section A_i is given the bundle coordinates

$$u = (x_{\mathcal{W}(i)}^{st}(u) + l_0^{st}(t(u)) + \dots + l_i^{st}(t(u); t(u)).$$

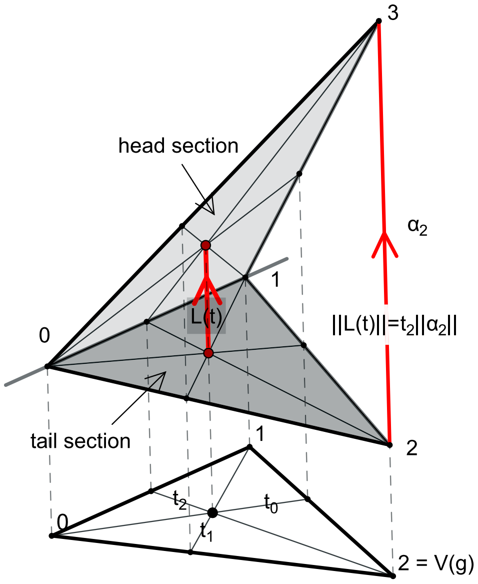

Fig. 4. The bundle $\Delta^3 \xrightarrow{|\sigma_2|} \Delta^2$.

The ith (with respect to the orientation and the fixed section S_0) 0-section S_i becomes the graph of the function $S_i(t) = \sum\limits_{b=0}^{i-1} l_b^{st}(t)$.

21. **Circle bundle coordinates.** Let us normalize the standard flat metric on $|\mathfrak{e}|$. Any elementary s.c. subbundle of $\mathfrak{e}$ over a vertex v_j of the base is a simplicial oriented circle $\mathfrak{R}_j \xrightarrow{\mathfrak{e}_{v_j}} \langle 0 \rangle$, $|\mathfrak{R}_j| \approx S^1$. Let the circle $\mathfrak{R}_j$ have m_j arcs. We assign to any arc in $|\mathfrak{R}_j|$ the length equal to $\frac{1}{m_j}$. This induces a new flat metric on all 1-sections A_i, since the collapsing edge of A_i belongs to $|\mathfrak{R}_{\mathcal{W}(i)}|$. So we obtain a new flat metric "gp" on $|\mathfrak{R}|$ ("gp" stands for "geometric proportions"). In the new metric gp, the vector of the lengths of the intervals in the fiber over $t \in \Delta^k$ is expressible using the normalized matrix of the word $\overline{\mathcal{L}}(\mathcal{W})$ (see (10) in p. **13**). Namely,

$$\sum_{a=0}^{k} \overline{\mathcal{L}}_i^a t_a = \frac{1}{m_{\mathcal{W}(i)}} t_{\mathcal{W}(i)} = l_i^{gp}(t). \tag{15}$$

Then
$$\overline{\mathcal{L}}(t_0 \dots t_k) = (l_0^{gp}(t), \dots, l_k^{gp}(t)), \tag{16}$$
and we have the identity
$$\sum_{i=0}^{n} l_i^{gp}(t) = \sum_{j=0}^{k} \frac{m_j}{m_j} t_j \equiv 1.$$
These observations show that $\overline{\mathcal{L}}$ is a linear operator
$$\Delta^k \xrightarrow{\overline{\mathcal{L}}} \Delta^n. \tag{17}$$
We have normalized the metric on $|\mathfrak{R}|$ in such a way that all fiber circles obtain unit lengths and face maps are isometries.

Now any point u in $|\mathfrak{R}|$, lying in the ith 1-section A_i of $|\mathfrak{R}|$, obtains the trivial $\mathbb{T}$-circle bundle coordinates
$$\mathfrak{T}_{S_0}^{gp}(u) = \left(\exp\left(\frac{1}{m_{\mathcal{W}(i)}} x_{\mathcal{W}(i)}^{st}(u) + \ S_i^{st}(t) \right) |t \right) \in \mathbb{T} \times \Delta^k.$$
Let $\boldsymbol{T}^k$ be a trivial circle bundle $\mathbb{T} \times \Delta^k \to \Delta^k$. The map $\mathfrak{T}_{S_0}^{gp}$ is a bundle homeomorphism
$$|(\mathfrak{e}, S_0)| \xrightarrow{\mathfrak{T}_{S_0}^{gp}} (\boldsymbol{T}^k, 0),$$
sending S_0 to a 0-section and S_i to the graph of the $\mathbb{T}$-valued function
$$\exp(S_i^{gp}(t))$$
on Δ^k, where
$$S_i^{gp}(t) = \sum_{a=0}^{i} l_a^{gp}(t). \tag{18}$$
Thus, introducing the metric of geometric proportions on the bundle $|\mathfrak{e}|$ with fixed 0-section S_0 produced $\mathbb{T}$-trivialization, controlled by the data of the reduced matrix of the word $\overline{\mathcal{L}}(\mathcal{W}(\mathfrak{e}, S_0))$.

22. **0-Sections as $\mathbb{T}$-transition functions; the PD circle bundle $\boldsymbol{T}^{gp}(\mathcal{W}(\mathfrak{p}, \mathbf{S}^0))$.** Let us consider a s.c. bundle $\mathfrak{p}$ with a system $\mathbf{S}_0$ of 0-sections over the simplices of the base, as in p. **18**. Let us fix the lengths of the edges of the circles $\mathfrak{E}$ over the vertices of $\mathfrak{B}$, as in p. **21**; then the geometric realizations of all elementary subbundles obtain the geometric proportions metric and $\mathbb{T}$-trivialization relative to the fixed sections. For a boundary $U^m \xrightarrow{\delta} V^k$ of the simplices in the base, the trivializations

$\boldsymbol{T}^m$ and $\boldsymbol{T}^k$ are related by the $\mathbb{T}$-gauge transformation determined by the change of 0-sections (see Fig. 6) followed by the trivial face embedding:

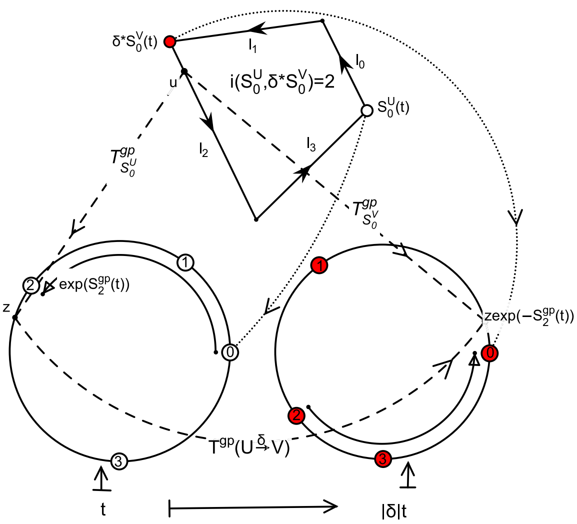

Fig. 5.

$$\begin{array}{ccc} |(\mathfrak{p}_U, S_0^U)| & \xrightarrow{\delta_*} & |(\mathfrak{p}_V, S_0^V)| \\ {\scriptstyle \mathcal{T}^{gp}_{S_0^U}}\downarrow & & \downarrow{\scriptstyle \mathcal{T}^{gp}_{S_0^V}} \\ (\boldsymbol{T}^m, 0) & \xrightarrow{\boldsymbol{T}^{gp}(U \xrightarrow{\delta} V)} & (\boldsymbol{T}^k, 0), \end{array} \tag{19}$$

$$\boldsymbol{T}^{gp}(U \xrightarrow{\delta} V)(z|t) = (z \exp(-S^{gp}_{\imath(S_0^U, \delta^* S_0^V)}(t))||\delta|(t)).$$

Here $\imath$ was defined in p. **16**. The transition transformation $\boldsymbol{T}^{gp}(U \xrightarrow{\delta} V)$ is determined by the cyclic morphism of words $\mathcal{W}(\mathfrak{p}, \mathbf{S}_0)(U \xrightarrow{\delta} V)$ (see (11) in p. **18**). Generally, the diagram of trivial circle bundles and linear boundary gauge transformations is a function of $\mathcal{W}(\mathfrak{p}, \mathbf{U}_0)$. The diagram $\boldsymbol{T}^{gp}(\mathcal{W}(\mathfrak{p}, \mathbf{S}_0))$ assembles to a PD circle bundle over $|\mathfrak{B}|$, and the map $\mathcal{T}^{gp}$ assembles to a triangulation $|\mathfrak{p}| \xrightarrow{\mathcal{T}^{gp}} \boldsymbol{T}^{gp}(\mathcal{W}(\mathfrak{p}, \mathbf{S}_0))$. Changing the system of sections $\mathbf{S}_0 \to \mathbf{S}_0'$ causes a global gauge transformation of $\boldsymbol{T}^{gp}(\mathcal{W}(\mathfrak{p}, \mathbf{S}_0)) \to \boldsymbol{T}^{gp}(\mathcal{W}(\mathfrak{p}, \mathbf{S}_0'))$.

5.2. The cyclic cosimplex and the universal cyclic circle bundle. We equip the half of Connes' cyclic cosimplex [21, Chap. 7.1.3] with a $\underline{\boldsymbol{\Delta}}C$-diagram of trivial $\mathbb{T}$-bundles $\boldsymbol{TC}$ having cyclic linear transition maps. The diagram $\boldsymbol{TC}$ itself does not admit a colimit that assembles to any sort of fiber bundle, but still we can use it as a universal object for the fiber bundles $\boldsymbol{T}^{gp}(\mathfrak{p}, \mathbf{S}_0)$.

23. **The circle bundle over the half of the cyclic cosimplex.** Let $\underline{\boldsymbol{\Delta}} \xrightarrow{\triangle} \mathbf{Top}$ be a canonical semi-cosimplex with barycentric coordinates. Below we will slightly change the notation for boundary operators and coordinates on it and denote

$$\triangle([n]) = \Delta^n = \{l_0, \dots, l_n | \sum l_i = 1\} \subset \mathbb{R}^{n+1};$$

$$\triangle(\partial_i) = |\partial_i|(l_0, \dots, l_{n-1}) = (l_0, \dots, l_{i-1}, 0, l_i, \dots, l_{n-1}).$$

The cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$ acts by cyclic permutations of $[n]$, the standard generator τ_n being represented by the left shift by one:

$$\tau_n(i) = i - 1 \bmod (n+1).$$

A *cyclic semi-cosimplex* is a functor $\underline{\boldsymbol{\Delta}}C \xrightarrow{\Delta C} \mathbf{Top}$; it is an extension of $\triangle$ which incorporates the cyclic shifts of barycentric coordinates. For $[n] \xrightarrow{\tau_n^i} [n]$, the map $\triangle C(\tau_n^i) = (\Delta^n \xrightarrow{|\tau_n^j|} \Delta^n)$ is defined by the standard coordinate representation of $\mathbb{Z}/(n+1)\mathbb{Z}$ in $\mathbb{R}^{n+1}$:

$$|\tau_n^j|(l_0, \dots, l_n) = (l_{\tau_n^j(0)}, \dots, l_{\tau_n^j(n)}) = (l_{n+1-j} \dots, l_n, l_0 \dots l_{n-j}).$$

Consider the following family of linear functions:

$$\Delta^n \xrightarrow{S_i^n} [0,1], \quad i = 0, \dots, n; \quad S_i^n = \sum_{a=0}^{i-1} l_a.$$

Then the family

$$\exp(S_i^n(l)), \quad i = 0, \dots, n,$$

of $\mathbb{T}$-valued functions on simplices can be viewed as a family of sections of the trivial circle bundle $\boldsymbol{T}^n = (\mathbb{T} \times \Delta)^n \to \Delta^n$ over Δ^n. Geometrically, the sections $\exp(S_i^n(l))$, $i = 0, \dots, n$, at a point $l = (l_0, \dots, l_n) \in \Delta^n$ decompose the fiber over each point, which is a circle $\mathbb{T}$ of unit perimeter with a fixed zero point, into intervals $L_i(l)$ of lengths l_i, i.e., the intervals $L_i(l) = [\exp(S_i^n(l)), \exp(S_{i+1}^n(l))] \subseteq \mathbb{T}$. This object can be regarded as Kontsevich's [19] oriented metric n-polygon $mp(l)$ (Fig. 6) of unit perimeter with fixed zero vertex.

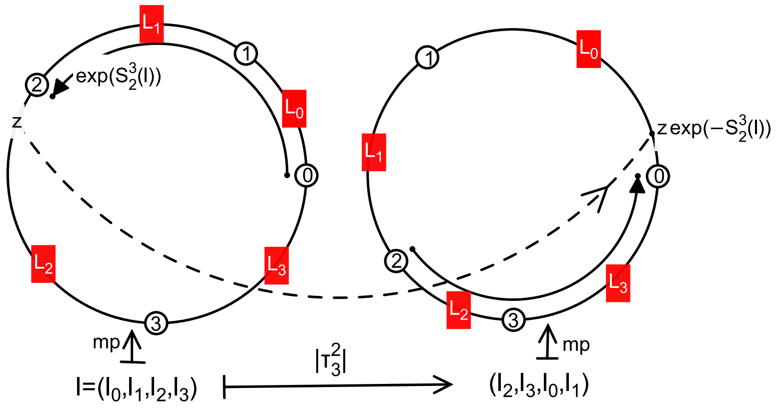

Fig. 6. The bundle map $\boldsymbol{T}C(\tau_3^2)$.

Consider the diagram $\boldsymbol{T}C$ of trivial $\mathbb{T}$-bundles and nontrivial gauge transformations over a cyclic semi-cosimplex $\triangle C$. For $[n-1] \xrightarrow{\partial_i} [t]$, put

$$\boldsymbol{T}^{n-1} \xrightarrow{\boldsymbol{T}C(\partial_i)} \boldsymbol{T}^n : \boldsymbol{T}C(\partial_i)(z|l) = (z||\partial_i|(l)). \tag{20}$$

For $[n] \xrightarrow{\tau_n^i} [n]$, put

$$\boldsymbol{T}^n \xrightarrow{\boldsymbol{T}C(\tau_n^i)} \boldsymbol{T}^n : \boldsymbol{T}C(\tau_n^i)(z|l) = \left(z \cdot \exp(-S_i^n(l))||\tau_n^i|(l)\right). \tag{21}$$

One can imagine the bundle maps $\boldsymbol{T}C(\tau_n^i)$ being constructed as follows: first we rotate the fiber of $\boldsymbol{T}^n$ over $l \in \Delta^n$ by the angle $\exp(-S_i^n(l))$, sending $\exp(S_i^n(l))$ to the zero section, after which we make the cyclic permutation $|\tau_n^i|$ of coordinates in the base, causing a proper cyclic renumbering of the intervals and sections in the fiber (see Fig. 6).

The diagram $\boldsymbol{T}C$ is a correct $\underline{\boldsymbol{\Delta}}C$-diagram of trivial $\mathbb{T}$-bundles over simplices and gauge transformations over the cyclic semi-cosimplex $\triangle C$. This fact can be proved by checking the composition law vs. cyclic identities.

24. The circle bundle $\boldsymbol{T}^{gp}(\mathcal{W}(\mathfrak{p}, \mathbf{S}^0))$ is the pullback of the universal bundle $\boldsymbol{T}C$ by a map composed from the matrices of words $\overline{\mathcal{L}}(\dots)$. The circle bundle $\boldsymbol{T}^{gp}(\mathcal{W}(\mathfrak{p}, \mathbf{S}^0))$ was defined in p. **22**, its linear $\mathbb{T}$-transition maps are given by (19), using a boundary change of 0-sections in the geometric proportions metric. What remains of the geometric constructions now, is just to observe that the bundle $\boldsymbol{T}^{gp}(\mathcal{W}(\mathfrak{p}, \mathbf{S}^0))$ can be canonically regarded as a pullback of the universal cyclic bundle $\boldsymbol{T}C$ so that the linear operators of the normalized matrices of the words $\overline{\mathcal{L}}(\dots)$ compose into the classifying map $\overline{\mathcal{L}}\mathcal{W}(\mathfrak{p}, \mathbf{S}_0))$.

Let us make a few simple observations on linear operators that come from the reduced matrices of words. Consider a boundary morphism and a cyclic shift of words:

$$
\begin{tikzcd}
{[n_0]} \arrow[r, "\partial_\delta"] \arrow[d, "w_0"'] & {[n]} \arrow[d, "w"] \\
{[k_0]} \arrow[r, "\delta"] & {[k],}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[n]} \arrow[rr, "\tau_n^i"] \arrow[dr, "w_0"'] & & {[n]} \arrow[dl, "w"] \\
& {[k].} &
\end{tikzcd}
$$

Then the following diagrams of linear maps in barycentric coordinates are commutative:

$$
\begin{tikzcd}
\Delta^{n_0} \arrow[r, "|\partial_\delta|"] & \Delta^n \\
\Delta^{k_0} \arrow[r, "|\delta|"] \arrow[u, "\overline{\mathcal{L}}(w_0)"] & \Delta^k, \arrow[u, "\overline{\mathcal{L}}(w)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\Delta^n \arrow[rr, "|\tau_n^i|"] & & \Delta^n \\
& \Delta^k. \arrow[ul, "\overline{\mathcal{L}}(w_0)"] \arrow[ur, "\overline{\mathcal{L}}(w)"'] &
\end{tikzcd}
\tag{22}
$$

These diagrams induce pullback diagrams of boundary gauge transformations:

$$
\begin{tikzcd}
\boldsymbol{T}^{n_0} \arrow[r, "\boldsymbol{T}C(\partial_\delta)"] & \boldsymbol{T}^n \\
\boldsymbol{T}^{k_0} \arrow[r, "\boldsymbol{T}_\delta"] \arrow[u, "\overline{\mathcal{L}}_*(w_0)"] & \boldsymbol{T}^k, \arrow[u, "\overline{\mathcal{L}}_*(w)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\boldsymbol{T}^n \arrow[r, "\boldsymbol{T}C(\tau_n^i)"] & \boldsymbol{T}^n \\
\boldsymbol{T}^k \arrow[r, "\boldsymbol{T}^{gp}(\tau_n^i)"] \arrow[u, "\overline{\mathcal{L}}_*(w_0)"] & \boldsymbol{T}^k. \arrow[u, "\overline{\mathcal{L}}_*(w)"']
\end{tikzcd}
\tag{23}
$$

Assume that we are given a coloring of the locally ordered simplicial complex $\mathfrak{B}$ by cyclic diagrams of words $\int^{\boldsymbol{\Delta}} \mathfrak{B} \xrightarrow{\mathcal{W}} C\mathcal{W}$ (see p. **18**). We have two functors $\int^{\boldsymbol{\Delta}} \mathfrak{B} \xrightarrow{\triangle\mathcal{W}, \triangle\mathfrak{B}} \mathbf{Top}$:

$$
\begin{aligned}
\triangle\mathcal{W} &= \left(\int^{\boldsymbol{\Delta}} \mathfrak{B} \xrightarrow{\mathcal{W}} C\mathcal{W} \xrightarrow{\mathrm{Dom}} \underline{\boldsymbol{\Delta}}C \xrightarrow{\triangle C} \mathbf{Top} \right), \\
\triangle\mathfrak{B} &= \left(\int^{\boldsymbol{\Delta}} \mathfrak{B} \to \underline{\boldsymbol{\Delta}} \xrightarrow{\triangle} \mathbf{Top} \right).
\end{aligned}
$$

The diagrams (22) are the data of the natural transformation $\triangle\mathfrak{B} \xrightarrow{\mathcal{L}} \triangle\mathcal{W}$, and the diagrams (23) are the data of the pullback $\boldsymbol{T}\mathcal{W} \xrightarrow{\mathcal{L}_*} \boldsymbol{T}C$ of the bundle diagram $\boldsymbol{T}C$ to the PD circle bundle $\boldsymbol{T}\mathcal{W}$ on $|\mathfrak{B}|$, defined by the transition functions encoded in $\mathcal{W}$. Therefore, starting from $\mathcal{W}(\mathfrak{p}, \mathbf{S}_0)$ we obtain the circle bundle $\boldsymbol{T}^{gp}(\mathcal{W}(\mathfrak{p}, \mathbf{S}_0))$ as a pullback of $\boldsymbol{T}C$ defined by the normalized matrices of the words corresponding to elementary subbundles with fixed 0-sections.

§6. Some linear algebra

6.1. Universal cyclic invariant characteristic forms. We wish to find a PD connection form on $\boldsymbol{T}C$. Observe that, strictly speaking, the bundle $\boldsymbol{T}C$ over a cyclic semi-cosimplex $\triangle C$ is not a bundle, but a diagram of trivial $\mathbb{T}$-bundles over simplices and transition gauge transformations. So the natural definition for a PD connection form on $\boldsymbol{T}C$ is a family of connections $\gamma_n \in \Omega^1\boldsymbol{T}^n$, $n = 0, 1, \ldots$, such that

$$\boldsymbol{T}C(\tau_n^i)^*\gamma_n = \gamma_n, \quad i = 0, \ldots, n;$$
$$\boldsymbol{T}C(\partial)^*\gamma_n = \gamma_m \quad \text{for every } ([m] \xrightarrow{\partial} [n]).$$

One may call such a connection a "cyclic invariant connection." This property holds, for instance, for the connection form "α" on metric polygons, see [19, p. 8], which we slightly recompile. If now γ_n is a cyclic invariant connection, then any power of its curvature form $\omega^h = \wedge^h d\gamma_n$ should be a PD cyclic invariant form on $\triangle C$.

Lemma 6.1. *The family of connection forms $\alpha_n \in \Omega^1\boldsymbol{T}^n$ defined in local coordinates $(x; l_0, \ldots, l_n)$ on $\boldsymbol{T}^k = \mathbb{T} \times \Delta^k$ by the expression*

$$\alpha_n = -dx - \sum_{0\leqslant i<j\leqslant n} l_i dl_j \tag{24}$$

is a universal cyclic invariant. Its curvature form has the expression

$$d\alpha_n = \omega_n = -\sum_{0\leqslant i<j\leqslant n} dl_i \wedge dl_j \in \Omega^2\Delta^n. \tag{25}$$

The power of the curvature form has the expression

$$\omega_n^h = (-1)^h h! \sum_{0\leqslant i_1<i_2<\cdots<i_{2h}\leqslant n} dl_{i_1} \wedge dl_{i_2} \cdots \wedge dl_{i_{2h}} \in \Omega^{2h}\Delta^n. \tag{26}$$

Proof. The form α_n is a Bott-style constructed universal form. The group of bundle automorphisms $\boldsymbol{T}^n C(\mathbb{Z}/(n+1)\mathbb{Z})$ of $\boldsymbol{T}^n$ acts on connection forms lying in $\Omega^1\boldsymbol{T}^n$, and we wish to find a connection invariant under this action. We may try to construct an invariant convex combination of elements of the orbit of the Maurer–Cartan horizontal connection $-dx$: for $i = 1, \dots, n+1$, put $\boldsymbol{T}C(\tau_n^i)^*(-dx) = -dx - dl_{k-i+1} - \dots - dl_n$, and take a smooth convex combination of connections in the orbit:

$$\begin{array}{rlll} & l_n & \times & (-dx) \\ & l_{n-1} & \times & (-dx - dl_n) \\ + & l_{n-2} & \times & (-dx - dl_{n-1} - dl_n) \\ & \dots & \dots & \dots \\ & l_0 & \times & (-dx - dl_1 - \dots - dl_n) \\ \hline & \multicolumn{3}{l}{-dx - \sum\limits_{0 \leqslant i < j \leqslant n} l_i dl_j.} \end{array} \tag{27}$$

The sum is the connection form $\alpha_n = -dx - \sum\limits_{0 \leqslant i < j \leqslant n} l_i dl_j$.

The fact that $\boldsymbol{T}C(\partial)^*\alpha_n = \alpha_m$ for any boundary $m \xrightarrow{\partial} n$ follows from the definition of $\boldsymbol{T}C(\partial)$ in $(z|l)$ coordinates (20). We now need to check that α_n is cyclic-invariant: $\boldsymbol{T}C(\tau_n^i)^*\alpha_n = \alpha_n$, see (21). It is sufficient to ensure that this is true for the generator, i.e., that $\boldsymbol{T}C(\tau_n)^*\alpha_n = \alpha_n$. We calculate:

$$\begin{aligned} \boldsymbol{T}C(\tau_n)^*\alpha_n &= -dx - dl_n - l_n(dl_0 + \dots + dl_{n-1}) - \sum_{0 \leqslant i < j \leqslant n-1} l_i dl_j, \\ \alpha_n &= -dx - \sum_{0 \leqslant i < j \leqslant n-1} l_i dl_j - (l_0 + \dots l_{n-1}) dl_n. \end{aligned} \tag{28}$$

Taking into account that the coordinates are barycentric and substituting $l_0 = 1 - l_1 - \dots - l_n$ in (28), we obtain equal expressions in both cases:

$$\begin{aligned} \boldsymbol{T}C(\tau_n)^*\alpha_n &= -dx - dl_n + l_n dl_n - \sum_{0 \leqslant i < j \leqslant n-1} l_i dl_j, \\ \alpha_n &= -dx - \sum_{0 \leqslant i < j \leqslant n-1} l_i dl_j - dl_n + l_n dl_n. \end{aligned} \tag{29}$$

The transgression of α_n is the simplectic 2-form

$$\omega_n = d\alpha_n = -\sum_{0 \leqslant i < j \leqslant n} dl_i \wedge dl_j$$

on the base, which is a $\underline{\mathbf{\Delta}}C$-form, since α is a $\boldsymbol{T}C$-form; its pullback to the bundle is the curvature of α_n (it is obvious that we can consider it instead of the curvature). The power of ω is obtained by a standard Grassmann algebra calculation as the power of the Grassmann quadratic form, providing the factor $(-1)^h h!$. □

6.2. The sum of minors Pfaffian identity and the "matrix parity" rational function. Let $X = X^{[n]\times[k]}$ be an $[n]\times[k]$ matrix of variables x^i_j, $i \in [n]$, $j \in [k]$. We suppose that $n \geqslant k$. Let $\left[{n+1 \atop k+1}\right]$ be the set of all $(k+1)$-element subsets of $[n]$. Let $D^{\mathbf{a}} \in \mathbb{Z}[X]$ be the maximal minor of X with rows numbered by $\mathbf{a} \in \left[{n+1 \atop k+1}\right]$, regarded as a polynomial. Consider the polynomial $\boldsymbol{s} = \sum_{\mathbf{a}\in\left[{n \atop k}\right]} D^{\mathbf{a}}(X) \in \mathbb{Z}[X]$, the sum of all maximal minors. The polynomial $\boldsymbol{s}$ can be expressed as the Pfaffian polynomial of an even skew-symmetric matrix in the variables X. This is Okada's sum of minors Pfaffian identity [25, Theorem 3], [15]. Assume that $\mathbf{u} \subseteq [k]$ and denote by $X_{\mathbf{u}}$ the submatrix of the variables X formed by the columns with numbers in $\mathbf{u}$, and by $\boldsymbol{s}_{\mathbf{u}} \in \mathbb{Z}[X]$, the sum of the maximal minors polynomial for $X_{\mathbf{u}}$.

If $k+1$ is odd, then

$$\boldsymbol{s} = \mathrm{Pf}\begin{pmatrix} 0 & \boldsymbol{s}_{\{0\}} & \boldsymbol{s}_{\{1\}} & \boldsymbol{s}_{\{2\}} & \cdots & \boldsymbol{s}_{\{k\}} \\ -\boldsymbol{s}_{\{0\}} & 0 & \boldsymbol{s}_{\{0,1\}} & \boldsymbol{s}_{\{0,2\}} & \cdots & \boldsymbol{s}_{\{0,k\}} \\ -\boldsymbol{s}_{\{1\}} & -\boldsymbol{s}_{\{0,1\}} & 0 & \boldsymbol{s}_{\{1,2\}} & \cdots & \boldsymbol{s}_{\{1,k\}} \\ -\boldsymbol{s}_{\{2\}} & -\boldsymbol{s}_{\{0,2\}} & -\boldsymbol{s}_{\{1,2\}} & 0 & \cdots & \boldsymbol{s}_{\{2,k\}} \\ & & & \cdots & & \\ -\boldsymbol{s}_{\{k\}} & -\boldsymbol{s}_{\{0,k\}} & -\boldsymbol{s}_{\{1,k\}} & -\boldsymbol{s}_{\{2,k\}} & \cdots & 0 \end{pmatrix}. \tag{30}$$

If $k+1$ is even, then

$$\boldsymbol{s} = \mathrm{Pf}\begin{pmatrix} 0 & \boldsymbol{s}_{\{0,1\}} & \boldsymbol{s}_{\{0,2\}} & \cdots & \boldsymbol{s}_{\{0,k\}} \\ -\boldsymbol{s}_{0,1} & 0 & \boldsymbol{s}_{\{1,2\}} & \cdots & \boldsymbol{s}_{\{1,k\}} \\ -\boldsymbol{s}_{\{0,2\}} & -\boldsymbol{s}_{\{1,2\}} & 0 & \cdots & \boldsymbol{s}_{\{2,k\}} \\ & & \cdots & & \\ -\boldsymbol{s}_{\{0,k\}} & -\boldsymbol{s}_{\{1,k\}} & -\boldsymbol{s}_{\{2,k\}} & \cdots & 0 \end{pmatrix}. \tag{31}$$

We will use the defining recursive identity for the Pfaffian of a skew-symmetric $2m \times 2m$ matrix M:

$$\mathrm{Pf}(M) = \sum_{j=2}^{2m} (-1)^j a_{1,j} \mathrm{Pf}(M_{\hat{1},\hat{j}}), \tag{32}$$

where $M_{\hat{1},\hat{j}}$ denotes the matrix M with both the 1st and the jth rows and columns removed.

Denote by $\delta_j^* X$ the $(n+1) \times k$ matrix obtained by deleting the jth column from X. Applying (32) to the right-hand side of (30), and replacing the Pfaffians by the sums of minors from the left-hand side of (30), (31), we obtain the following identity if $k+1$ is odd:

$$\boldsymbol{s} = \sum_{i=0}^{k} (-1)^j \boldsymbol{s}_{\{j\}} \boldsymbol{s}(\delta_j^* X)). \tag{33}$$

Define a rational function of a matrix, the "matrix rational parity function," by the formula

$$P = \frac{\boldsymbol{s}}{\prod_{j=0}^{k} \boldsymbol{s}_{\{j\}}}. \tag{34}$$

Important properties of the matrix rational parity are given by the following lemma.

Lemma 6.2.
(a) *If $k+1$ is odd, then $P(X)$ is invariant under cyclic permutations of rows.*
(b)

$$\sum_{j=0}^{k} (-1)^j P(\delta_j^* X) = \begin{cases} P(X) & \text{if } k+1 \text{ is odd,} \\ 0 & \text{if } k+1 \text{ is even.} \end{cases} \tag{35}$$

Proof. (a) The determinant of an odd-dimensional matrix is invariant under cyclic permutations. Therefore, if $k+1$ is odd, then the sum of the maximal minors of X is invariant under cyclic permutations of rows.

(b) If $k+1$ is odd, then we can take expression (33) and divide both sides by $\prod_{i=0}^{k+1} \boldsymbol{s}_{\boldsymbol{j}}$, resulting in the required identity. If $k+1$ is even, then $\delta_j^* X$ has an odd number of columns; hence, by the odd case, we have the cocycle condition on the parity. $\square$

6.3. The pullback of the univesal cyclic characteristic forms by a matrix map.

25. Let us have an $[n]\times[2h]$ matrix $A=\{a_i^j\}$, $i\in 0,1,\dots,n$, $j\in 0,\dots,2h$, of nonnegative reals. We suppose that $n\geqslant 2h$ and $\sum\limits_{i=0}^{n} a_i^j=1$, $j=0,\dots,2h$. Regard A as a linear map in barycentric coordinates $\Delta^{2h}\xrightarrow{A}\Delta^n$, where $t_0,\dots,t_{2h}$ are the barycentric coordinates on Δ^{2h} and $l_0,\dots,l_n$ are coordinates on Δ^n:

$$t=(t_0,\dots,t_{2h})\overset{A}{\mapsto}(l_0(t),\dots l_n(t)), l_i(t)=a_i^0t_0+a_i^1t_1+\dots+a_i^{2h}t_{2h}.$$

We wish to compute the pullback $A^*\omega^h$ of the hth power of the curvature (or transgression) form (26) in the standard coordinates $t_0,t_1,\dots,t_{2h-1}$ on Δ^{2h}. Denote by $\boldsymbol{s}(A)$ the sum of the maximal minors of the matrix A.

Lemma 6.3.

$$A^*\omega_n^h=(-1)^h h!\boldsymbol{s}(A)dt_0\wedge dt_1\wedge\dots\wedge dt_{2h-1}.$$

Proof. We compute the summands in the sum

$$\sum_{0\leqslant i_1<i_2<\dots<i_{2h}\leqslant n} dl_{i_1}(t)\wedge dl_{i_2}(t)\cdots\wedge dl_{i_{2h}}(t) \tag{36}$$

from (26) corresponding to all $(2h)\times(2h+1)$ submatrices of A, and then we apply the identity of Lemma 6.2. To describe a summand, we first assume that $n=2h-1$ and compute $dl_0(t)\wedge dl_2\cdots\wedge dl_{2h-1}(t)$. Let δ_j^*A, $j=0,\dots,2h$, be the square $2h\times 2h$ matrix obtained from A by deleting the jth column. We denote $\delta_j^*dt=dt_0\wedge dt_{j-1}\wedge dt_{j+1}\wedge\dots\wedge dt_{2h}$. Then, by the Grassmann algebra rules,

$$dl_0(t)\wedge dl_2(t)\cdots\wedge dl_{2h-1}(t)=\sum_{j=0}^{2h}\det(\delta_j^*A)\delta_j^*dt.$$

Substituting $t_{2h}=1-t_0-\dots-t_{2h-1}$ into the right-hand side, we obtain

$$\begin{aligned} &dl_0(t)\wedge dl_2(t)\wedge\dots\wedge dl_{2h-1}(t)\\ &\qquad=\Bigl(\sum_j(-1)^j\det(\delta_j^*A)\Bigr)dt_0\wedge\dots\wedge dt_{2h-1}. \end{aligned}\tag{37}$$

Now we assume that $n\geqslant 2h$ and apply (37) to each summand of (36). Finally, using Lemma 6.2 in the odd case and keeping in mind the condition that the sums of elements in columns of A are equal to 1 (and hence the denominator in expression (34) for the matrix parity is equal to 1), we obtain the desired expression for $A^*\omega_n^h$. □

§7. Proof of Theorem 4.1

We check that ${}^{p}C_1^{2h}$ satisfies the definition of a rational simplicial local formula from p. **8**.

26. First, we check that ${}^{p}C_1^{2h}$ is a rational simplicial $2h$-cocycle on $\mathfrak{R}^c(\overrightarrow{\mathrm{S}})$. Let $\mathfrak{N}$ be the semi-simplicial set of isomorphism classes of necklaces. Then $\mathfrak{N}_k$ (see p. **12**) is the set of isomorphism classes of all finite necklaces with beads colored by $[k]$. The boundary map ∂_i^* is induced from the corresponding boundary on words, i.e., by deleting all beads with color i. Let $K_\Delta^\bullet(\mathfrak{N};\mathbb{Q})$ be the rational simplicial cochain complex of $\mathfrak{N}$. Then for a word $w \in C\mathcal{W}_{2h}$ (i.e., an "odd word," a word in the alphabet of $2h+1$ letters), the rational parity $P(w)$ is an invariant of the isomorphism class of the oriented "odd necklace" defined as the cyclic orbit of w. Therefore, the rational parity of odd necklaces is a function $(\mathfrak{N}_{2h} \xrightarrow{P^{2h}} \mathbb{Q}) \in K_\Delta^{2h}(\mathfrak{N};\mathbb{Q})$. The rational cochain P^{2h} is a simplicial cocycle, this follows from Lemma 6.2 (even case) applied to the matrix representation of the rational parity function (12) and the matrix representation of the boundary of a word, see p. **13**. Associating the necklace $\mathcal{N}(\mathfrak{e})$ to an elementary s.c. bundle $\mathfrak{e}$ (see p. **16**) sends the isomorphism class of the bundle to the isomorphism class of the necklace and the boundary to the boundary. Hence it defines a map of semi-simplicial sets $\mathfrak{R}^c(\overrightarrow{\mathrm{S}}) \xrightarrow{\mathcal{N}} \mathfrak{N}$. So we get the pullback $2h$-cocycle $P^{2h}(\mathcal{N}(-)) \in K^{2h}(\mathfrak{R}^c;\mathbb{Q})$. The $2h$-chain ${}^{p}C_1^{2h}$ (33) is proportional to the cocycle $P^{2h}(\mathcal{N}(-))$; therefore, it is a rational simplicial $2h$-cocycle on $\mathfrak{R}^c(\overrightarrow{\mathrm{S}})$.

27. We need to prove that for a s.c. bundle $\mathfrak{R} \xrightarrow{\mathfrak{p}} \mathfrak{B}$ the pullback of ${}^{p}C_1^{2h}$ by the map $\mathfrak{G}_\mathfrak{p}$ is a simplicial cochain on $\mathfrak{B}$ representing $c_1^{2h}(\mathfrak{p})$. For the first Chern class, the formula can be guessed and then checked on the Madahar–Shakaria triangulation of the Hopf bundle [24]. For higher classes, we can be sure only that ${}^{p}C_1^{2h}$ are some universal cocycles. We are not sure about the homotopy class of $\mathfrak{N}$ or $\mathfrak{R}^c(\overrightarrow{\mathrm{S}})$, we have no good series of examples of triangulated circle bundles to check. The latter fact is related to the well-known problem of triangulating the complex projective spaces $\mathbb{C}P^n$. It is very hard to triangulate $\mathbb{C}P^n$ ([29]), it is much harder to triangulate Hopf circle bundles over them. Also, it is diffucult to compare the formulas with the simplicial cup product of the first class; this is related to well-known problems on formulas for the cup product.

28. What we can do now, is to use the Chern–Weil homomorphism for Kontsevich's connection form α on metric polygons and then use de Rham's theorem. To this end, we have discussed cyclic bundle geometry in Sec. 5 and linear algebra in Sec. 6.

(1) The piecewise differential Chern–Weil homomorphism for piecewise differential principal bundles exists as a byproduct of the Chern–Weil homomorphism for simplicial manifold principal bundles ([5]).
(2) We choose a system $\mathbf{S}_0$ of 0-sections of elementary s.c. subbundles of $\mathfrak{p}$, and obtain the cyclic diagram of words $\mathcal{W}(\mathfrak{p}, \mathbf{S}_0)$ on $\mathfrak{B}$ (see p. **18**).
(3) We choose the geometric proportions metric gp on $|\mathfrak{E}|$. The normalized matrix of a word $\overline{\mathfrak{L}}(\mathcal{W}(\mathfrak{R} \xrightarrow{\mathfrak{e}} \langle k \rangle, S_0))$ of an elementary bundle $\mathfrak{e}$ with fixed combinatorial section S_0 applied as a linear operator to a point of the base simplex Δ^k produces the vector of distances between the 0-sections of $|\mathfrak{R}|$ in the metric gp ordered by the orientation (see p. **21**). Changing the section S_0 results in a cyclic permutation of this vector. This is a point of communication between simplicial and cyclic geometry.
(4) With $\mathcal{W}(\mathfrak{p}, \mathbf{S}_0)$ we associate the PD circle bundle $\mathbf{T}^{gp}(\mathcal{W}(\mathfrak{E}, \mathbf{S}_0))$ on $|\mathfrak{B}|$ defined as a diagram of trivial bundles over simplices and transition gauge transformations defined by changing combinatorial sections (see p. **22**). The circle bundle $\mathbf{T}^{gp}(\mathcal{W}(\mathfrak{p}, \mathbf{S}_0))$ is canonically triangulated by $|\mathfrak{p}|$.
(5) In p. **24** we obtained the bundle $\boldsymbol{T}^{gp}(\mathcal{W}(\mathfrak{p}, \mathbf{S}_0))$ as a diagram pullback
$$\mathbf{T}^{gp}(\mathcal{W}(\mathfrak{p}, \mathbf{S}_0)) \xrightarrow{\overline{\mathfrak{L}}_*} \boldsymbol{T}C$$
of the universal cyclic bundle diagram $\boldsymbol{T}C$ over the cyclic semi-co-simplex $\triangle C$. The diagram morphism $|\mathfrak{B}| \approx \triangle \mathfrak{B} \xrightarrow{\overline{\mathfrak{L}}(\mathcal{W}(\mathfrak{p}, \mathbf{S}_0))} \triangle C$ on the simplex $\triangle(U^k)$ of the base is a linear operator
$$\Delta^k \xrightarrow{\overline{\mathfrak{L}}(\mathcal{W}(\mathfrak{p}_U, S_0^U))} \Delta^n.$$
Here $\mathfrak{p}_U$ is the elementary s.c. subbundle of $\mathfrak{p}$ over U and S_0^U is its fixed section, while $n+1$ is the number of combinatorial 0-sections of $\mathfrak{p}_U$, the same as the total number of letters in the word $\mathcal{W}(\mathfrak{p}_U, S_0^U)$.

(6) The cyclic invariance of Kontsevich's connection α on $\boldsymbol{TC}$ (Lemma 6.1) means that its indivdulal pullbacks

$$\alpha_U = \overline{\mathcal{L}}(\mathcal{W}(\mathfrak{p}_U, S_0^U)^* \alpha \in \Omega^1(\boldsymbol{T}^{gp}(\mathcal{W}(\mathfrak{p}_U, S_0^U); \mathbb{Q}) \approx \Omega^1(\mathbb{T} \times \Delta^k; \mathbb{Q}), \quad U \in \mathfrak{B},$$

are invariant under changing the fixed section S_0^U and, therefore, under all transition gauge transformations. Hence the pullbacks α_U, $U \in \mathfrak{B}$, assemble into a rational PD connection on the PD circle bundle $\boldsymbol{T}^{gp}(\mathcal{W}(\mathfrak{p}, \mathbf{S}_0))$ invariant under all gauge tranformations caused by changing systems of sections $\mathbf{S_0}$. We can apply the PD Chern–Weil homomorphism and deduce that the powers

$$\omega_U^h = \overline{\mathcal{L}}(\mathcal{W}(\mathfrak{p}_U, S_0^U)^* \omega^h \in \Omega^{2h}(\triangle U; \mathbb{Q})$$

of the curvature $\omega_U = d\alpha_U \in \Omega^2(\triangle U; \mathbb{Q})$ assemble into a rational PD form in $\Omega^{2h}(|\mathfrak{B}|; \mathbb{Q})$ representing the rational h-power of the first Chern class

$$c_1^{2h}(\boldsymbol{T}^{gp}(\mathcal{W}(\mathfrak{p}, \mathbf{S}_0)); \mathbb{Q}) = c_1^{2h}(\mathfrak{p}; \mathbb{Q}) \in H^{2h}(|\mathfrak{B}|; \mathbb{Q}).$$

(7) We can now apply the de Rham–Weyl–Dupont–Sullivan homotopy between $\Omega_{PD}(|\mathfrak{B}|; \mathbb{Q})$ and $K_\Delta(\mathfrak{B}; \mathbb{Q})$, obtaining the simplicial cocycles representing $c^h(\mathfrak{p}; \mathbb{Q})$, by integrating the forms ω_U^h over the base simplices. This gives zero if the dimension of the base simplex is not equal to $2h$. Thus we arrive at computing the pullbacks of the universal cyclic characteristic form

$$\omega^h(\mathcal{W}(\mathfrak{e}, S_0)) = \overline{\mathcal{L}}(\mathcal{W}(\mathfrak{e}, S_0))^* \omega_n^h$$

for an elementary c.s. bundle $\mathfrak{e}$ over $2h$-simplices having $n+1$ 0-sections and integrating them over the base simplices. The form $\omega^h(\mathcal{W}(\mathfrak{e}, S_0))$ is invariant under changing the base section S_0, therefore, the resulting number is an invariant of the necklace $\mathcal{N}(\mathfrak{e})$. The pullback of the cyclic form ω_n^h by the matrix map on Δ^{2h} was computed in Lemma 6.3 of p. **25** using the sum of minors Pfaffian identity. The result is

$$\omega^h(\mathcal{W}(\mathfrak{e}, S_0)) = (-1)^h h! \mathbf{s}(\overline{\mathcal{L}}(\mathcal{W}(\mathfrak{e}, S_0)) dt_0 \wedge \cdots \wedge dt_{2h-1}.$$

Here $s(\overline{\mathcal{L}}(\mathcal{W}(\mathfrak{e}, S_0))$ is the sum of the maximal minors of the normalized matrix of an odd word. This number is equal to the rational parity of the necklace (12):

$$s(\overline{\mathcal{L}}(\mathcal{W}(\mathfrak{e}, S_0)) = P(\mathcal{N}(\mathfrak{e})).$$

The factor $h!$ appears from the power of the Grassmann quadratic form, and $(-1)^h$ comes from the change of coordinates rule for the universal cyclic connection. What remains is to integrate the constant $2h$-form over the $2h$-simplex, which adds the volume $\frac{1}{2h!}$ of the standard $2h$ simplex as a factor, and the promised local simplicial expression (13) for $c_1^h(\mathfrak{p};\mathbb{Q})$ as ${}^{p}C_1^h(\mathfrak{e})$ is ready.

§8. Notes

29. Here we swept under the carpet an appropriate version of PL simplicial bundle theory. Although we need it only in an elementary form and in the one-dimensional case, it still requires space for the setup. Simplicial bundle theory is a parametric extension of the simple homotopy theory of families of simple maps. It was presented in [31] and commented on in the lectures [22]. Simple maps pop up in the description of the boundary of an elementary s.c. bundle (see p. **17**) and in the one-dimensional case relate simplicial bundle combinatorics to cyclic category, this is what we are actually investigating. In our case, the adequate variant would be semi-simplicial, which has not yet been fixed. A semi-simplicial bundle is a singular map of semi-simplicial sets (the same as a map of "n.d.c. simplicial sets" of [26], or a "trisps map" of [20]). The semi-simplicial circle bundles on a given base are in a one-to-one correspondence with the cyclic decorations of the base by words.

30. Modulo the hidden semi-simplicial setup, we can formulate a couple of facts which we hope to write out somewhere in future.

Since Chern classes are integer classes, the corresponding simplicial cochains, represented by any rational local formulas, should have integer simplicial periods, i.e., they are integrated to integer numbers over all integer $2h$-simplicial cycles in the base. When the base is some triangulation of an oriented closed surface, this is a version of the combinatorial Gauss–Bonnet theorem. This fact coupled with the simple expressions ${}^{p}C_1(\mathfrak{e})$ provides some understanding which bundles have or have not a triangulation over a particular simplicial base:

> Let $|\mathfrak{B}|$ be an oriented two-dimensional closed surface triangulated by a classical simplicial complex $\mathfrak{B}$, and let the complex $\mathfrak{B}$ have F triangles. In this situation, F is always even. Then the Chern number of a classically triangulated

circle bundle over $|\mathfrak{B}|$ having $\mathfrak{B}$ as the base complex belongs to the integer interval $[-\frac{1}{2}F + 1, \dots, \frac{1}{2}F - 1]$.

Moreover, the Chern numbers of semi-simplicially triangulated circle bundles over $|\mathfrak{B}|$ having $\mathfrak{B}$ as the base complex fill the entire integer interval $[-\frac{1}{2}F, \dots, \frac{1}{2}F]$. In this situation, $\mathfrak{B}$ can be assumed to be a finite semi-simplicial set, and $|\mathfrak{B}|$ is a "Δ-complex" in the sense of [10].

The only concrete example of a triangulated circle bundle observable in the literature is the triangulation of the Hopf bundle over the boundary of the tetrahedron $\partial\Delta^3$, constructed in [24]. The parity local formulas allow one to deduce that the cited result is the best possible. From the above statement one may conclude that over $\partial\Delta^3$ one can triangulate only the trivial bundle and the Hopf bundle using a map of classical simplicial complexes. If one can use semi-simplicial triangulations, then over $\partial\Delta^3$ one can additionally triangulate the circle bundle associated to the tangent bundle of the 2-sphere, and this is the complete list of circle bundles allowing a triangulation over $\partial\Delta^3$. Classical triangulations are fundamental, but have their own additional degree of interesting arithmetical complexity relative to semi-simplicial triangulations, see [16]. The semi-simplicial category is related to the classical simplicial category by functorial double normal ($\approx$ double barycentric) subdivision.

There is a somewhat strange more general statement which requires as a premise an integer combinatorial formula for the first Chern class:

If a circle bundle p has a triangulation with a simplicial locally ordered base $\mathfrak{B}$, then $c_1(p)$ can be represented by a simplicial 2-cocycle on $\mathfrak{B}$ having values 0 and 1 on 2-simplices. The inverse is true for semi-simplicial triangulations of circle bundles and not true for triangulations by classical simplicial complexes.

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Поступило 16 августа 2016 г.